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l_∞ to l_∞ Performance Of Slowly Varying Spatiotemporal Systems

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***Abstract*— Performance analysis for slowly varying spatiotemporal systems is presented in the case when the controller design is based on frozen spatially and temporally invariant descriptions of the plant. This approach generalizes the results developed for the standard case for slowly time-varying systems. We show that the l_∞ to l_∞ performance of such systems cannot be much worse than that of the frozen spatially and temporally invariant systems.**

I. INTRODUCTION

In this paper we restrict our focus to a certain class of discrete distributed systems that have slowly varying dynamics in time as well as in space. In particular, we focus on the recursively computable spatiotemporal systems. Recursively computable spatiotemporal systems arise naturally in system identification and adaptive control of systems characterized by partial differential equations e.g. [1], [2]. Recursibility is a property of certain difference equations which allows one to iterate the equation by choosing an indexing scheme so that every output sample can be computed from outputs that have already been found from initial conditions and from samples of the input sequence.

This paper analyzes the results presented in [3] where a characterization of stability for slowly varying spatiotemporal systems based on input-output description of the plant and controller is presented. The controller design presented in [3] is based on frozen spatially and temporally invariant descriptions of the plant. In other words the controllers are not necessarily adjusted for every instance in space and time, and hence are used for some fixed window in time and space before new controllers are implemented. In this paper we aim to show that the l_∞ to l_∞ performance of such systems cannot be much worse than that of the frozen spatially and temporally invariant systems. Our result is a generalization of the results on slowly time-varying systems presented in [4]. The organization of this paper is as follows: Section II presents mathematical preliminaries. Section III elaborates on the frozen space-time control law as presented in [3]. Performance analysis is presented in Section IV. We present the conclusion of our discussion in Section V.

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II. PRELIMINARIES

A. Notation

The set of reals is denoted by $\mathbb{R}$, and the set of integers is denoted by $\mathbb{Z}$. The set of non-negative integers is denoted by $\mathbb{Z}^+$. We use l_∞^e to denote the set of all real double sequences $f = \{f_i(t)\}_{i=-\infty,\ t=0}^{i=\infty,\ t=\infty}$. These sequences correspond to spatiotemporal signals with a 2-sided spatial support ($-\infty \le i \le \infty$) and one sided temporal ($0 \le t \le \infty$). We use l_∞ to denote the space of such sequences with $\|f\|_\infty := \sup_{i,t} |f_i(t)| < \infty$. Note that for $f \in l_\infty^e$, we can represent it as a one-sided (causal) temporal sequence as $f = \{f(0), f(1), \cdots\}$, where

$$f(t) = \left(\ \cdots,\ f_{-1}(t),\ f_0(t),\ f_{+1}(t),\ \cdots\ \right)', \quad t \in \mathbb{Z}^+$$

and each $f_j(t) \in \mathbb{R}$, with $j \in \mathbb{Z}$.

B. Spatiotemporal Varying Systems

Linear spatiotemporal varying systems (LSTV) are systems $M : u \to y$ on l_∞^e given by the convolution

$$y_i(t) = \sum_{\tau=0}^{\tau=t} \sum_{j=-\infty}^{j=\infty} m_{i,i-j}(t, t-\tau) u_j(\tau)$$

where $\{m_{i,j}(t,\tau)\}$ is the kernel representation of M. These systems can be viewed as an infinite interconnection of different linear time varying systems. For simplicity, we assume that each of these subsystems is single-input-single-output (SISO). Let $y_i = (y_i(0), y_i(1), y_i(2), \cdots)'$, then the corresponding input-output relationship of the i_{th} block can be given as follows:

$$\begin{pmatrix} y_i(0) \\ y_i(1) \\ y_i(2) \\ \vdots \end{pmatrix} = \begin{pmatrix} m_{i,0}(0,0) & 0 & 0 & \cdots \\ m_{i,0}(1,0) & m_{i,0}(1,1) & 0 & \cdots \\ m_{i,0}(2,0) & m_{i,0}(2,1) & m_{i,0}(2,2) & \cdots \\ \cdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_i(0) \\ u_i(1) \\ u_i(2) \\ \vdots \end{pmatrix}$$
$$\sum_{\substack{j=-\infty \\ j\neq i}}^{\infty} + \begin{pmatrix} m_{i,j}(0,0) & 0 & 0 & \cdots \\ m_{i,j}(1,0) & m_{i,j}(1,1) & 0 & \cdots \\ m_{i,j}(2,0) & m_{i,j}(2,1) & m_{i,j}(2,2) & \cdots \\ \cdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_{i+j}(0) \\ u_{i+j}(1) \\ u_{i+j}(2) \\ \vdots \end{pmatrix} + \cdots$$

