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Optimal Boundary Control of Kuramoto-Sivashinsky Equation

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Abstract— **This work focuses on optimal boundary control of highly dissipative Kuramoto-Sivashinsky equation (KSE) which describes the long-wave motions of a thin film over vertical plane. A standard transformation is initially used to reformulate the original boundary control problem as an abstract boundary control problem of the KSE partial differential equation (PDE) in an appropriate functional space setting. Low dimensional representation of the KSE is used in the synthesis of a finite dimensional linear quadratic regulator (LQR) in the full-state feedback control realization and in a compensator design with a Luenberger-type observer. The proposed control problem formulation and the performance and robustness of the closed-loop system in the full state-feedback, output-feedback and in the output-feedback with the presence of noise controller realization have been evaluated through simulations.**

Key words: Distributed-Parameter Systems, Kuramoto-Sivashinsky Equation, Boundary control, LQR, State/Output Feedback Control

I. Introduction

This paper focuses on optimal boundary control of highly dissipative fourth-order PDEs given by the Kuramoto-Sivashinsky equation, which describes a variety of physicochemical phenomena like long-wave motions of the liquid thin film over a vertical plate, or evolution of laminar fronts [1], or which serves as a model for description of the phase turbulence in reaction-diffusion systems [2]. Due to its ability to describe a large variety of physical phenomena and due to its complexity, the Kuramoto-Sivashinsky equation (KSE) has been extensively studied. In particular, the pioneering works of Nicolaenko et al. [3] and Foias et al. [4] provided first description of complex dynamical features such as global attractors and inertial manifold of the KSE.

Although there is a significant amount of research devoted to the description of complex dynamical features of the KSE, a lot remains to be done in the control problem formulations associated with the KSE [5]. Recently, the problem of stabilization of the Kuramoto-Sivashinsky equation with the actuators placed within the system's domain has been addressed within the output feedback formulation by Christofides and Armaou [6], [7]. In the same vein, Lee and Tran [8] explored two reduced-order methods, the approximate inertial manifolds and the proper orthogonal decomposition to obtain a reduced-order models which are utilized in synthesis of linear and nonlinear quadratic regulators for the distributed control of the KSE. In all aforementioned controller realizations, an important issue of actuation applied at the domains' boundary has not been explored. Namely, in a large number of implementable controlled systems, the implementation specifications rarely permit placement of the actuation devices within the systems' domain, but more frequently the control implementation is achieved by a finite number of actuators places on the boundary of the system [9], see Fig.1. Along the line

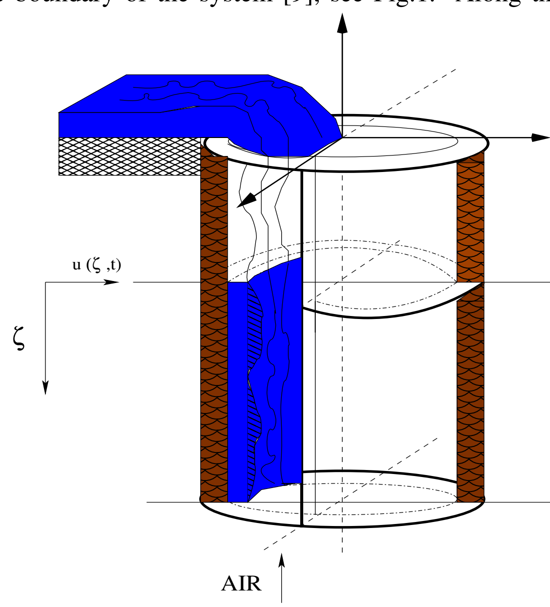

Fig. 1. Schematic representation of the two-phase annular flow in vertical pipes in which the evolution of the thin film layer height is described by the Kuramoto-Sivashinsky Equation.

of boundary applied actuation in the context of control of the KSE, Liu and Krstic [5] explored the problem of global boundary control in the case when control appears under Dirichlet and Neumann boundary conditions. However, an important notion of the optimality within boundary control of the KSE setting has not been explored due to following two complexities, the model complexity associated with highly dissipative fourth-order spatial operator and distributed nature of the KSE on one side, and due to the complexity of formulating optimal boundary control problem, on the other. The latter exactly accounts for the point actuation applied at the domains' boundary and leads possibly to the large scale nonlinear programing control problems which are difficult to be realized and implemented in practice.

In order to address the issue of boundary control of spatially distributed systems, a significant research work has focused on the development of a general framework from the stand point of necessary conditions under which a system can be stabilized by the state feedback controller (see e.g., [10],[11],[12],[13]). Commonly, the issue of stabilization for a parabolic system is resolved by the state space decomposition based on system modes, as the "relocation" of finite number of unstable modes using well-known finite dimensional algorithms stabilizes the original system. This implies that the dominant dynamics described by a finite number of possibly unstable modes, once stabilized, and along with the exponential stable infinite dimensional modal complement, renders asymptotic stability to the entire infinite dimensional system. Along this paradigm, the class of results that follows explores boundary identification and control of distributed parameter systems using singular functions [14], boundary static and dynamic output regulation of nonlinear distributed parameter systems [15], comprehensive development of the state feedback boundary control laws based on the backstepping methodology by Krstic and coworkers [16], [5], [17], stabilization by the control Lyapunov function and application of finite-dimensional LQR [18], and development of model predictive methodology that includes input and state constraints in the boundary control design [19]. The model predictive control methodology within the boundary control setting [19], has been based on the development of a general framework for the synthesis of low-order controllers for parabolic PDE systems and other highly dissipative PDE systems that arise in the modeling of spatially-distributed systems on the basis of low-order ODE models derived through the Galerkin method.

