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Properties of the Parametric Lyapunov Equation Based Low Gain Design with Applications in Stabilization of Time-Delay Systems

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Abstract— **This paper studies some properties of the recently developed parametric Lyapunov equation based low gain feedback design method. As applications of these new properties, alternative and simpler solutions are proposed to the (global) stabilization problem for a class of linear systems with input delay and the semi-global stabilization problem when the systems are in addition subject to actuator saturation. Besides the simplicity in their construction, the new solutions can also be easily scheduled online to achieve global result in the presence of input saturation.**

I. Introduction

Low gain feedback refers to a family of feedback gains that approach zero as a parameter, referred to as the low gain parameter, is tuned toward zero. It was originally developed to achieve semi-global stabilization of linear systems subject to actuator saturation [9], [12] and has found applications in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control [12], nonlinear stabilization [12], and, more recently, in stabilization of time-delayed systems [8].

The low gain feedback laws of [9] are constructed by eigenstructure assignment. Alternative ways of constructing low gain feedback laws were later proposed based on the solution of a parameterized $\mathcal{H}_2$ algebraic Riccati equation (ARE) [11] and $\mathcal{H}_\infty$ ARE [17], respectively. See [12] for a comparison of different low gain design approaches. Between the eigenstructure assignment approach and the ARE-based approach to low gain feedback design, the biggest advantage of the former is that it results in feedback gains that are matrix polynomials in the low gain parameter. Thus the design is non-repetitive in the sense that if the value of the low gain parameter is required to change, the design process need not be repeated. The ARE-based approach is however conceptually appealing and directly results in a Lyapunov function along with the feedback gain. However, the resulting feedback gain is indirectly dependent on the low gain parameter. For every different value of the low gain parameter, the solution of a new ARE is required. The solution of these AREs may become numerically ill-conditioned as the value of the low gain parameter becomes small. This is the case, for example, when the value of the low gain parameter is adjusted on line to achieve global results, instead of semi-global ones.

Recently, we proposed a new low gain design approach, which we referred to as the parametric Lyapunov equation based low gain design approach ([21]). This new approach possesses the advantages of both the eigenstructure assignment approach and the ARE-based approach. It leads to a feedback gain that is explicitly a function of the low gain parameter as the closed-form solution to the Lyapunov matrix equation can be easily obtained. It also directly results in a quadratic Lyapunov function for the closed-loop system.

Low gain design is important for controller design for systems with input saturation which is commonly encountered in practice. On the other hand, time-delay is also commonly encountered in practical control systems. Therefore, many control problems for systems with both input saturation and time-delay have been investigated and a lot of methods has been proposed in the literature. For example, global asymptotic stabilization of linear systems by nonlinear feedback is considered in [19], stabilization and estimation of domain of attraction for saturated and delayed linear systems is considered in [16] by using LMI's technique, output feedback stabilization of uncertain time-varying state-delayed systems with saturating actuators is studied in [4] and L_2 anti-windup design of delayed and saturated system is solved in [20]. Furthermore, as a byproduct of more general results about global stabilization of feedforward nonlinear systems, the problem of stabilization of a chain of integrators with input delays is addressed in [6] from the perspective of providing a low gain and low amplitude design.

In this paper, we further study some intricate properties of the parametric Lyapunov equation based low gain design feedback laws. As applications of these new properties, we propose alternative and simpler solutions to the (global) stabilization problem for a class of linear systems with input delay and the semi-global stabilization problem when the systems are subject to actuator saturation as well. The class of linear systems we consider have all their poles either at the origin or in the open left half plane. Solutions to the above stabilization problems for this class of systems are given in [8] by using eigenstructure assignment approach. Besides the simplicity in their construction that is inherited from the simplicity of the parametric Lyapunov equation based low gain design, the new solution to be proposed in this paper can be easily scheduled online by using the idea in [13]

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and [15] to achieve global results in the presence of actuator saturation.

A key feature of the solutions in this paper and the corresponding solutions in [8] is that the feedback laws we construct do not depend on the precise knowledge of the amount of delay, as long as an upper bound on the delay is known. As explained in [8], this is possible because of only the assumption that there are no non-zero imaginary axis open loop poles. We note that, in the presence of non-zero imaginary axis open loop poles, stabilization in the presence of input delay and global/semi-global stabilization in presence of both input delay and actuator saturation are still possible if we allow to use the time delay information in the construction of feedback laws (see, [1], [8], [14], [18]).

Notation: Throughout this paper, we use A^{T} and $\mathrm{tr}\,(A)$ to denote respectively the transpose and the trace of matrix A. For a positive scalar τ, let $\mathscr{C}_{n,\tau} = \mathscr{C}\left([-\tau, 0], \mathbf{R}^n\right)$ denote the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbf{R}^n$ with the topology of uniform convergence.