where $\{u_i(t)\}$ is the input applied at the i_{th} block with $u_i(t) \in \mathbb{R}$ and $t \in \mathbb{Z}^+$ is the time index, and $\{m_{i,j}(t,\tau)\}$ is the kernel representation of M. Also, $\{y_i(t)\}$ is the output sequence of the i_{th} block, with $y_i(\cdot) \in \mathbb{R}$.

We can write the overall input-output relationship for a LSTV system as follows:

$$y(t) = \begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} M^{00} & & & & \\ M^{10} & M^{11} & & & \\ M^{20} & M^{21} & M^{22} & & \\ M^{30} & M^{31} & M^{32} & M^{33} & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \\ \vdots \end{pmatrix}$$

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Where, $u(t) = (\cdots, u_{-1}(t), u_0(t), u_{+1}(t), \cdots)'$ and

$$M^{t\tau} = \begin{pmatrix} \cdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & m_{i-1,0}(t,\tau) & m_{i-1,1}(t,\tau) & m_{i-1,2}(t,\tau) & \cdots \\ \cdots & m_{i,-1}(t,\tau) & m_{i,0}(t,\tau) & m_{i,1}(t,\tau) & \cdots \\ \cdots & m_{i+1,-2}(t,\tau) & m_{i+1,-1}(t,\tau) & m_{i+1,0}(t,\tau) & \cdots \\ \cdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

where $t, \tau \in \mathbb{Z}^+$. The l_∞ induced operator norm on M in this case is given as

$$\|M\| = \sup_{i,t} \sum_{\tau=0}^{t} \sum_{i=-\infty}^{i=\infty} |m_{i,j}(t,\tau)|$$

The space of l_∞ bounded LSTV systems will be denoted as $\mathscr{L}_{STV}$

C. Spatially Invariant Systems

Linear spatially invariant systems are spatiotemporal systems $M : u \to y$ on l_∞^e given by the convolution

$$y_i(t) = \sum_{\tau=0}^{\tau=t} \sum_{j=-\infty}^{j=\infty} m_{i-j}(t-\tau) u_j(\tau)$$

where $\{m_i(t)\}$ is the pulse response of M. These systems can be viewed as an infinite array of interconnected linear time invariant (LTI) systems. The subspace of $\mathscr{L}_{STV}$ that contains the stable LSTI systems will be denoted as $\mathscr{L}_{STI}$. The induced l_∞ operator norm on M in this case is given as

$$\|M\| = \sum_{t=0}^{\infty} \sum_{i=-\infty}^{i=\infty} |m_i(t)|$$

D. Local and Global Product

For a LSTV system M, we can associate a LSTI system $M_{i,t}$ for any given pair (i,t) (where $i \in \mathbb{Z}$ represents a spatial coordinate, and $t \in \mathbb{Z}^+$ represents time). The input-output time domain description corresponding to the LSTI system $M_{i,t}$ can be given as follows:

$$\begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} M_{i,t}^0 & & & & \\ M_{i,t}^1 & M_{i,t}^0 & & & \\ M_{i,t}^2 & M_{i,t}^1 & M_{i,t}^0 & & \\ M_{i,t}^3 & M_{i,t}^2 & M_{i,t}^1 & M_{i,t}^0 & \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \\ \vdots \end{pmatrix}$$

where, $u(t) = (\cdots, u_{-1}(t), u_0(t), u_{+1}(t), \cdots)'$ and

$$M_{i,t}^{\tau} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & m_{i,-1}(t,\tau) & m_{i,0}(t,\tau) & m_{i,1}(t,\tau) & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

where $\tau \in \mathbb{Z}^+$. We will refer to $M_{i,t}$ as the local or frozen system corresponding to the pair (i,t). The interpretation is that, $y_i(t) = (Mu)_i(t) = (M_{i,t}u)_i(t)$. For LSTV systems A, B associated with the families $A_{i,t}$, $B_{i,t}$ of frozen LSTI operators, we define a global product $A_{i,t} \cdot B_{i,t}$ to mean an operator associated to the usual composition AB in the sense that, if $u \in l_\infty^e$, then $((A_{i,t} \cdot B_{i,t})u)_i(t) = (ABu)_i(t)$. Given a pair (i,t), the local product of operators A, B corresponds to the product (composition) of the LSTI systems $A_{i,t}$, and $B_{i,t}$, i.e. $A_{i,t}B_{i,t}$.