Building on these results associated with the control of parabolic PDE systems, in this work optimal boundary control of Kuramoto-Sivashinsky equation is developed. In particular, the evolution of the fourth-order highly dissipative PDE state of the KSE is given by an abstract evolution equation which is appropriately defined on the Hilbert space. A standard transformation is initially used to reformulate the original boundary control problem as an abstract boundary control problem. Low dimensional representation of the KSE is used in the synthesis of a finite dimensional linear quadratic regulator (LQR) in the full-state feedback control realization and in a compensator design with a Luenberger-type observer. As an example of the proposed controller synthesis methodology, we consider the optimal stabilization of spatially-uniform unstable steady state of the Kuramoto-Sivashinsky equation subject to Dirichlet boundary conditions. A boundary control actuation in this work appears as the control of the flux at the boundary, as one possible boundary control realization; hence the synthesis in the case of Neumann or any other combination of the boundary applied actuation would follow in the same steps as the ones presented in this work. The proposed control problem formulation has been evaluated through simulations in the case of full-state feedback control, output-feedback control and the influence of output noise on close-loop stability is investigated.

II. Preliminaries

A. Kuramoto-Sivashinsky equation

In this work, we consider the following uncontrolled Kuramoto-Sivashinsky equation given in the following form:

$$u_t + \nu u_{\zeta\zeta\zeta\zeta} + u_{\zeta\zeta} + u u_\zeta = 0 \tag{1}$$

$$y(t) = \int_0^l c_{d_j}(\zeta) u(\zeta, t) d\zeta \tag{2}$$

with the following boundary and initial conditions:

$$u(0,t) = 0 \qquad u(l,t) = 0 \tag{3}$$

$$u_\zeta(0,t) = 0 \qquad u_\zeta(l,t) = v(t) \tag{4}$$

$$u(\zeta, 0) = u_0(\zeta) \tag{5}$$

where $u(\zeta,t) \in \mathcal{H}$ denotes the state variable, $\zeta \in [0,l]$ is the coordinate, $t \in [0,\infty)$ is the time, $v(t) \in U$ denotes the manipulated input, $y(t) \in Y$ is the output variable obtained by d_j-th sensor. The terms $u_{\zeta\zeta\zeta\zeta}$ and $u_{\zeta\zeta}$ denote the fourth-order and second-order spatial derivative of $u(\zeta,t)$ and $u_0(\zeta)$ is a sufficiently smooth function of ζ. The functions $c_{d_j}(\zeta) \in L_2(0,l)$ are square integrable functions of ζ that describe how the sensing is distributed within the spatial interval $[0,l]$. The state space of interest is $\mathcal{H} = L_2(0,l)$, with the standard inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ defined on it. Additional restriction on the space $\mathcal{H}$, which is coming from the unique conservation property of the KS PDE model [20], is given by the property that any function $\psi(\zeta) \in \mathcal{H}$ also satisfies,

$$\int_0^l \psi(\zeta) d\zeta = 0 \tag{6}$$

We consider a linearized form of Eq.1 around the spatially uniform unstable steady state $u(\zeta,t) = 0$, and in order to reformulate it in the abstract equation setting, the following operator is defined:

$$\mathcal{A}_0 = \frac{d^4}{d\zeta^4} \tag{7}$$

with its dense domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}_0) = \{\psi(\zeta) \in L_2(0,l) | \psi, \psi_\zeta, \psi_{\zeta\zeta}, \psi_{\zeta\zeta\zeta}, \\ are\ abs.\ cont., \psi_{\zeta\zeta\zeta\zeta} \in L_2(0,l), \psi(0) = 0 = \psi(l), \\ \psi_\zeta(0) = 0\} \end{aligned} \tag{8}$$

The operator Eq.7 with its domain Eq.8 belongs to the class of nonsymmetric linear differential operators whose eigenfunction expansions converge in much the same way as Fourier series [21] and thus enjoy many of the properties of systems generated by self-adjoint operators. The input to the system is given by the boundary operator $\mathcal{B} : L_2(0,l) \rightarrow \mathbb{R}$,

$$\mathcal{B}\psi(\zeta) = \frac{d\psi}{d\zeta}(l),\ with\ \mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{D}(\mathcal{B}) \tag{9}$$

and the output operator as:

$$\mathcal{C}u(t) = (c_{d_j}(\cdot), u(\zeta,t)) \tag{10}$$

We define an associated operator A as:

$$\mathcal{A}\psi(\zeta) = \mathcal{A}_0\psi(\zeta) \; and \; \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_0) \cap ker(\mathcal{B}) = \{\psi \in L_2(0,l) : \psi, \psi_\zeta, \psi_{\zeta\zeta}, \psi_{\zeta\zeta\zeta}, are \; abs. \; cont., \mathcal{A}\psi(\zeta) \in L_2(0,l), \psi(0) = 0 = \psi(l), \psi_\zeta(0) = 0 = \psi_\zeta(l)\} \tag{11}$$