II. Properties of the Parametric Lyapunov Equation Based Low Gain Feedback

Parametric Lyapunov equation based low gain design relies on the solution to the following parametric ARE:

$$A^{\mathrm{T}}P + PA - PBB^{\mathrm{T}}P = -\gamma P, \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$ are two given constant matrices and γ is a scalar. With the solution $P(\gamma)$, a state feedback law can be constructed as follows

$$u = -B^{\mathrm{T}}P(\gamma)x. \tag{2}$$

As Proposition 1 shows, the feedback gain in (2) decreases to zero as the parameter goes to zero if all eigenvalues of A have zero real parts.

The following properties of the ARE (1) are essential in the low gain design of [21].

Proposition 1: Assume that (A, B) is controllable. Let

$$\gamma > -2\min\left\{\mathrm{Re}\left(\lambda(A)\right)\right\}, \tag{3}$$

where $\mathrm{Re}\left(\lambda(A)\right)$ denotes the set of the real parts of the eigenvalues of A. Then the ARE (1) has a unique positive definite solution $P(\gamma) = W^{-1}(\gamma)$, where $W(\gamma)$ is the unique positive solution to the following Lyapunov equation

$$W\left(A + \frac{\gamma}{2}I\right)^{\mathrm{T}} + \left(A + \frac{\gamma}{2}I\right)W = BB^{\mathrm{T}}. \tag{4}$$

Moreover, $\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma) > 0$, and

$$\mathrm{tr}\left(B^{\mathrm{T}}P(\gamma)B\right) = 2\mathrm{tr}\,(A) + n\gamma. \tag{5}$$

If, in addition, all eigenvalues of A have zero real parts, then, $\lim_{\gamma\to 0^+} P(\gamma) = 0$.

Proof: All properties, except (5), of this proposition can be found in [21]. We only need to prove (5). Since $P(\gamma)$ is positive definite, it follows from (1) that

$$A^{\mathrm{T}} + P(\gamma)AP^{-1}(\gamma) - P(\gamma)BB^{\mathrm{T}} = -\gamma I. \tag{6}$$

Taking trace on both sides of (6) gives

$$\mathrm{tr}\,(A) + \mathrm{tr}\left(P(\gamma)AP^{-1}(\gamma)\right) - \mathrm{tr}\left(P(\gamma)BB^{\mathrm{T}}\right) = -\mathrm{tr}\,(\gamma I). \tag{7}$$

By the fact $\mathrm{tr}\,(XY) = \mathrm{tr}\,(YX)$, equation (5) follows from (7). ■

Remark 1: It follows from equation (5) that

$$\begin{aligned}
& P(\gamma)BB^{\mathrm{T}}P(\gamma) \\
= \;& P^{\frac{1}{2}}(\gamma)P^{\frac{1}{2}}(\gamma)BB^{\mathrm{T}}P^{\frac{1}{2}}(\gamma)P^{\frac{1}{2}}(\gamma) \\
\leq \;& P^{\frac{1}{2}}(\gamma)\,\mathrm{tr}\left(P^{\frac{1}{2}}(\gamma)BB^{\mathrm{T}}P^{\frac{1}{2}}(\gamma)\right)P^{\frac{1}{2}}(\gamma) \\
= \;& \mathrm{tr}\left(B^{\mathrm{T}}P(\gamma)B\right)P(\gamma) \\
= \;& \left(2\mathrm{tr}\,(A) + n\gamma\right)P(\gamma),
\end{aligned} \tag{8}$$

which will be frequently used in this paper.

Based on Proposition 1, we can establish the following more intricate property of the Lyapunov equation based low gain feedback design.

Theorem 1: Assume that (A, B) is controllable and γ satisfies (3). Let $P(\gamma)$ be the unique positive definite solution to parametric ARE (1). Then

$$A_{\mathrm{c}}^{\mathrm{T}}(\gamma)P(\gamma)A_{\mathrm{c}}(\gamma) \leq \varpi(\gamma)P(\gamma), \tag{9}$$

holds true for any γ satisfying (3), where $\varpi(\gamma) = 1/2\left(n\gamma + 2\mathrm{tr}\,(A)\right)\left((n+1)\gamma + 2\mathrm{tr}\,(A)\right) + \gamma\mathrm{tr}\,(A) - \mathrm{tr}\left(A^2\right)$, and

$$A_{\mathrm{c}}(\gamma) = A - BB^{\mathrm{T}}P(\gamma). \tag{10}$$

Proof: Form (1), we have

$$P(\gamma)BB^{\mathrm{T}}P(\gamma) = A^{\mathrm{T}}P(\gamma) + P(\gamma)A + \gamma P(\gamma),$$