E. Support of m

We define the support of a sequence $\{m_i(t)\}$ by $Supp(m)$, i.e.

$$Supp(m) = \{[i,t] \in \mathbb{Z}^2 : m_i(t) \neq 0\}$$

F. Slowly Varying Spatiotemporal System

A LSTV system A is said to be slowly space-time varying if given two pairs (i,t), and $(\mathtt{i}, \tau)$, we have

$$\|A_{i,t} - A_{\mathtt{i},\tau}\| \leq \gamma(|i - \mathtt{i}| + |t - \tau|)$$

where $\gamma \in \mathbb{Z}^+$ is a constant. Such systems will be denoted by SSTV(γ)

G. Integral Time and Space Absolute Error

Given a LSTI system M, the integral time and space absolute error (ITSAE) is defined as

$$ITSAE(M) = \sum_{t=0}^{\infty} \sum_{i=-\infty}^{i=\infty} (|i| + |t|)|m_i(t)|$$

H. z, λ Transform

We define the z, λ transform for a LSTI SISO system M as

$$\hat{M}(z,\lambda) = \sum_{t=0}^{\infty} \sum_{k=-\infty}^{\infty} (m_k(t) z^k) \lambda^t$$

with the associated spectral or $\mathscr{H}_\infty$ norm

$$\left\|\hat{M}\right\|_\infty := \sup_{\theta,\omega} |\ \hat{M}(e^{i\theta}, e^{j\omega})\ |$$

It is well known (see e.g. [5]) that for a M in $\mathscr{L}_{STI}$, M^{-1} is in $\mathscr{L}_{STI}$ if and only if

$$\inf_{|z|=1, |\lambda| \leq 1} |\hat{M}(z,\lambda)| > 0$$

III. Frozen Space-Time Control

We briefly review the frozen space-time control as presented in [3]. Consider the general form of closed loop system given in Figure 1. The plant P is a LSTV recursively computable spatiotemporal system with the input-output relationship defined by an equation of the form

$$(A_{i,t} y_1)_i(t) = (B_{i,t} y_4)_i(t)$$

with $\{a_{i,j}(t,\tau)\}, \{b_{i,j}(t,\tau)\}$, being the kernel representations of the operators $A_{i,t}$, $B_{i,t}$ in $\mathscr{L}_{STI}$ respectively. The above equation can be explicitly written as follows;

$$\sum_{\substack{j \\ (j,\tau) \in I_{a(i,t)}}} \sum_{\tau} a_{i,j}(t,\tau) y_{1,i-j}(t-\tau) = \sum_{\substack{j \\ (j,\tau) \in I_{b(i,t)}}} \sum_{\tau} b_{i,j}(t,\tau) y_{4,i-j}(t-\tau) \quad (1)$$

where $I_{a(i,t)}$ (output mask) and $I_{b(i,t)}$ (input mask) denote, respectively, the area region of support for $\{a_{i,j}(t,\tau)\}$ and $\{b_{i,j}(t,\tau)\}$. The system in (1) is well defined if $\{a_{i,0}(t,0)\} \neq 0$, and $\{a_{i,j}(t,\tau)\} \neq 0$ for some (j,τ), and $Supp(\{a_{i,j}(t,\tau)\})$ is a subset of the lattice sector with vertex $(0,0)$ of angle less than 180°, for every pair (i,t) [5]. We will assume that all the spatiotemporal systems under consideration are well defined.

Given an instance in space and time, the plant is thought of as a LSTI system, with the defining operators fixed at that time and space. The controllers are designed for the corresponding frozen LSTI system. Allowing the flexibility of using a designed controller for several instances in time and space, the controller is designed every T steps in time and every S steps in space. Define $n_t = nT$ and $k_i = kS$, where n and k are smallest integers such that t and i lie in the interval $[nT,\ (n+1)T]$ and $[kS,\ (k+1)S]$ respectively. The controller is designed at intervals of nT, and kS in time and space respectively. The closed loop is stable if the map

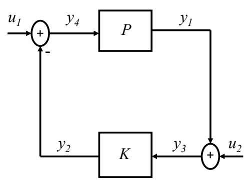

Fig. 1. General form of closed loop.