with an assumption that the operator $\mathcal{A}$ is the infinitesimal generator of a strongly continuous C_0-semigroup. One can easily check that the operator $\mathcal{A}$ is a self-adjoint, positive and boundedly invertible, with discrete spectrum. In order to completely characterized operator $\mathcal{A}$ one needs to solve the following eigenvalue problem:

$$\begin{aligned} \frac{d^4\psi}{d\zeta^4} &= \lambda\psi, \quad \zeta \in [0,l] \\ \psi(0) = \; & 0 \; = \psi(l) \\ \psi_\zeta(0) = \; & 0 \; = \psi_\zeta(l) \end{aligned} \tag{12}$$

and the characteristic equation of Eq.12 is $s^4 = \lambda$, with its solutions given as $s_1 = \alpha$, $s_2 = -\alpha$, $s_3 = i\alpha$, $s_4 = -i\alpha$, where $\alpha = (\lambda)^{1/4}$. Thus the general solution of Eq.12 is given by

$$\psi(\zeta) = C_1 e^{\alpha\zeta} + C_2 e^{-\alpha\zeta} + C_3 cos(\alpha\zeta) + C_4 sin(\alpha\zeta) \tag{13}$$

which, by boundary conditions, renders the following system of linear equations:

$$\begin{gathered} C_1 + C_2 + C_3 = 0 \\ C_1 e^{\alpha l} + C_2 e^{-\alpha l} + C_3 cos(\alpha l) + C_4 sin(\alpha l) = 0 \\ C_1 - C_2 + C_4 = 0 \\ C_1 e^{\alpha l} - C_2 e^{-\alpha l} - C_3 sin(\alpha l) + C_4 cos(\alpha l) = 0 \end{gathered} \tag{14}$$

Setting the determinant of the coefficient matrix of the above system to zero, renders the expression for the nonzero eigenvalues of the problem Eq.12:

$$\frac{1}{cos(\alpha l)} = cosh(\alpha l) \tag{15}$$

The Eq.15 for the α_n has an infinite number of solutions which can be obtained by plotting the graphs of the functions $\frac{1}{cos(\alpha l)}$ and $cosh(\alpha l)$. The set of eigenvalues $\lambda_n = (\alpha_n)^4$, for all $n \geq 1$ with the corresponding eigenfunctions is given as,

$$\psi_n(\zeta) = \kappa_n \left[\frac{N_1(\alpha_n, l)}{P_1(\alpha_n, l)} e^{\alpha_n\zeta} + e^{-\alpha_n\zeta} + \frac{N_2(\alpha_n, l)}{P_2(\alpha_n, l)} cos(\alpha_n\zeta) + \frac{N_3(\alpha_n, l)}{P_3(\alpha_n, l)} sin(\alpha_n\zeta) \right] \tag{16}$$

where the normalizing factor κ_n is given so that $(\psi_n, \psi_n) = \int_0^l \psi_n^2 d\zeta = 1$ holds, and functions $P_1(\alpha_n, l)$, $P_2(\alpha_n, l)$, and $P_3(\alpha_n, l)$ are given as follows, $P_1(\alpha_n, l) = 2\cos(\alpha_n l) - 2\,\mathrm{e}^{\alpha_n l} + \cos(\alpha_n l)\,\mathrm{e}^{2\alpha_n l} - \cos(2\alpha_n l)\,\mathrm{e}^{\alpha_n l}$, $P_2(\alpha_n, l) = 2\cos(\alpha_n l) - \mathrm{e}^{\alpha_n l} + \cos(\alpha_n l)\,\mathrm{e}^{2\alpha_n l} - 2\cos(\alpha_n l)^2\,\mathrm{e}^{\alpha_n l}$, $P_3(\alpha_n, l) = \cos(\alpha_n l)\,\mathrm{e}^{\alpha_n l} - 1$, and $N_1(\alpha_n, l) = -(\sin(\alpha_n l) - \frac{\sin(2\alpha_n l)}{\mathrm{e}^{\alpha_n l}})$, $N_2(\alpha_n, l) = \mathrm{e}^{\alpha_n l} + \frac{2}{\mathrm{e}^{\alpha_n l}} - \sin(\alpha_n l) + \cos(\alpha_n l)\left(\frac{2\sin(\alpha_n l)}{\mathrm{e}^{\alpha_n l}} - 3\right) - \frac{2\sin(\alpha_n l)^2}{\mathrm{e}^{\alpha_n l}}$ and $N_3(\alpha_n, l) = \frac{\sin(\alpha_n l)}{\mathrm{e}^{\alpha_n l}} - \frac{\cos(\alpha_n l)}{\mathrm{e}^{\alpha_n l}} + 1$. The eigenvalue $\alpha_0 = 0$ with the corresponding eigenfunction $\psi = 1$ does not satisfy Eq.6.