from which it follows that

$$\begin{aligned}
& A_{\mathrm{c}}^{\mathrm{T}}(\gamma)P(\gamma)A_{\mathrm{c}}(\gamma) \\
= \;& A^{\mathrm{T}}P(\gamma)A - A^{\mathrm{T}}P(\gamma)BB^{\mathrm{T}}P(\gamma) \\
& -P(\gamma)BB^{\mathrm{T}}P(\gamma)A + P(\gamma)BB^{\mathrm{T}}P(\gamma)BB^{\mathrm{T}}P(\gamma) \\
= \;& P(\gamma)BB^{\mathrm{T}}P(\gamma)BB^{\mathrm{T}}P(\gamma) - A^{\mathrm{T}}P(\gamma)BB^{\mathrm{T}}P(\gamma) \\
& +A^{\mathrm{T}}P(\gamma)A - \left(A^{\mathrm{T}}P(\gamma) + P(\gamma)A + \gamma P(\gamma)\right)A \\
= \;& -P(\gamma)A^2 - \gamma P(\gamma)A + P(\gamma)ABB^{\mathrm{T}}P(\gamma) \\
& +\gamma P(\gamma)BB^{\mathrm{T}}P(\gamma) \\
= \;& -P(\gamma)AA_{\mathrm{c}}(\gamma) - \gamma P(\gamma)A \\
& +\gamma P(\gamma)BB^{\mathrm{T}}P(\gamma).
\end{aligned} \tag{11}$$

Rewrite equation (1) as

$$A_{\mathrm{c}}^{\mathrm{T}}(\gamma) = \left(-\gamma P(\gamma) - P(\gamma)A\right)P^{-1}(\gamma),$$

substitution of which in (11) and using (5) give

$$\begin{aligned}
& A_{\mathrm{c}}^{\mathrm{T}}(\gamma)P(\gamma)A_{\mathrm{c}}(\gamma) \\
= \;& P(\gamma)AP^{-1}(\gamma)\left(A^{\mathrm{T}}P(\gamma) + \gamma P(\gamma)\right) \\
& -\gamma P(\gamma)A + \gamma P(\gamma)BB^{\mathrm{T}}P(\gamma) \\
= \;& P^{\frac{1}{2}}(\gamma)\Big(P^{\frac{1}{2}}(\gamma)AP^{-1}(\gamma)A^{\mathrm{T}}P^{\frac{1}{2}}(\gamma) \\
& +\gamma P^{\frac{1}{2}}(\gamma)BB^{\mathrm{T}}P^{\frac{1}{2}}(\gamma)\Big)P^{\frac{1}{2}}(\gamma) \\
\leq \;& P^{\frac{1}{2}}(\gamma)\,\mathrm{tr}\Big(P^{\frac{1}{2}}(\gamma)AP^{-1}(\gamma)A^{\mathrm{T}}P^{\frac{1}{2}}(\gamma) \\
& +\gamma P^{\frac{1}{2}}(\gamma)BB^{\mathrm{T}}P^{\frac{1}{2}}(\gamma)\Big)P^{\frac{1}{2}}(\gamma)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\mathrm{tr}\left(P^{\frac{1}{2}}(\gamma) A P^{-1}(\gamma) A^{\mathrm{T}} P^{\frac{1}{2}}(\gamma)\right)\right. \\
&\quad \left.+\gamma \mathrm{tr}\left(B^{\mathrm{T}} P(\gamma) B\right)\right) P(\gamma) \\
&= \left(\mathrm{tr}(P(\gamma) A P^{-1}(\gamma) A^{\mathrm{T}})\right. \\
&\quad \left.+\gamma(n\gamma + 2\mathrm{tr}(A))\right) P(\gamma). \qquad (12)
\end{aligned}
$$

Multiplying to the right of both sides of equation (1) by $P^{-1}(\gamma) A^{\mathrm{T}}$ gives

$$
(A^{\mathrm{T}})^2 + P(\gamma) A P^{-1}(\gamma) A^{\mathrm{T}} - P(\gamma) B B^{\mathrm{T}} A^{\mathrm{T}} = \gamma A^{\mathrm{T}}, \tag{13}
$$

from which

$$
\begin{aligned}
&\mathrm{tr}\left(P(\gamma) A P^{-1}(\gamma) A^{\mathrm{T}}\right) + \mathrm{tr}\left(A^2\right) \\
&= \mathrm{tr}\left(P(\gamma) B B^{\mathrm{T}} A^{\mathrm{T}}\right) + \gamma \mathrm{tr}(A)
\end{aligned}. \tag{14}
$$

On the other hand, by using equation (1) again, we obtain

$$
\begin{aligned}
&\mathrm{tr}\left(P(\gamma) B B^{\mathrm{T}} A^{\mathrm{T}}\right) \\
&= \mathrm{tr}\left(B B^{\mathrm{T}}\left(P(\gamma) B B^{\mathrm{T}} P(\gamma) - \gamma P(\gamma) - P(\gamma) A\right)\right) \\
&= \mathrm{tr}\left(\left(B^{\mathrm{T}} P(\gamma) B\right)^2\right) - \gamma \mathrm{tr}\left(B^{\mathrm{T}} P(\gamma) B\right) \\
&\quad -\mathrm{tr}\left(A^{\mathrm{T}} P(\gamma) B B^{\mathrm{T}}\right).
\end{aligned}
$$