from $u_1,\ u_2$ to $y_1,\ y_2$ is bounded. The dynamics of the control law are given by

$$(L_{k_i,n_t} y_2)_i(t) = (M_{k_i,n_t} y_3)_i(t)$$

where $L_{k_i,n_t},\ M_{k_i,n_t} \in \mathscr{L}_{STI}$ for each pair of indices (k_i,n_t). The evolution of these operators is given by

$$\begin{aligned}(L_{k_i,n_t} y_2)_i(t) &= \sum_{\tau=0}^{\tau=t} \sum_{j=-\infty}^{j=\infty} l_{k_i,i-j}(n_t, t-\tau) y_{2,j}(\tau) \\ (M_{k_i,n_t} y_3)_i(t) &= \sum_{\tau=0}^{\tau=t} \sum_{j=-\infty}^{j=\infty} m_{k_i,i-j}(n_t, t-\tau) y_{3,j}(\tau)\end{aligned}$$

The frozen space and time operator that defines the above control law satisfies the following Bezout identity

$$L_{k_i,n_t} A_{k_i,n_t} + M_{k_i,n_t} B_{k_i,n_t} = G_{k_i,n_t}$$

where $G^{-1}_{k_i,n_t} \in \mathscr{L}_{STI}$ for each fixed pair (k_i,n_t). That is, for every frozen plant given by $A_{k_i,n_t},\ B_{k_i,n_t}$, the control generated by $L_{k_i,n_t},\ M_{k_i,n_t}$ is such that the "frozen" closed loop map $G^{-1}_{k_i,n_t}$ is stable. Note that the frozen plant is LSTI, and hence a frozen LSTI controller that satisfies the frozen closed loop can be obtained using various methods, e.g. [6], [7]. Here, the interest does not lie in any specific method as long as K operates as described above and provides frozen stability.

The fact that the controller is updated only every T steps in time and after every S number of plants in space introduces a new parameter in the stability analysis. From Figure 1, the closed loop equations for the controlled system can be written as follows:

$$(A_{i,t} y_1)_i(t) = (B_{i,t}(u_1 - y_2))_i(t) \tag{2}$$

$$(L_{k_i,n_t} y_2)_i(t) = (M_{k_i,n_t}(u_2 + y_1))_i(t) \tag{3}$$

$$L_{k_i,n_t} A_{k_i,n_t} + M_{k_i,n_t} B_{k_i,n_t} = G_{k_i,n_t} \tag{4}$$

In the following a relation that connects the input sequences $\{u_{1,i}(t)\}$, $\{u_{2,i}(t)\}$ to the outputs $\{y_{1,i}(t)\}$ and $\{y_{2,i}(t)\}$ is obtained. Operating on equation (2) by L_{k_i,n_t}, we get

$$(L_{k_i,n_t} \cdot A_{i,t} y_1)_i(t) = (L_{k_i,n_t} \cdot B_{i,t} u_1)_i(t) - (L_{k_i,n_t} \cdot B_{i,t} y_2)_i(t)$$

Adding, subtracting, and grouping certain terms yields the following:

$$\begin{aligned}&\{(L_{k_i,n_t} A_{k_i,n_t} + B_{k_i,n_t} M_{k_i,n_t}) y_1 + (L_{k_i,n_t} \nabla A_{i,t} + (L_{k_i,n_t} A_{i,t} \\ &- L_{k_i,n_t} A_{k_i,n_t}) + B_{i,t} \nabla M_{k_i,n_t} + (B_{i,t} M_{k_i,n_t} - B_{k_i,n_t} M_{k_i,n_t})) y_1 \\ &+ (L_{k_i,n_t} \nabla B_{i,t} - B_{i,t} \nabla L_{k_i,n_t}) y_2\}(i,t) \\ &= (L_{k_i,n_t} \cdot B_{i,t} u_1)(i,t) - (B_{i,t} \cdot M_{k_i,n_t} u_2)(i,t)\end{aligned}$$

where the notation; $A_{i,t} \nabla B_{i,t} = A_{i,t} \cdot B_{i,t} - A_{i,t} B_{i,t}$ has been used, i.e. $A_{i,t} \nabla B_{i,t}$ is the difference between the global and local product of operators given a pair (i,t). To obtain a second closed loop equation, operate on equation (2) by M_{k_i,n_t}:

$$(M_{k_i,n_t} \cdot A_{i,t} y_1)_i(t) = (M_{k_i,n_t} \cdot B_{i,t} u_1)_i(t) - (M_{k_i,n_t} \cdot B_{i,t} y_2)_i(t)$$