We make an assumption that function B exists, that $B \in \mathcal{D}(\mathcal{A}_0)$ so that the following holds:

$$\mathcal{B}Bv(t) = v(t) \tag{17}$$

In other words, an ansatz for the function B can be written as follows:

$$B(\zeta) = a\zeta^3 + b\zeta^2 + c\zeta + d \tag{18}$$

and the conditions given by Eq.17 and $B \in \mathcal{D}(\mathcal{A}_0)$ render the following expression:

$$B(\zeta) = \frac{1}{l^2}\zeta^3 - \frac{1}{l}\zeta^2 \tag{19}$$

In order to completely define abstract boundary problem, we define another operator $\tilde{\mathcal{A}}_0 = -\frac{d^2}{d\zeta^2}$ with its domain,

$$\mathcal{D}(\tilde{\mathcal{A}}_0) = \{\psi(\zeta) \in L_2(0,l) | \psi, \psi_\zeta, are \; abs.cont., \psi_{\zeta\zeta} \in L_2(0,l), \psi(0) = 0 = \psi(l)\} \tag{20}$$

The new operator $\tilde{\mathcal{A}}_0$ is self-adjoint, positive and boundedly invertible, so that the following holds:

$$\tilde{\mathcal{A}}_0^2\psi = \mathcal{A}\psi, \;\; \psi \in \mathcal{D}(\mathcal{A}) \tag{21}$$

and so that $\tilde{\mathcal{A}}_0 = \mathcal{A}^{\frac{1}{2}}$, the square root of $\mathcal{A}$.

The linearized system of the PDE system of Eqs.1-2-3-4-5 can be equivalently written in the following form of the abstract boundary control problem:

$$\begin{aligned} \dot{u}(t) &= -\nu\mathcal{A}_0 u(t) + \mathcal{A}^{\frac{1}{2}} u(t), \quad u(0) = u_0 \\ \mathcal{B}u(t) &= v(t) \\ y(t) &= \mathcal{C}u(t) \end{aligned} \tag{22}$$

where $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subset \mathcal{H} \mapsto \mathcal{H}$ and $\mathcal{B}$ is a boundary operator $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subset \mathcal{H} \mapsto U$, with $\mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{D}(\mathcal{B})$, and output operator $\mathcal{C} : \mathcal{H} \mapsto \mathbb{R}$. This requirement on existence of $B(\zeta)$ together with an assumption that the input $v(t) \in L_2([0,t]; U)$, allow us to define the following transformation $p(t) = u(t) - Bv(t)$, which renders the well posed abstract differential equation,

$$\begin{gathered} \dot{p}(t) = (-\nu\mathcal{A} + \mathcal{A}^{\frac{1}{2}})p(t) + (-\nu\mathcal{A}_0 + \mathcal{A}^{\frac{1}{2}})Bv(t) - B\dot{v}(t) \\ p(0) = p_0 \in \mathcal{D}(\mathcal{A}) \end{gathered} \tag{23}$$

which has a well defined mild solution. The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup and due to bounded property of linear operators B, $\mathcal{A}_0 B$ and $\mathcal{A}^{\frac{1}{2}}B$ equation Eq.23 is well defined in the infinite dimensional state space setting. However, the abstract evolutionary equation Eq.23 includes in its expression a derivative of the control term, which requires to reformulate the problem on the extended state space $\mathcal{H}^e := \mathcal{H} \bigotimes U$, as

$p^e(t) = [v(t)\ p(t)]'$ together with ($\bar{v}(t) = \dot{v}(t)$) yields,

$$\begin{aligned} \dot{p}^e(t) &= \begin{pmatrix} 0 & 0 \\ (-\nu\mathcal{A}_0 + \mathcal{A}^{\frac{1}{2}})B & (-\nu\mathcal{A} + \mathcal{A}^{\frac{1}{2}}) \end{pmatrix} p^e(t) + \\ &\quad + \begin{pmatrix} I \\ -B \end{pmatrix} \bar{v}(t) \\ p^e(0) &= [p_1^e(0)\ \ p_2^e(0)]' = [v(0)\ \ p(0)]' \end{aligned} \tag{24}$$

The new extended space operator $\mathcal{A}^e$ is defined as $\mathcal{A}^e = \left(0\ 0; (-\nu\mathcal{A}_0 + \mathcal{A}^{\frac{1}{2}})B\ \ (-\nu\mathcal{A} + \mathcal{A}^{\frac{1}{2}})\right)$ and with its domain $\mathcal{D}(\mathcal{A}^e) = U \bigotimes \mathcal{D}(\mathcal{A})$ is the infinitesimal generator of a C_0-semigroup on $\mathcal{H}^e$. The operator $\mathcal{A}^e$ and operator $\mathcal{B}^e = [I \quad -B]'$ define well posed linear infinite dimensional system with the mild solution of Eq.24 given by the following expression:

$$\begin{aligned} p^e(t) &= \begin{pmatrix} I & 0 \\ \mathcal{S}(t) & \mathcal{T}(t) \end{pmatrix} p^e(0) + \\ &+ \int_0^t \begin{pmatrix} I & 0 \\ \mathcal{S}(t-s) & \mathcal{T}(t-s) \end{pmatrix} \begin{pmatrix} I \\ -B \end{pmatrix} \bar{v}(s) ds \\ p^e(0) &= [p_1^e(0)\ \ p_2^e(0)]' = [v(0)\ \ p(0)]' \end{aligned} \tag{25}$$