It follows from the above equation, identity (5), Proposition 1 and Lemma 3 (in the appendix) that

$$
\begin{aligned}
&\mathrm{tr}\left(P(\gamma) B B^{\mathrm{T}} A^{\mathrm{T}}\right) \\
&= \frac{1}{2}\mathrm{tr}\left(\left(B^{\mathrm{T}} P(\gamma) B\right)^2\right) - \frac{1}{2}\gamma \mathrm{tr}\left(B^{\mathrm{T}} P(\gamma) B\right) \\
&\leq \frac{1}{2}\left(\mathrm{tr}\left(B^{\mathrm{T}} P(\gamma) B\right)\right)^2 - \frac{1}{2}\gamma \mathrm{tr}\left(B^{\mathrm{T}} P(\gamma) B\right) \\
&= \frac{1}{2}(n\gamma + 2\mathrm{tr}(A))^2 - \frac{1}{2}\gamma(n\gamma + 2\mathrm{tr}(A)). \qquad (15)
\end{aligned}
$$

Substituting (15) into (14), we get

$$
\begin{aligned}
&\mathrm{tr}\left(P(\gamma) A P^{-1}(\gamma) A^{\mathrm{T}}\right) \\
&= \mathrm{tr}\left(P(\gamma) B B^{\mathrm{T}} A^{\mathrm{T}}\right) + \gamma \mathrm{tr}(A) - \mathrm{tr}\left(\left(A^{\mathrm{T}}\right)^2\right) \\
&\leq \frac{1}{2}(n\gamma + 2\mathrm{tr}(A))^2 - \frac{1}{2}\gamma(n\gamma + 2\mathrm{tr}(A)) \\
&\quad +\gamma \mathrm{tr}(A) - \mathrm{tr}\left(\left(A^{\mathrm{T}}\right)^2\right),
\end{aligned}
$$

from which the inequality in (12) can be continued as follows

$$
\begin{aligned}
&A_{\mathrm{c}}^{\mathrm{T}}(\gamma) P(\gamma) A_{\mathrm{c}}(\gamma) \\
&\leq \left(\frac{1}{2}(n\gamma + 2\mathrm{tr}(A))^2 + \gamma \mathrm{tr}(A) - \mathrm{tr}\left(A^2\right)\right. \\
&\quad \left.-\frac{1}{2}\gamma(n\gamma + 2\mathrm{tr}(A)) + \gamma(n\gamma + 2\mathrm{tr}(A))\right) P(\gamma) \\
&= \left(\frac{1}{2}(n\gamma + 2\mathrm{tr}(A))^2 + \gamma \mathrm{tr}(A)\right. \\
&\quad \left.+\frac{1}{2}\gamma(n\gamma + 2\mathrm{tr}(A)) - \mathrm{tr}\left(A^2\right)\right) P(\gamma),
\end{aligned}
$$

which completes the proof. ■

Corollary 1: Let all the eigenvalues of A be zero, (A, B) be controllable and $P(\gamma)$ be the unique positive definite solution to the parametric ARE (1). Then for all $\gamma \geq 0$, there holds

$$
A_{\mathrm{c}}^{\mathrm{T}}(\gamma) P(\gamma) A_{\mathrm{c}}(\gamma) \leq \frac{1}{2}\left(n^2 + n\right)\gamma^2 P(\gamma). \tag{16}
$$

Moreover, by Proposition 1, $\lim_{\gamma \to 0^+} P(\gamma) = 0$.

Proof: Since all eigenvalues of A are zero, $\mathrm{tr}(A) = \mathrm{tr}\left(A^2\right) = 0$. Then, (16) follows from (9). ■

III. Stabilization of Time-Delay Systems

Consider the following linear system with input delay

$$
\left\{
\begin{array}{l}
\dot{x}(t) = Ax(t) + Bu(t - \tau(t)), \\
y(t) = Cx(t),
\end{array}
\right. \tag{17}
$$

where $x(t) \in \mathbf{R}^n, u(t) \in \mathbf{R}^m$ and $y(t) \in \mathbf{R}^p$ are respectively the state, the input and the output vectors, and $\tau(t) : [0, \infty) \to \mathbf{R}^+$ is a continuous function representing the delay in the input. As shown in [8], the system (17) can be (globally) stabilized provided all the poles of A are on the closed left-half plane, (A, B) is stabilizable and (A, C) is detectable.

In the special situation that A does not have any nonzero imaginary axis eigenvalues, *i.e.*, all eigenvalues of A are zero, it is shown in [8] that not only can system (17) be (globally) stabilized with linear state or output feedback, but also the feedback gains are independent of the time delay $\tau(t)$. And, if the system is also subject to actuator saturation, semi-global stabilization can be achieved with delay-independent linear feedback laws. In this section, we will show, by using the properties of the parametric Lyapunov equation based low gain design we established in Section II, simpler feedback laws can be constructed that achieve these stabilization results. Moreover, the delay $\tau(t)$ is allowed to be time-varying while it is assumed to be constant in [8].

