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Dynamic Dual Decomposition for Distributed Control

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***Abstract*— We show how dynamic price mechanisms can be used for decomposition and distributed optimization of feedback systems.**

A classical method to handle large scale optimization problems is dual decomposition, where the coupling between sub-problems is relaxed using Lagrange multipliers. These variables can be interpreted as prices in a market mechanism serving to achieve mutual agreement between solutions of the sub-problems. In this paper, the same idea is used for decomposition of feedback systems, with dynamics in both decision variables and prices. We show how the prices can be used for a decentralized test, to verify that the global feedback system stays within a prespecified distance from optimality.

I. Background

Decision making when the decision makers have access to different information concerning underlying uncertainties has been studied since the late 1950s [11], [13]. The subject is sometimes called team theory, sometimes decentralized or distributed control. The theory was originally static, but work on dynamic aspects was initiated by Witsenhausen [19], who also pointed out a fundamental difficulty in such problems. Some special types of team problems were solved in the 1970's [17], [9], but the problem area has recently gain renewed interest. Spatial invariance was exploited in [2], [3], conditions for closed loop convexity were derived in [16] and methods using linear matrix inequalities were given in [10], [14].

Dual decomposition has been used in large-scale optimization since the early 1960s [7] and a closely related tool is Usawa's algorithm [1]. Decomposition was applied to linear quadratic optimal control in [18] and more general methods for decomposition and coordination of dynamic systems were introduced in [12], [8], [5], [6]. The purpose of this paper is to investigate how the same methods can be used for analysis and synthesis of distributed feedback controllers.

In our previous paper [15], we used dual decomposition for iterative decentralized feedback synthesis in a vehicle formation. Here we generalize the approach to coupled dynamic systems and combine distributed performance validation with control synthesis.

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II. Dual Decomposition and the Saddle Algorithm

The following example explains the idea of dual decomposition. Suppose that the minimization problem

$$\widehat{J} = \min_{z_i}[V_1(z_1, z_2) + V_2(z_2) + V_3(z_3, z_2)]$$

is to be solved by three computers working in parallel, with one computer devoted to each term of the objective function. If V_1, V_2 and V_3 are all convex [4], the problem can be rewritten as

$$\widehat{J} = \max_{p_i} \min_{z_i, v_i} \Big[V_1(z_1, v_1) + V_2(z_2) + V_3(z_3, v_3) + p_1(z_2 - v_1) + p_3(z_2 - v_3)\Big]$$

This decomposes the problem into five separate optimization problems:

Computer 1:	$\min_{z_1,v_1} \left[V_1(z_1, v_1) - p_1 v_1\right]$
Computer 2:	$\min_{z_2} \left[V_2(z_2) + (p_1 + p_3)z_2\right]$
Computer 3:	$\min_{z_3,v_3} \left[V_3(z_3, v_3) - p_3 v_3\right]$
Between 1 and 2:	$\max_{p_1} \left[p_1(z_2 - v_1)\right]$
Between 2 and 3:	$\max_{p_3} \left[p_3(z_2 - v_3)\right]$

The decomposition has a natural interpretation in economic terms: The three functions V_1, V_2 and V_3 can be interpreted as costs that arise for each of three agents given certain values of the variables z_1, z_2 and z_3. When all agents try to minimize their own cost, they arrive at different opinions about the desirable value of z_2. With introduction of prices, the agents can pay each other to modify the values and find a common equilibrium. This is what happens at the saddle point, where the prices p_1, p_3 create a consensus among the three agents about the desirable values of z_2.

In game theoretic terms, one can say that the original minimization problem is a *team problem* where three different agent are acting to optimize the common objective function $\widehat{J}$. After decomposition, we are instead dealing with non-cooperative game of five players. In addition to the three computers, there are two "market makers" who adjust the price variables p_1 and p_3 to take advantage of any violations of the constraints $z_2 = v_1$, $z_2 = v_3$. A Nash equilibrium of the five player game corresponds to a global optimum of the original optimization problem. In fact, also the search for optimal values of the variables can be decomposed, using a gradient search:

Computer 1:	$\begin{cases} \dot{z}_1 = -\partial V_1/\partial z_1 \\ \dot{v}_1 = -\partial V_1/\partial z_2 + p_1 \end{cases}$
Computer 1 and 2:	$\dot{p}_1 = z_2 - v_1$
Computer 2:	$\dot{z}_2 = -\partial V_2/\partial z - p_1 - p_3$
Computer 2 and 3:	$\dot{p}_3 = z_2 - v_3$
Computer 3:	$\begin{cases} \dot{z}_3 = -\partial V_3/\partial z_3 \\ \dot{v}_3 = -\partial V_3/\partial z_2 + p_3 \end{cases}$

A remarkable theorem from 1958 proves global convergence towards the saddle point under general conditions:

Proposition 1 (Arrow, Hurwicz, Usawa [1]): Assume that $V \in C^1(\mathbf{R}^n)$ is strictly convex with gradient ∇V, while G and H are positive definite and R has full row rank. Then, all solutions to

$$\dot{z} = -G\left[(\nabla V)^T - R^T p\right] \tag{1}$$

$$\dot{p} = -HRz \tag{2}$$

converge to the unique saddle point (z_*, p_*) attaining

$$\max_p \min_z \left[V(z) - p^T Rz\right] \tag{3}$$

Proof. Let $\phi(z,p) = V(z) - p^T Rz$. Then

$$\dot{z} = -G\left[\nabla_z \phi(z,p)\right]^T \qquad \dot{p} = H\left[\nabla_p \phi(z,p)\right]^T$$

where G and H are positive definite. Define the Lyapunov function

$$W(z,p) = \frac{1}{2}\left(|z - z_*|^2_{G^{-1}} + |p - p_*|^2_{H^{-1}}\right)$$

Then convexity of ϕ implies that

$$\begin{aligned} \dot{W} &= \dot{z}^T G^{-1}(z - z_*) + \dot{p}^T H^{-1}(p - p_*) \\ &= \left[\nabla_z \phi(z,p)\right](z_* - z) + \left[\nabla_p \phi(z,p)\right](p - p_*) \\ &\leq \left[\phi(z_*,p) - \phi(z,p)\right] + \left[\phi(z,p) - \phi(z,p_*)\right] \\ &= \left[\phi(z_*,p_*) - \phi(z,p_*)\right] + \left[\phi(z_*,p) - \phi(z_*,p_*)\right] \leq 0 \end{aligned}$$

with equality if and only if $z = z_*$. Hence, by LaSalle's theorem, $(z(t), p(t))$ tends towards M, the largest invariant set in the subspace $z = z_*$. Invariance means that z is constant. Hence $\nabla V(z)^T = R^T p$, so also p is constant and the only point in M is (z_*, p_*). This completes the proof. □

Yet another important feature of dual decomposition is that strict upper and lower bounds on the optimal cost are obtained even before optimum has been reached. In particular, if $p_1, p_3, \bar{z}_1, \bar{z}_2, \bar{z}_3$ satisfy the distributed test

$$V_1(\bar{z}_1, \bar{z}_2) - p_1 \bar{z}_2 \leq \alpha \min_{z_1, v_1} \left[V_1(z_1, v_1) - p_1 v_1\right]$$

$$V_2(\bar{z}_2) + (p_1 + p_3)\bar{z}_2 \leq \alpha \min_{z_2} \left[V_2(z_2) + (p_1 + p_3) z_2\right]$$

$$V_3(\bar{z}_3, \bar{z}_2) - p_3 \bar{z}_2 \leq \alpha \min_{z_3, v_3} \left[V_3(z_3, v_3) - p_3 v_3\right]$$

for some $\alpha \geq 1$, then the globally optimal cost J^* is bounded as

$$J^* \leq V_1(\bar{z}_1, \bar{z}_2) + V_2(\bar{z}_2) + V_3(\bar{z}_3, \bar{z}_2) \leq \alpha J^*$$

The first inequality follows trivially from the definition of J^*. The second follows by adding up the three previous inequalities and noting that the resulting right hand side has more freedom in the minimization than the definition of J^*.