Again adding, subtracting, and grouping certain terms yields the following:

$$\begin{aligned}&\{(M_{k_i,n_t} B_{k_i,n_t} + A_{k_i,n_t} L_{k_i,n_t}) y_2 + (M_{k_i,n_t} \nabla B_{i,t} + (M_{k_i,n_t} B_{i,t} \\ &- M_{k_i,n_t} B_{k_i,n_t}) + A_{i,t} \nabla L_{k_i,n_t} + (A_{i,t} L_{k_i,n_t} - A_{k_i,n_t} L_{k_i,n_t})) y_2 \\ &+ (A_{i,t} \nabla M_{k_i,n_t} - M_{k_i,n_t} \nabla A_{i,t}) y_1\}(i,t) \\ &= (M_{k_i,n_t} \cdot B_{i,t} u_1)(i,t) + (A_{i,t} \cdot M_{k_i,n_t} u_2)(i,t)\end{aligned}$$

For $t \in \mathbb{Z}^+,\ i \in \mathbb{Z}$, define the following

$$\begin{aligned}X_{i,t} &= L_{k_i,n_t} \nabla A_{i,t} + (L_{k_i,n_t} A_{i,t} - L_{k_i,n_t} A_{k_i,n_t}) \\ &\quad + B_{i,t} \nabla M_{k_i,n_t} + (B_{i,t} M_{k_i,n_t} - B_{k_i,n_t} M_{k_i,n_t}) \\ Y_{i,t} &= L_{k_i,n_t} \nabla B_{i,t} - B_{i,t} \nabla L_{k_i,n_t} \\ Z_{i,t} &= M_{k_i,n_t} \nabla A_{i,t} - A_{i,t} \nabla M_{k_i,n_t} \\ W_{i,t} &= M_{k_i,n_t} \nabla B_{i,t} + (M_{k_i,n_t} B_{i,t} - M_{k_i,n_t} B_{k_i,n_t}) \\ &\quad + A_{i,t} \nabla L_{k_i,n_t} + (A_{i,t} L_{k_i,n_t} - A_{k_i,n_t} L_{k_i,n_t})\end{aligned}$$

Using (4) the closed loop equation can be written as follows:

$$\begin{aligned}&\begin{bmatrix} G_{k_i,n_t} + X_{i,t} & Y_{i,t} \\ -Z_{i,t} & G_{k_i,n_t} + W_{i,t} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} (i,t) \\ &\qquad = \begin{bmatrix} L_{k_i,n_t} \cdot B_{i,t} & -B_{i,t} M_{k_i,n_t} \\ -M_{k_i,n_t} \cdot B_{i,t} & A_{i,t} M_{k_i,n_t} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} (i,t)\end{aligned} \tag{5}$$

Denote by X, Y, Z, W, G the spatiotemporal varying operators associated with the families $X_{i,t},\ Y_{i,t},\ Z_{i,t},\ W_{i,t},\ G_{k_i,n_t}$ respectively. The idea is to analyze the above system by considering the operators $X,\ Y,\ Z,\ W$ as perturbations. We state below the main result of [3] regarding stability of the system given in (5) without proof.

Theorem 1: Assume the following for system (5):

A1. The operators defining the plant are slowly time and space varying with rates γ_A and γ_B, i.e. $A_{i,t} \in$ SSTV(γ_A), and $B_{i,t} \in$ SSTV(γ_B).

A2. The sequence of controllers are slowly time and space varying, i.e. $M_{k_i,n_t} \in \text{SSTV}(\gamma_M)$ and $L_{k_i,n_t} \in \text{SSTV}(\gamma_L)$.

A3. The l_∞ induced norms and the ITSAE of the operators $A_{i,t}$, $B_{i,t}$, L_{k_i,n_t}, M_{k_i,n_t} are uniformly bounded in i, and t. From this and A1, A2, and the Bezout identity it follows that the operator G_{k_i,n_t} will also be slowly varying in space and time and, hence, we can write $G_{k_i,n_t} \in \text{SSTV}(\gamma_G)$

A4. The l_∞ to l_∞ norms and the ITSAE of the LSTI operators $G^{-1}_{k_i,n_t}$ are bounded uniformly in i, and t.