Without loss of generality, we assume that (A, B) is given in the following form

$$
A = \left[\begin{array}{cc} A_0 & 0 \\ 0 & A_- \end{array}\right], \quad B = \left[\begin{array}{c} B_0 \\ B_- \end{array}\right], \tag{18}
$$

where $A_- \in \mathbf{R}^{n_1 \times n_1}$ contains all eigenvalues of A that have negative real parts and $A_0 \in \mathbf{R}^{n_0 \times n_0}$ contains all eigenvalues of A that are on the imaginary axis. Then $n_1 + n_0 = n$. The stabilizability of (A, B) then implies that (A_0, B_0) is controllable. Clearly, with state feedback, the subsystem (A_-, B_-) does not affect the stabilizability property of the system. In what follows, we will further assume, without loss of generality, that (A, B) is controllable with all the eigenvalues of A on the imaginary axis.

Theorem 2: Assume that all the eigenvalues of $A \in \mathbf{R}^{n \times n}$ are zero and (A, B) is controllable. Let $P(\gamma)$ be the unique positive definite solution to the parametric ARE (1). Then the following state feedback law

$$
u(t) = -B^{\mathrm{T}} P(\gamma) x(t), \quad \gamma \in \left(0, \frac{1}{3\sqrt{3}n\sqrt{n}\bar{\tau}}\right], \tag{19}
$$

(globally) stabilizes the linear time-delay system (17) for all values of delay satisfying

$$
0 \leq \tau(t) \leq \bar{\tau}, \quad 0 \leq t < \infty, \tag{20}
$$

where $\bar{\tau}$ is an arbitrarily large and bounded scalar.

Proof: For an arbitrary initial condition $\psi(\theta) \in \mathscr{C}_{n,\bar{\tau}}$, we construct an artificial initial condition as

$$\overline{\psi}(\theta) = \begin{cases} \psi(\theta), & \forall\theta \in [-\bar{\tau}, 0] \\ 0, & \forall\theta \in [-2\bar{\tau}, -\bar{\tau}). \end{cases} \tag{21}$$

Denote

$$\overline{\mathscr{C}}_{n,2\bar{\tau}} \triangleq \left\{\overline{\psi}(\theta) \mid \psi(\theta) \in \mathscr{C}_{n,\bar{\tau}}\right\} \subset \mathscr{C}_{n,2\bar{\tau}}. \tag{22}$$

Note that the solution to the time-delay system (17) with initial condition $\psi(\theta), \forall\theta \in [-\bar{\tau}, 0]$ coincides with the solution to the time-delay system (17) with initial condition $\overline{\psi}(\theta), \forall\theta \in [-2\bar{\tau}, 0]$. Therefore, we need only to consider the stability of closed-loop system

$$\dot{x}(t) = Ax(t) - BB^{\mathrm{T}}P(\gamma)x(t - \tau(t)),$$

with initial condition (21) (see, for example, [3]).

Rewrite the closed-loop system as follows:

$$\dot{x}(t) = A_{\mathrm{c}}(\gamma)x(t) + BK(\gamma)\Delta x(t), \tag{23}$$

where $\Delta x(t) = x(t) - x(t - \tau(t))$ and $K(\gamma) = B^{\mathrm{T}}P(\gamma)$. In view of (1) and (8), the time derivative of $V(x(t)) = x^{\mathrm{T}}(t)P(\gamma)x(t)$ along the trajectories of system (23) can be evaluated as follows:

$$\begin{aligned} &\dot{V}(x(t)) \\ \le\ & -\gamma x^{\mathrm{T}}(t)P(\gamma)x(t) + (\Delta x(t))^{\mathrm{T}}K^{\mathrm{T}}(\gamma)K(\gamma)\Delta x(t) \\ \le\ & -\gamma x^{\mathrm{T}}(t)P(\gamma)x(t) + n\gamma(\Delta x(t))^{\mathrm{T}}P(\gamma)\Delta x(t). \end{aligned} \tag{24}$$

Integrating both sides of system (23) from $t - \tau(t)$ to t gives

$$\Delta x(t) = \int_{t-\tau(t)}^{t} (A_{\mathrm{c}}(\gamma)x(s) + BK(\gamma)(\Delta x(s)))\,\mathrm{d}s.$$

By using the above relation, Lemmas 1 and 2 in the appendix, we get the inequality in (25), shown at the top of the next page, where $r > 0$ is to be specified and

$$w_1(t) = \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s)H(\gamma)x(s)\,\mathrm{d}s, \tag{26}$$

$$w_2(t) = \int_{t-\tau(t)}^{t} \Delta x^{\mathrm{T}}(s)T(\gamma)\Delta x(s)\,\mathrm{d}s, \tag{27}$$

with $H(\gamma) = A_{\mathrm{c}}^{\mathrm{T}}(\gamma)P(\gamma)A_{\mathrm{c}}(\gamma)$ and $T(\gamma) = K^{\mathrm{T}}(\gamma)B^{\mathrm{T}}P(\gamma)BK(\gamma)$. By Corollary 1,

$$w_1(t) \le \frac{1}{2}\left(n^2 + n\right)\gamma^2 \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s)P(\gamma)x(s)\,\mathrm{d}s. \tag{28}$$