III. Dynamic Dual Decomposition

With notation $|x|^2_Q = x^T Q x$, define

$$\ell_i(x_i, u_i) = |x_i|^2_{Q_i} + |u_i|^2_{R_i}$$

with $Q_i, R_i > 0$ for $i = 1, \ldots, J$. Consider the stochastic optimal control problem

$$\widehat{J} = \min_\mu \mathbf{E} \sum_{i=1}^J \ell_i(x_i, u_i) \tag{4}$$

with minimization over control laws $u_i(t) = \mu_i(x(t))$ and stationary solutions $x_i(t)$ to the state equations

$$x_i(t+1) = \sum_{j=1}^J A_{ij} x_j(t) + B_i u_i(t) + w_i(t) \tag{5}$$

where $i = 1, \ldots, J$ and $w_1, \ldots, w_J$ are independent white noise processes. The problem has an associated graph, with one node for every i and an edge connecting j and i if and only if A_{ij} and A_{ji} are not both zero.

To decompose this problem, we introduce variables v_i as in [6] and write the state equations as

$$x_i(t+1) = A_{ii} x_i(t) + B_i u_i(t) + v_i(t) + w_i(t) \tag{6}$$

with the additional constraints that

$$v_i(t) = \sum_{j \neq i} A_{ij} x_j(t) \qquad \text{for all } t \tag{7}$$

The constraints are then relaxed by introduction of corresponding Lagrange multipliers in the cost function:

$$\begin{aligned} &\max_p \min_{\mu, \eta} \sum_{i=1}^J \mathbf{E}\left[\ell_i(x_i, u_i) + 2p_i^T \left(v_i - \textstyle\sum_{j \neq i} A_{ij} x_j\right)\right] \\ &= \max_p \sum_i \underbrace{\min_{\mu_i, \eta_i} \mathbf{E}\left[\ell_i(x_i, u_i) + 2p_i^T v_i - 2\left(\textstyle\sum_{j \neq i} p_j^T A_{ji}\right) x_i\right]}_{J_i(x_i, u_i, v_i, p)} \end{aligned}$$

The prices $p_i(t)$ are stationary processes and minimization is over control laws $u_i = \mu_i(x)$, $v_i = \eta_i(x)$.

As in the previous section, the introduction of dual variables decomposes the optimization problem into separate criteria for every node in the graph. The objective of the agent in node i is to minimize

$$\underbrace{\mathbf{E}\ell_i(x_i, u_i)}_{\text{his own cost}} \quad \overbrace{2\mathbf{E}p_i^T v_i}^{\text{what he expects others to charge him}} \quad \underbrace{-2\mathbf{E}\left(\textstyle\sum_{j \neq i} p_j^T A_{ji}\right) x_i}_{\text{what he receives from others}}$$

The variable v_i can be interpreted as the expected influence of other agents in the update of x_i.

The following theorem, a standard application of duality theory, shows how bounds on the global distance from optimality can be obtained from corresponding bounds for individual agents.

Theorem 1: Consider control laws $\bar{u}_i = -\sum_j \bar{L}_{ij}\bar{x}_j$ and corresponding stationary solutions to the state equations (5). For given white noise processes w_i, suppose there exist price processes p_i such that

$$J_i(\bar{x}_i, \bar{u}_i, \textstyle\sum_{j\neq i} A_{ij}\bar{x}_j, p) \leq \alpha \min_{\mu_i,\eta_i} J_i(x_i, u_i, v_i, p) \qquad (8)$$

when minimizing over control laws

$$u_i(t) = \mu_i(x(t)) \qquad v_i(t) = \eta_i(x(t))$$

and stationary solutions of (6), (7). Then

$$\widehat{J} \leq \mathbf{E}\sum_{i=1}^{J} \ell_i\big(\bar{x}(t), \bar{u}(t)\big) \leq \alpha \widehat{J}$$

Remark 1. The left hand side of (8) can be interpreted as a the cost for agent i under the *actual* influence of other agents, while the minimum on the right hand side is the cost for agent i under the *most desirable* behavior of other agents.

Remark 2. Even if $\bar{u}_i$ are given by a distributed control law, i.e. $\bar{L}_{ij} \neq 0$ only when j and i are neighbors, the right hand side of (8) still needs to be evaluated for control laws with full state information. In a future publication, we hope to state a more advanced version of the theorem, where each agent instead compares his current performance with the performance that would be achievable with access also to the information that his neighbors now use.