where $\mathcal{S}(t)p = \int_0^t \mathcal{T}(t-s)(-\nu\mathcal{A}_0 + \mathcal{A}^{\frac{1}{2}})Bpds = \int_0^t \mathcal{T}(s)(-\nu\mathcal{A}_0 + \mathcal{A}^{\frac{1}{2}})Bpds$, and $\mathcal{T}(t)$ is the operator associated with the infinitesimal generator $\mathcal{A}$ (see for details [12], [22]). Therefore, by the necessary assumption that $u \in L_2([0,t];U)$ and by the continuity of initial conditions (that is, $u(0) = p(0) + Bv(0)$), a mild solution of Eg.22 is given by:

$$\begin{aligned} u(t) = Bv(t) - \mathcal{T}(t)Bv(0) + \mathcal{T}(t)u(0) - \\ - \textstyle\int_0^t \mathcal{T}(t-s)B\dot{v}(s)ds + \\ + \textstyle\int_0^t \mathcal{T}(t-s)(-\nu\mathcal{A}_0 + \mathcal{A}^{\frac{1}{2}})Bv(s)ds \end{aligned} \tag{26}$$

The Riesz spectral operator $\mathcal{A}$ generates a C_0–strongly continuous semigroup $\mathcal{T}(t)$ given by:

$$\mathcal{T}(t) = \sum_{n=0}^{\infty} e^{\lambda_n t} \left(\cdot, \phi_n(\zeta)\right) \psi_n(\zeta) \tag{27}$$

so that $\sup_{n \geq 1} Re(\lambda_n) \leq \infty$, and spectrum of $\mathcal{A}$ is given as $\Omega(\mathcal{A}) = \{\lambda_n, n \geq 1\}$ where $\lambda_n = \alpha_n^4, \forall n \geq 1$, are simple eigenvalues of $\mathcal{A}$ and where α_n is obtained, as it is given in Eq.15, ψ_n are corresponding eigenfunctions of $\mathcal{A}$ given by Eq.16 and the ϕ_n is the biorthogonal vector so that $(\psi_m, \phi_n) = \delta_{mn}$. The equation Eq.26 suggests that the type of the boundary control action does not provide insight into a precise generalization of the finite dimensional form of the state-feedback control law that can be used to stabilize the KSE. However, one can deduce a features of the finite dimensional controller on the premises of the Eq.24 by exploring the number of unstable eigenvalues of the extended space operator $\mathcal{A}^e$, which leads to a more practical controller synthesis for boundary controlled the KSE.

Remark 1: The proposed procedure for determination of function $B(\zeta)$ is of general nature and it is not associated with the boundary conditions considered in Preliminaries. Namely, one could easily consider Neumann boundary conditions or more general cases when the actuation appears on both domains boundaries.

Remark 2: The approximate controllability of boundary controlled system of Eq.24 can be assured by checking that the following condition holds for all $n \geq 1$,

$$rank[(-\nu\mathcal{A}_0 + \mathcal{A}^{\frac{1}{2}})B(\zeta) - \lambda_n B(\zeta), \psi_n(\zeta)] = 1 \tag{28}$$

In the same vein, the condition of approximate observability for the boundary controlled problem holds if the $rank[(\mathcal{C}(B(\zeta) + I), \psi_n(\zeta))] = 1$ holds for $n \geq 1$. The approximate controllability and observability conditions of boundary controlled system are transformed from their standard forms due to the boundary transformation [22].

III. Optimal controller design

The synthesis of the finite dimensional controller stabilizing the KSE is based on the extended space operator $\mathcal{A}^e$, whose spectrum is partitioned into a finite dimensional unstable part $\Omega^+(\mathcal{A}^e)$ and an infinite dimensional stable complement $\Omega^-(\mathcal{A}^e)$, $\Omega(\mathcal{A}^e) = \Omega^+(\mathcal{A}^e) \cup \Omega^-(\mathcal{A}^e)$. The finite dimensional LQR problem for the finite dimensional state given by $\bar{p}^e(t) = [v(t)\ \ p(t)]'$ is formulated in the following form:

$$\min_{\bar{v}} J(\bar{p}^e(0); \bar{v}) = \int_0^\infty \left(\bar{p}^e(t)'Q\bar{p}^e(t) + \bar{v}(t)'R\bar{v}(t)\right) dt \tag{29}$$

$$s.t.\ \dot{\bar{p}}^e(t) = \mathcal{A}_u \bar{p}^e(t) + \mathcal{B}_u \bar{v}(t) \tag{30}$$

where $\mathcal{A}_u$ and $\mathcal{B}_u$ are matrices that correspond by their dimensions to the dimensions of an unstable eigenspace $\Omega^+(\mathcal{A}^e)$. Matrices Q and R are positive semidefinite and definite, respectively. The resulting LQR control law is $\bar{v}(t) = -\frac{1}{2}R^{-1}\mathcal{B}_u' P\bar{p}^e(t) = -\mathcal{K}\bar{p}^e(t)$, where P is a positive definite solution to the LQR-ARE:

$$0 = \mathcal{A}_u' P + P\mathcal{A}_u + Q - P\mathcal{B}_u R^{-1}\mathcal{B}_u' P \tag{31}$$

Lyapunov based analysis of stabilization of unstable modes $\bar{p}^e(t)$ by LQR state feedback can be demonstrated by considering the following standard control Lyapunov function (CLF), $V(t) = \bar{p}^e(t)'P\bar{p}^e(t)$, so that:

$$\begin{aligned} \dot{V}(t) &= \tfrac{d}{dt}[\bar{p}^e(t)'P\bar{p}^e(t)] \\ &= \bar{p}^e(t)'\left(\mathcal{A}_u' P + P\mathcal{A}_u - P\mathcal{B}_u R^{-1}\mathcal{B}_u' P\right)\bar{p}^e(t) \\ &= -\bar{p}^e(t)'Q\bar{p}^e(t) < 0 \end{aligned} \tag{32}$$