Then there exists a non-zero constant γ such that if $\gamma_A,\ \gamma_B,\ \gamma_M,\ \gamma_L,\ \gamma_G \leq \gamma$, the closed loop system is internally stable.

IV. Performance Analysis

In this section we seek a relationship between the performance of the frozen-time pair $(P_{i,t}, K_{k_i,n_t})$ and the actual time-varying feedback pair (P, K). This is addressed in the following theorem.

Theorem 2: Let S^{kl} $(k = 1,2,\ l = 1,2,3,4)$ represent the map from u_k to y_l in the system of Figure 1 and $S^{kl}_{i,t}$ the LSTI map from u_k to y_l for the frozen system $(P_{i,t}, K_{k_i,n_t})$. Now, let the assumptions of Theorem 1 hold. Given $\varepsilon > 0$, there exists a nonzero constant γ_p with $\gamma \leq \gamma_p$ such that

$$(1-\varepsilon)\left\|S^{kl}\right\| \leq \sup_{i,t}\left\|S^{kl}_{i,t}\right\| + \varepsilon$$

Proof: Let $u_1 = 0$ and $\|u_2\| \leq 1$. From the system equations we get

$$\begin{aligned} y_{1,i}(t) &= -(H_{k_i,n_t}(G-G_{k_i,n_t})y_1)_i(t) - (H_{k_i,n_t}Xy_1)_i(t) \\ - &\quad (H_{k_i,n_t}Yy_2)_i(t) - (H_{k_i,n_t}(B_{i,t}\cdot M_{k_i,n_t}u_2))_i(t) \end{aligned} \tag{6}$$

Consider now the frozen LSTI feedback system given a pair (i,t) and subjected to the same input u_2. Let $\hat{y}_1$ denote the output that corresponds to y_1 in the time varying loop. Evaluating $\hat{y}_1$ at (i,t) we have

$$\hat{y}_{1,i}(t) = -(H_{k_i,n_t}B_{i,t}M_{k_i,n_t}u_2)_i(t) \tag{7}$$

Subtracting (6) from (7) we obtain

$$\begin{aligned} \hat{y}_{1,i}(t) - y_{1,i}(t) &= (H_{k_i,n_t}(G-G_{k_i,n_t})y_1)_i(t) + (H_{k_i,n_t}Xy_1)_i(t) \\ &+ (H_{k_i,n_t}Yy_2)_i(t) + (H_{k_i,n_t}(B_{i,t}\cdot M_{k_i,n_t} - B_{i,t}M_{k_i,n_t})u_2)_i(t) \end{aligned}$$

The idea is to bound $|(H_{k_i,n_t}(B_{i,t}\cdot M_{k_i,n_t} - B_{i,t}M_{k_i,n_t})u_2)_i(t)|$ by some constant. For this purpose, define the operator $Q \in \mathscr{L}_{STV}$ as

$$(Qz)_{\mathtt{i}}(\tau) = (B_{\mathtt{i},\tau}M_{k_{\mathtt{i}},n_\tau}z)_{\mathtt{i}}(\tau), \qquad \mathtt{i} \in \mathbb{Z},\ \tau \in \mathbb{Z}^+$$

then

$$\begin{aligned} &(H_{k_i,n_t}(B_{i,t}\cdot M_{k_i,n_t} - B_{i,t}M_{k_i,n_t})u_2)_i(t) \\ &= (H_{k_i,n_t}(B_{i,t}\cdot M_{k_i,n_t} - Q)u_2)_i(t) + (H_{k_i,n_t}(Q - B_{i,t}M_{k_i,n_t})u_2)_i(t) \end{aligned}$$

By Lemma 1 in [3], and the fact that H_{k_i,n_t} has uniformly bounded norm, it follows that

$$|(H_{k_i,n_t}(B_{i,t}\cdot M_{k_i,n_t} - Q)u_2)_i(t)| \leq \gamma c_1$$

where c_1 is a positive constant. We have the following for the term $(H_{k_i,n_t}(Q - B_{i,t}M_{k_i,n_t})u_2)_i(t)$:

$$\begin{aligned} \left\|B_{\mathtt{i},\tau}M_{k_{\mathtt{i}},n_\tau} - B_{i,t}M_{k_i,n_t}\right\| &\leq \|B_{\mathtt{i},\tau}\|\left\|M_{k_{\mathtt{i}},n_\tau} - M_{k_i,n_t}\right\| \\ &+ \left\|M_{k_i,n_t}\right\|\|B_{i,t} - B_{\mathtt{i},\tau}\| \\ &\leq \|B_{\mathtt{i},\tau}\|\gamma_M(|i-\mathtt{i}| + |t-\tau|) \\ &+ \left\|M_{k_i,n_t}\right\|\gamma_B(|i-\mathtt{i}| + |t-\tau|) \\ &\leq \gamma c_2(|i-\mathtt{i}| + |t-\tau|) \end{aligned}$$