On the other hand, by using relations (5) and (8), we get

$$\begin{aligned} & w_2(t) \\ \le\ & (n\gamma)^2 \int_{t-\tau(t)}^{t} (\Delta x(s))^{\mathrm{T}}P(\gamma)\Delta x(s)\,\mathrm{d}s \\ \le\ & 2(n\gamma)^2 \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s-\tau(s))P(\gamma)x(s-\tau(s))\,\mathrm{d}s \\ & +2(n\gamma)^2 \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s)P(\gamma)x(s)\,\mathrm{d}s. \end{aligned} \tag{29}$$

Therefore, it follows from inequality (25) that

$$\begin{aligned} & (\Delta x(t))^{\mathrm{T}}P(\gamma)\Delta x(t) \\ \le\ & \alpha(\gamma)\int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s)P(\gamma)x(s)\,\mathrm{d}s + 2\left(1 + \frac{1}{r}\right)(n\gamma)^2 \\ & \times\tau(t)\int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s-\tau(s))P(\gamma)x(s-\tau(s))\,\mathrm{d}s, \end{aligned} \tag{30}$$

where

$$\alpha(\gamma) = \frac{1}{2}(1+r)\tau(t)\left(n^2+n\right)\gamma^2 + 2\left(1+\frac{1}{r}\right)(n\gamma)^2\tau(t).$$

Let $V(x(t+\theta)) < \phi V(x(t)), \forall\theta \in [-2\bar{\tau}, 0]$, where $\phi > 1$ is to be specified. Then it follows from inequality (30) that

$$(\Delta x(t))^{\mathrm{T}}P(\gamma)\Delta x(t) < \widehat{\varphi}(\gamma\tau(t), r)\phi V(x(t)), \tag{31}$$

where

$$\widehat{\varphi}(s, r) = \left(\frac{1}{2}(1+r)\left(n^2+n\right) + 4\left(1+\frac{1}{r}\right)n^2\right)s^2.$$

It is easy to show that

$$\varphi(s) \triangleq \min_{r>0}\{\widehat{\varphi}(s,r)\} = \left(\sqrt{\frac{1}{2}(n^2+n)} + 2n\right)^2 s^2.$$

Let $r = \sqrt{\frac{8n}{n+1}}$ which is such that $\varphi(s) = \widehat{\varphi}(s, r)$. Then, inequality (31) reduces to

$$(\Delta x(t))^{\mathrm{T}}P(\gamma)\Delta x(t) \le \phi\varphi(\gamma\tau(t))V(x(t)). \tag{32}$$

Substituting (32) into (24) gives

$$\dot{V}(x(t)) < -\gamma V(x(t))(1 - \phi n\varphi(\gamma\tau(t))). \tag{33}$$

Let $\phi = \frac{3}{2}$. Then it follows from (19) and (20) that

$$\begin{aligned} \gamma &\le \frac{1}{\sqrt{3}\left(\sqrt{\frac{1}{2}(n^2+n)} + 2n\right)\tau(t)} \\ &\Leftrightarrow\ 1 - \phi n\varphi(\gamma\tau(t)) \ge \frac{1}{2}, \end{aligned}$$

with which inequality (33) can be continued as

$$\dot{V}(x(t)) < -\frac{1}{2}\gamma V(x(t)).$$

The global stability is guaranteed by the Razumikhin Stability Theorem (Theorem 5 in the appendix). ■

We next consider output feedback stabilization. In this case, it is no longer without loss of generality by assuming that all the eigenvalues of A are on the imaginary axis. In what follows, we assume that (A, B) is in the form of (18) and (A, C) is detectable.

Theorem 3: Consider the delayed linear system (17) where $\tau(t)$ is an exactly known function and is such that (20) is satisfied with $\bar{\tau}$ arbitrarily large and bounded. Assume that (A, B) is given by (18) where A_- is asymptotically stable, all the eigenvalues of $A_0 \in \mathbf{R}^{n_0 \times n_0}$ are zero, (A_0, B_0) is controllable and (A, C) is detectable. Then there exists a

$$
\begin{aligned}
& (\Delta x(t))^{\mathrm{T}} P(\gamma)\Delta x(t) \\
= & \left(\int_{t-\tau(t)}^{t}\left(A_{\mathrm{c}}(\gamma)x(s)+BK(\gamma)(\Delta x(s))\right)\mathrm{d}s\right)^{\mathrm{T}} P(\gamma)\left(\int_{t-\tau(t)}^{t}\left(A_{\mathrm{c}}(\gamma)x(s)+BK(\gamma)(\Delta x(s))\right)\mathrm{d}s\right) \\
\leq & (1+r)\left(\int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s)A_{\mathrm{c}}^{\mathrm{T}}(\gamma)\mathrm{d}s\right)P(\gamma)\left(\int_{t-\tau(t)}^{t} A_{\mathrm{c}}(\gamma)x(s)\mathrm{d}s\right) \\
& +\left(1+\frac{1}{r}\right)\left(\int_{t-\tau(t)}^{t}(\Delta x(s))^{\mathrm{T}} K^{\mathrm{T}}(\gamma)B^{\mathrm{T}}\mathrm{d}s\right)P(\gamma)\left(\int_{t-\tau(t)}^{t}\left(BK(\gamma)(\Delta x(s))\right)\mathrm{d}s\right) \\
\leq & (1+r)\tau(t)w_1(t)+\left(1+\frac{1}{r}\right)\tau(t)w_2(t),
\end{aligned} \tag{25}
$$