From Eq.32, it can be concluded that the unstable modes are optimally stabilized and due to the cascaded interconnection between unstable and stable modal states, once the unstable states are stabilized under the stabilizing feedback law, $\bar{p}^e(t) \to 0$ and $\bar{v}(t) \to 0$, the stable infinite modal states evolution is only driven by the zero-input dynamics which renders asymptotic stability of the infinite

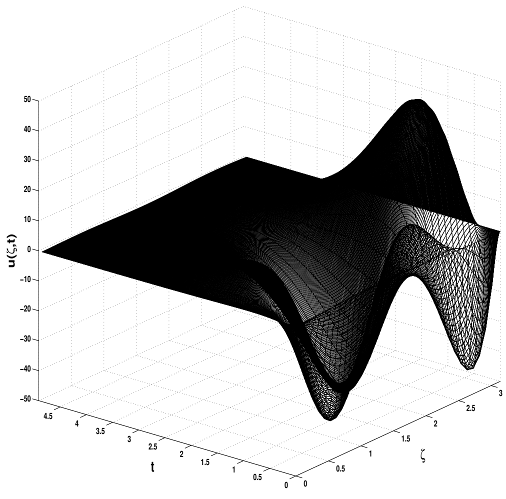

Fig. 2. Boundary stabilization of the linearized KS PDE Eqs.1-2-3-4-5 under the full state-feedback LQR control law Eq.36 and with initial condition $p_1(0) = 1.5$, $p_2(0) = 0.4$.

dimensional closed-loop system. The associated weights given by Q and R matrices in the formulation of the LQR control law given by Eqs.29-30, represent weights on the state evolution $p(t)$, control input evolution $v(t)$, and derivative of boundary control input $\bar{v}(t)$. The term R represents the weight on the derivative of the input while the first diagonal term in the matrix Q represents the weight that is associated with $v(t)$ and the remaining nonzero terms are weights on modal states $p(t)$. The motivation for the synthesis of finite dimensional LQR problem on the premises of the dimensions of unstable eigenspace stems from the high dissipative nature of the KS equation. Namely, a few dominant modes are sufficient to capture the entire KS state while higher modes are neglected. Along the line of synthesis of infinite dimensional LQR that would encompasses entire KS equation state, one needs to solve set of infinite dimensional algebraic Riccati equations (ARE) which can provide a unique solution to the feedback problem under the assumption of exponential detectability and stabilizability. However, one has to be cautious if the solution to ARE is obtained by solving a significantly large system of algebraic equations which may invoke numerical errors.

In the case where state feedback control can not be realized, natural extension to the controller synthesis is to incorporate an observer in the feedback structure. A state observer of the Luenberger type is considered. Under the assumption that the approximate observability holds [22], the Luenberger observer is constructed as,

$$\dot{\hat{p}}^e = \mathcal{A}_u \hat{p}^e(t) + \mathcal{B}_u \bar{v}(t) - \mathcal{L}(y(t) - \mathcal{C}_u \hat{p}^e(t)) \qquad (33)$$

where $\mathcal{C}_u$ is the matrix of appropriate dimensions corresponding to the dimensions of the unstable eigenspace $\Omega^+(\mathcal{A})$ and the number of measurement sensors. Finally, under the assumption of exponential stabilizability and detectability of $(\mathcal{A}_u, \mathcal{B}_u)$ and $(\mathcal{A}_u, \mathcal{C}_u)$, respectively, there exist $\mathcal{K}$ and $\mathcal{L}$ so that $\mathcal{A}_u + \mathcal{B}_u\mathcal{K}$ and $\mathcal{A}_u + \mathcal{C}_u\mathcal{L}$ are exponentially stable. The resulting compensator enforces asymptotic stability in the linearized infinite-dimensional closed-loop system.

Remark 3: It is of importance to address the issue of noise in the framework of the compensator design. A small noise introduced in the output generates perturbations that propagate through the feedback and may form a standing wave solution, which is usually, a linear combination of the unstable eigenspace modes' eigenfunctions.

IV. Simulation study

Kuramoto-Sivashinsky PDE given by Eqs.1-2-3-4-5 with the parameter $\nu = 0.12$ is considered. The spectrum of the operator $\Omega(\mathcal{A})$ is calculated by solving the eigenvalue problem given by Eq.12 whereby the eigenvalues λ_n are obtained as the solution to Eq.15.The eigenvalues of the operator $\mathcal{A}$ are calculated as $\lambda_n^e = -\nu\lambda_n^4 + \lambda_n^2$ and the first three eigenvalues of the operator $\mathcal{A}^e$ are unstable ($\lambda_1^e = 0.0, \lambda_2^e = 1.6502, \lambda_3^e = 1.5631$), while the remaining infinite eigenvalues are stable. The distribution of eigenvalues demonstrates the dissipative nature of the underlying PDE, since the necessary "gap" condition providing that consecutive eigenvalues sufficiently differ among themselves holds.