Hence if $z_{\mathtt{i}}(\tau) = ((Q - B_{i,t}M_{k_i,n_t})u_2)_{\mathtt{i}}(\tau)$, then $|z_{\mathtt{i}}(\tau)| \leq \gamma c_2(|i-\mathtt{i}| + |t-\tau|)$, $\mathtt{i} \in \mathbb{Z},\ \tau \in \mathbb{Z}^+$, with $c_2 > 0$. However, from the fact that H_{k_i,n_t} has bounded (uniformly in t, and i) ITSAE, it follows that

$$\begin{aligned} |(H_{k_i,n_t}(Q - B_{i,t}M_{k_i,n_t})u_2)_i(t)| &= \left|\sum_{\tau=0}^{t}\sum_{\mathtt{i}=-\infty}^{\infty}\left(h_{k_i,i-\mathtt{i}}(n_t, t-\tau)\right)z_{\mathtt{i}}(\tau)\right| \\ &\leq \gamma c_2\sum_{\tau=0}^{t}\sum_{\mathtt{i}=-\infty}^{\infty}\left(h_{k_i,\mathtt{i}}(n_t,\tau)\right)(|\mathtt{i}| + |\tau|) \\ &\leq \gamma c_3,\quad c_3 > 0 \end{aligned}$$

Looking at the rest of the terms, and since $\|u_2\| \leq 1$, we have $|(H_{k_i,n_t}Xy_1)_i(t)| \leq \gamma c_4\left\|S^{12}\right\|$, $|(H_{k_i,n_t}Yy_2)_i(t)| \leq \gamma c_5\left\|S^{22}\right\|$ and $|(H_{k_i,n_t}(G-G_{k_i,n_t})y_1)_i(t)| \leq \gamma c_6\left\|S^{12}\right\|$. Putting every thing together, it follows that there are constants $c,\ c_{12},\ c_{22} > 0$ such that

$$|\hat{y}_{1,i}(t) - y_{1,i}(t)| \leq \gamma c + \gamma c_{12}\left\|S^{12}\right\| + \gamma c_{22}\left\|S^{22}\right\|$$

Since $\|u_2\| \leq 1$, we have $|\hat{y}_{1,i}(t)| \leq \left\|S^{12}_{i,t}\right\|$, and therefore

$$\sup_{i,t}|y_{1,i}(t)| \leq \sup_{i,t}\left\|S^{12}_{i,t}\right\| + \gamma c + \gamma c_{12}\left\|S^{12}\right\| + \gamma c_{22}\left\|S^{22}\right\|$$

Since u_2 is arbitrary

$$\left\|S^{12}\right\| \leq \sup_{i,t}\left\|S^{12}_{i,t}\right\| + \gamma c + \gamma c_{12}\left\|S^{12}\right\| + \gamma c_{22}\left\|S^{22}\right\|$$

Similarly working for $\left\|S^{22}\right\|$ we get

$$\left\|S^{22}\right\| \leq \sup_{i,t}\left\|S^{22}_{i,t}\right\| + \gamma k + \gamma k_{22}\left\|S^{22}\right\| + \gamma k_{12}\left\|S^{12}\right\|$$

Noting that $\left\|H_{k_i,n_t}\right\|$ is uniformly bounded, we have $\sup_{i,t}\left\|S^{12}_{i,t}\right\|, \sup_{i,t}\left\|S^{22}_{i,t}\right\| < \infty$, and hence by assuming γ_p sufficiently small, the proof of the theorem is complete. ∎

V. Conclusion

We have presented performance analysis for slowly varying spatiotemporal systems in the case when the controller design is based on frozen spatially and temporally invariant descriptions of the plant. We have shown that the l_∞ to l_∞ performance of global spatiotemporal varying system cannot be much worse than the worst frozen spatially and temporally l_∞ to l_∞ performance given that the rates of variation of the plant and the controller are sufficiently small.

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