---

positive scalar $\gamma^* > 0$ such that the family of linear output feedback

$$
\begin{cases} \dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(t-\tau(t)) + L(y(t) - C\widehat{x}(t)), \\ u(t) = -\begin{bmatrix} B_0^{\mathrm{T}} P_0(\gamma) & 0 \end{bmatrix}\widehat{x}(t), \quad \gamma \in (0,\gamma^*], \end{cases} \tag{34}
$$

where $P_0(\gamma)$ is the unique positive definite solution to the parametric ARE

$$
A_0^{\mathrm{T}} P_0 + P_0 A_0 - P_0 B_0 B_0^{\mathrm{T}} P_0 = -\gamma P_0, \tag{35}
$$

and $L \in \mathbf{R}^{p\times n}$ is such that $A - LC$ is asymptotically stable, globally stabilizes the system (17) at the origin.

Proof: Omitted due to space limitation. ■

If the input $u(t)$ in system (17) is subject to actuator saturation, namely, $\|u(t)\|_\infty \leq 1$, the family of state feedback law (19) will achieve semi-global stabilization of the time-delay system (17). In what follows, we will state and prove such a result in a theorem. The output feedback results can be stated and proven in the same manner.

Theorem 4: Consider the time-delayed linear system (17) with input saturation where $\tau(t)$ is such that (20) is satisfied with $\bar{\tau}$ arbitrarily large and bounded. Assume that all the eigenvalues of $A \in \mathbf{R}^{n\times n}$ are zero and (A, B) is controllable. Then the family of linear state feedback laws (19) semi-globally stabilizes system (17) at the origin, i.e., for any a priori given bounded set $\Omega \subset \mathscr{C}_{n,\bar{\tau}}$, there exists a $\gamma^* > 0$ such that, for any $\gamma \in (0,\gamma^*]$, the closed-loop system is asymptotically stable at the origin with Ω contained in the domain of attraction. Moreover, the scalar γ^* can be determined as

$$
\gamma^* = \min\left\{\frac{1}{3\sqrt{3}n\sqrt{n}\bar{\tau}}, \gamma_1^*\right\}, \tag{36}
$$

where

$$
\gamma_1^* = \inf_{\psi(\theta)\in\Omega,\theta\in[-\bar{\tau},0]} \gamma \quad \text{s.t. } n\gamma\psi^{\mathrm{T}}(\theta)P(\gamma)\psi(\theta) = 1. \tag{37}
$$

Proof: Omitted due to space limitation. ■

IV. A Numerical Example

In this section, we provide a numerical example to illustrate the effectiveness of the proposed results. Consider a linear time-delay system in the form of (17) with (borrowed from [8])

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ 0 & -1 \\ 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},
$$

and $\tau(t) = \sin^2(2t)$. Therefore, $\bar{\tau} = 1$. This system contains two chains of integrators with two coupled inputs and a time-varying input delay. Because of the space limitation, we only consider the state feedback case. By solving the parametric Lyapunov equation (4) and according to (19), the family of linear state feedback laws can be constructed as

$$
u\left(t - \sin^2(2t)\right) = -K(\gamma)x\left(t - \sin^2(2t)\right), \tag{38}
$$

where

$$
K(\gamma) = \begin{bmatrix} \frac{1}{5}\gamma^3 & \frac{7}{5}\gamma^2 & \frac{11}{5}\gamma & \frac{1}{5}\gamma^3 - 2\gamma^2 & k_{15}(\gamma) \\ \frac{2}{5}\gamma^3 & \frac{4}{5}\gamma^2 & \frac{2}{5}\gamma & \frac{2}{5}\gamma^3 + \gamma^2 & k_{25}(\gamma) \end{bmatrix},
$$

with $k_{15}(\gamma) = \frac{7}{5}\gamma^2 - \frac{2}{5}\gamma^3 - 4\gamma$ and $k_{25}(\gamma) = \frac{4}{5}\gamma^2 - \frac{4}{5}\gamma^3 + 2\gamma$. To verify that this family of linear feedback law globally stabilizes the system (17), we simulate the closed-loop system under a given nonzero initial condition for different values of γ satisfying $0 < \gamma \leq \frac{1}{15\sqrt{15}}$. The results are given in Fig. 1, which also indicates that such a family of control law semi-globally stabilizes the given system subject to actuator saturation because the magnitude of the control input decreases as the value of γ decreases.