A high-order finite-dimensional approximation of the infinite dimensional abstract boundary control problem given by Eq.24 is first obtained by considering $n = 30$ eigenfunctions $u(\zeta, t) = \sum_{n=1}^{30} u_n(t)\psi_n(\zeta)$, and it is given by:

$$\dot{p}^e(t) = \bar{\mathcal{A}}^e p^e(t) + \bar{\mathcal{B}}^e \bar{v}(t) \qquad (34)$$

$$y_i(t) = \bar{\mathcal{C}}^e p^e(t) \qquad (35)$$

where $\bar{\mathcal{A}}^e$, $\bar{\mathcal{B}}^e$ and $\bar{\mathcal{C}}^e$ are matrices of the following dimensions (31×31), (31×1), ((# of spatially distributed sensors) $\times$ 31), respectively, with 3 sensors used at $c(\zeta, \zeta_{ci}) = \frac{1}{2\epsilon}1_{[\zeta_{ci}-\epsilon, \zeta_{ci}+\epsilon]}(\zeta)$, where $\zeta_{ci} = [0.1225 \;\; 1.0964 \;\; 2.5101]$. Standard Galerkin method is applied, where modal finite dimensional approximation of Eqs.1-2-3-4-5 is obtained by taking an inner product on $L_2(0, l)$ with operators' eigenfunctions $(u(\zeta, t), \psi_n(\zeta))$, $n = 1, 2, \cdots, 30$. Function $B(\zeta) \in \mathcal{D}(\mathcal{A}_0)$ is selected to satisfy the following condition $\mathcal{B}Bv(t) = v(t)$ and it is chosen to be $B(\zeta) = \frac{1}{\pi}\zeta^3 - \frac{1}{\pi}\zeta^2$. In the extended space $\mathcal{D}(\mathcal{A}^e) = \mathcal{D}(\mathcal{A}) \bigoplus U$, the entries of finite dimensional matrices $\bar{\mathcal{A}}^e$, $\bar{\mathcal{B}}^e$ and $\bar{\mathcal{C}}^e$ are calculated as follows:

$$\Big((-\nu\mathcal{A}_0 + \mathcal{A}^{1/2})B\Big)_n = \left((-\nu\frac{d^4}{d\zeta^4} - \frac{d^2}{d\zeta^2})B(\zeta), \psi_n(\zeta)\right)$$

$$(-\nu\mathcal{A} + \mathcal{A}^{1/2})_n = \left((-\nu\frac{d^4}{d\zeta^4} - \frac{d^2}{d\zeta^2}), \psi_n(\zeta)\right)$$

$$B_n = \left(-(\frac{1}{\pi}\zeta^3 - \frac{1}{\pi}\zeta^2), \psi_n(\zeta)\right)$$

$$C_{in} = ((c(\zeta, \zeta_{ci}), B(\zeta)); \; (c(\zeta, \zeta_{ci}), \psi_n(\zeta)))$$

A simple linear model used for the practical controller synthesis has three states:

$$\bar{p}^e(t) = [v(t); p_1(t); p_2(t)]$$

with associated matrices of appropriate dimensions (3×3) in the case of $\bar{\mathcal{A}}_u$, (3×1) in the case of $\bar{\mathcal{B}}_u$, and (3×3) in the case of $\bar{\mathcal{C}}_u$, with $\bar{v}(t)$ being a derivative of $v(t)$. The finite dimensional LQR control law $\bar{v}(t) = -K\bar{p}^e(t)$ is the solution of the following optimal control problem:

$$\min_{\bar{v}} J(\bar{p}^e(0); \bar{v}) = \int_0^\infty (\bar{p}^e(t)' Q \bar{p}^e(t) + \bar{v}(t)' R \bar{v}(t))\, dt$$
$$s.t.\ \dot{\bar{p}}^e(t) = \bar{\mathcal{A}}_u \bar{p}^e(t) + \bar{\mathcal{B}}_u \bar{v}(t) \tag{36}$$

which yields the following stabilizing gain

$$K = \left(\begin{array}{ccc} 0.6578 & -0.8917 & 1.1715 \end{array} \right) 10^4$$

that places the unstable eigenmodes of a three-dimensional closed-loop system at the following locations $\lambda_{cl} = [-16.6960\ -1.5514\ -2.1172]$ for the following values of matrices $Q = [0.0001\ \mathbf{0}; \mathbf{0}\ I]$, where I is the unitary (2×2) matrix and $R = 0.001$. It is important to emphasize that the first entry in the Q matrix is associated with the weights on the evolution of the control input effort, while the R weight refers to the derivative of the control input. This adds another degree of freedom in the design of the controller that is realized as the boundary place actuation. Furthermore, the gain of the Luenberger observer of Eq.33 is calculated as the gain that shifts the observer eigenvalues at $\lambda_{\mathcal{LC}} = \lambda_{cl} - [10.5\ \ 2.5\ \ 3.5]$ in order to ensure faster convergence of the observer dynamics compared to the systems dynamics. The control law $\bar{v}(t) = -K\bar{p}_u^e(t)$ is first applied to the linear finite dimensional approximation of Eqs.34-35 with 30 eigenfunctions, and the solution is obtained by integrating the closed-loop system by an explicit Euler integration scheme, where the time step is taken as $\triangle t = \frac{1}{1.2max|eig\{\Omega(\mathcal{A})^e\}|}$, so that numerical stability is ensured. In the simulation study Fig.2, it is demonstrated

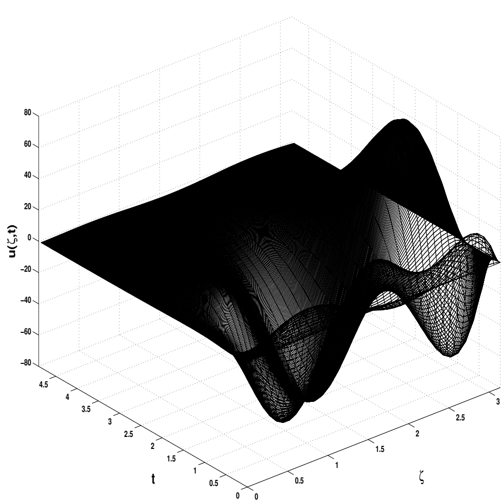

Fig. 3. Boundary stabilization of the linearized KS PDE Eqs.1-2-3-4-5 under the output state-feedback LQR control law Eq.36 and with initial condition $p_1(0) = 1.5$, $p_2(0) = 0.4$.

that the PDE state, close to the boundary where control is applied through the flux actuation, brings the state of the KS PDE to zero. This is achieved with the large excursion of the state even for a relatively small perturbation of initial conditions, which is mainly due to the necessity to have the three unstable modes from the boundary stabilized, see Figs.2-5. Complementary with Fig.2 is Fig.5 that shows the evolution of the control $v(t)$ applied at the boundary $\zeta = \pi$. The linear output feedback control realized with

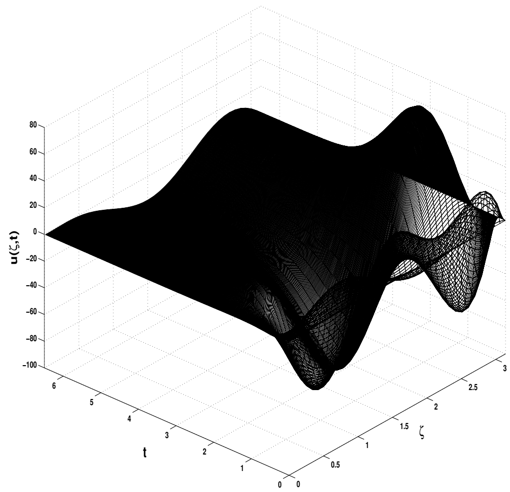

Fig. 4. Boundary stabilization of the linearized KS PDE Eqs.1-2-3-4-5 under the full state-feedback LQR control law Eq.36 with measurement noise $\varrho(t) \leq 0.01$ and with initial condition $p_1(0) = 1.5$, $p_2(0) = 0.4$.

three point measurements achieves successful stabilization of the KS PDE state in a similar manner as in the case of the state feedback stabilization, see Fig.3. It is observed that the state-feedback controller outperforms the output-feedback controller, see Figs.3-5. In Fig.5, as expected, both points and dashed lines converge to the same trajectory, as it takes initially some time for the state estimate to converge to the actual state. In addition, when the impact of noisy measurements is included in the output feedback controller implementation, in the simulation studies, using the linearized PDE model, it is demonstrated that relatively small noise level results in deviation of the state $u(\zeta, t)$ from the zero solution. Namely, for noise of magnitude $\varrho(t) \leq 0.01$ that is directly added to $y(t)$ in Eq.33, we observe, see Fig.4 and Fig.5, that $u(\zeta, t)$ behaves like a near standing wave in the space around $u(\zeta, t) = 0$ with respect to time.

V. SUMMARY

In this work, a practical boundary control approach of highly dissipative Kuramoto-Sivashinsky equation by optimal control is demonstrated. An original boundary control problem is exactly transformed into an abstract boundary control problem which provides a model basis for synthesis of a practical finite dimensional controller. A low-dimensional model representation of the KSE is

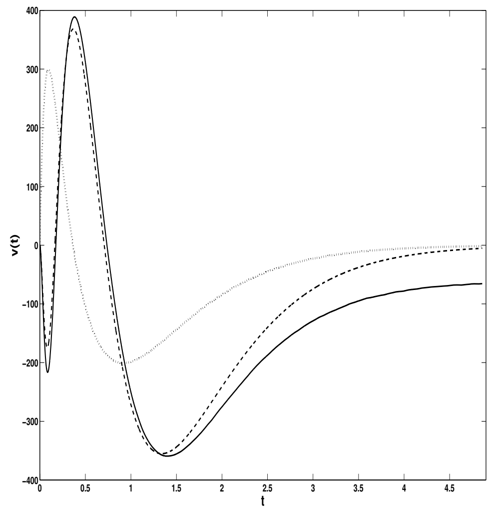

Fig. 5. Boundary input profiles in the case of the full state feedback-"dotted", output feedback-"dashed line" and output feedback with measurement noise-"solid-line".

utilized in the synthesis of a finite dimensional linear quadratic regulator (LQR) in the full-state feedback control realization and in a compensator design with a Luenberger-type observer. The proposed control problem formulation and the performance of the closed-loop system have been explored in the full-state feedback, output-feedback and in the output-feedback with the presence of noise controller realization.

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