V. Conclusions

In this paper, some intricate properties of the parametric Lyapunov equation based low gain design feedback laws were established. As applications of these properties, we proposed new solutions to the stabilization for a class of linear systems with input delay and the semi-global stabilization problem when the systems are also subject to actuator saturation. Our solutions are not only simpler to construct but can also be easily scheduled online to achieve global results in the presence of input saturation.

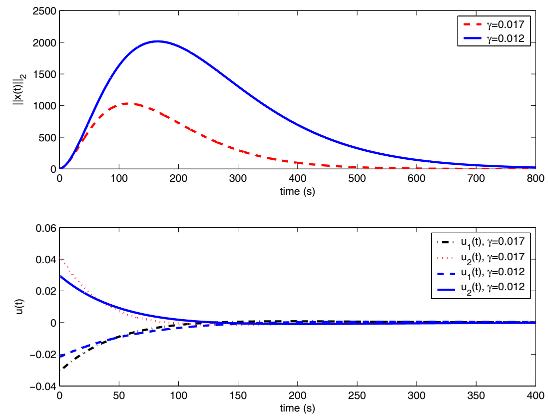

Fig. 1. Evolution of the state and the control of the closed-loop system for different values of γ.

Appendix

In this appendix, we provide some technical results that are used in the paper. The first result is the following well-known Razumikhin Stability Theorem.

Theorem 5: ([3] and [8]) Consider the functional differential equation

$$\begin{cases} \dot{x}(t) = f(t, x_t), & x(t) \in \mathbf{R}^n, \ t \geq t_0, \\ x_{t_0}(\theta) = \varphi(\theta), & \forall \theta \in [-\tau, 0], \end{cases} \tag{39}$$

where $x_t = x(t+\theta), \forall \theta \in [-\tau, 0]$. The function $f : \mathbf{R} \times \mathscr{C}_{n,\tau}$ is such that the image by f of $\mathbf{R} \times$ (a bounded subset of $\mathscr{C}_{n,\tau}$) is a bounded subset of $\mathbf{R}^n$ and the functions $u, v, w, p : \mathbf{R}^+ \to \mathbf{R}^+$ are continuous, nondecreasing and positive for all $s > 0, u(0) = v(0) = 0$, v is strictly increasing and $p(s) > s$. Suppose that there are a continuous function $V : \mathbf{R}^n \to \mathbf{R}$ and a positive number ρ, such that for all $x_t \in \mathscr{M}_V(\rho)$ where

$$\mathscr{M}_V(\rho, \tau) \triangleq \{\psi \in \mathscr{C}_{n,\tau} \mid V(\psi(\theta)) \leq \rho, \forall \theta \in [-\tau, 0]\},$$

the following conditions hold,

1) $u(\|x\|) \leq V(x) \leq v(\|x\|)$;
2) $\dot{V}(x) \leq -w(\|x\|)$, if $V(x(t+\theta)) < p(V(x(t)))$, $\forall \theta \in [-\tau, 0]$.

Then the trivial solution $x(t) \equiv 0$ of the differential equation (39) is asymptotically stable. Moreover, the set $\mathscr{M}_V(\rho, \tau)$ is an invariant set inside the domain of attraction. If Items 1–2 hold for all $x \in \mathbf{R}^n$ and $\lim_{s \to \infty} u(s) = \infty$, then the solution $x(t) \equiv 0$ is globally asymptotically stable.

The second result is the following well-known Jensen integral inequality whose proof can be found in [2].

Lemma 1: For any positive definite matrix $M > 0$, scalars γ_2 and γ_1 with $\gamma_2 \geq \gamma_1$, vector function $\omega : [\gamma_1, \gamma_2] \to \mathbf{R}^n$ such that the integrations in the following are well-defined, then

$$\begin{aligned} &\int_{\gamma_1}^{\gamma_2} \omega^{\mathrm{T}}(\beta)\, \mathrm{d}\beta M \int_{\gamma_1}^{\gamma_2} \omega(\beta)\, \mathrm{d}\beta \\ \leq\ & (\gamma_2 - \gamma_1) \int_{\gamma_1}^{\gamma_2} \omega^{\mathrm{T}}(\beta) M \omega(\beta)\, \mathrm{d}\beta. \end{aligned}$$

Finally, the following two simple results are also recalled.

Lemma 2: Let $x, y \in \mathbf{R}^n$ be two arbitrary vectors and $P \in \mathbf{R}^{n \times n}$ be a positive definite matrix. Then

$$x^{\mathrm{T}} y + y x^{\mathrm{T}} \leq x^{\mathrm{T}} P x + y^{\mathrm{T}} P^{-1} y.$$

Lemma 3: ([7]) Let $X \geq 0$ and $Y \geq 0$. Then,

$$\mathrm{tr}(XY) \leq \mathrm{tr}(X)\, \mathrm{tr}(Y).$$

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