Proof.

$$\begin{aligned}
\widehat{J} &\leq \sum_i \mathbf{E}\ell_i\big(\bar{x}_i, \bar{u}_i\big)\\
&= \sum_i J_i(\bar{x}, \bar{u}, \textstyle\sum_{j\neq i} A_{ij}\bar{x}_j, p)\\
&\leq \alpha \sum_i \min_{\mu_i,\eta_i} J_i(x, u, v_i, p)\\
&\leq \alpha \min_{\mu} \sum_i J_i(x, u, \textstyle\sum_{j\neq i} A_{ij}x_j, p)\\
&= \alpha \min_{\mu} \sum_i \mathbf{E}\ell_i\big(x_i, u_i\big) = \alpha\widehat{J}
\end{aligned}$$

□

For a converse result, existence of prices that allow for distributed verification of optimality can be proved by application of a discrete version of Pontryagin's maximum principle, introducing $p_i(t)$ through the adjoint equations

$$p_i(t-1) = \textstyle\sum_j (A_{ij} + B_i\bar{L}_{ij})^T p_j(t) - Q_i x_i(t) - \sum_j \bar{L}_{ji}^T R_j u_j(t) \qquad (9)$$

However, prices introduced this way will depend non-causally on the disturbances w, even though the anti-causal part is irrelevant for the evaluation of $\sum_i J_i(x_i, u_i, v_i, p)$ when x_i, u_i and v_i have causal w-dependence. As an alternative, we can introduce causal prices as follows:

Theorem 2: Suppose (6) and (7) have the form

$$x(t+1) = \bar{A}x(t) + Bu(t) + v(t) + w(t) \qquad (10)$$

and $v_i = \widetilde{A}_i x$. Let $A = \bar{A} + \widetilde{A}$ and let $P > 0$ and L, M be determined by

$$|x|_P^2 = \min_u \big(|Ax + Bu|_P^2 + |x|_Q^2 + |u|_R^2\big) \qquad (11)$$

$$L = (R + B^T PB)^{-1} B^T PA \qquad (12)$$

$$M = P(A - BL) \qquad (13)$$

Given the white noise w, let $\bar{x}$, $\bar{u}$ and p be defined by

$$\bar{x}(t+1) = A\bar{x}(t) + B\bar{u}(t) + w(t) \qquad (14)$$

$$\bar{u}(t) = -L\bar{x}(t) \qquad (15)$$

$$p(t) = -M\bar{x}(t) \qquad (16)$$

Then (8) holds with $\alpha = 1$ for $i = 1, \ldots, J$.

Proof. Combining (10) and $v = \widetilde{A}x$ gives

$$x(t+1) = Ax(t) + Bu(t) + w(t)$$

By standard theory, the LQ optimal control law is

$$u = -Lx = \arg\min_u \big(|Ax + Bu|_P^2 + |x|_Q^2 + |u|_R^2\big)$$

An alternative way of writing (11) is

$$|x|_P^2 = \max_p \min_{u,v} \Big[|\bar{A}x + Bu + v|_P^2 + |x|_Q^2 + |u|_R^2 + 2p^T(v - \widetilde{A}x)\Big]$$

where the saddle-point on right hand side is given by $v = \widetilde{A}x$ together with (12)-(13) and (15)-(16). Hence

$$\sum_i J_i(\bar{x}_i, \bar{u}_i, \widetilde{A}_i\bar{x}, p) = \min_{\mu,\eta} \sum_i J_i(x_i, u_i, v_i, p) \qquad (17)$$

At the same time we have by definition

$$\min_{\mu_i,\eta_i} J_i(x_i, u_i, v_i, p) \leq J_i(\bar{x}_i, \bar{u}_i, \widetilde{A}_i\bar{x}, p) \qquad (18)$$

for every i. Combining (17) and (18) gives (8) with $\alpha = 1$ for every i and the proof is complete. □

The section is concluded by an example with four agents connected in a one-dimensional graph:

Example 1 Theorem 1 and Theorem 2 will here be used to perform distributed performance validation of decentralized control laws for the linear system

$$x(t+1) = Ax(t) + u(t) + w(t) \qquad (19)$$

with

$$A = \begin{bmatrix} 0.6 & 0.1 & 0 & 0\\ 0.3 & 0.6 & 0.1 & 0\\ 0 & 0.3 & 0.6 & 0.1\\ 0 & 0 & 0.3 & 0.6 \end{bmatrix}$$

The decoupled dynamics (10) can be written

$$\begin{aligned} x_1(t+1) &= 0.6x_1(t) + u_1(t) + v_1(t) + w_1(t) \\ x_2(t+1) &= 0.6x_2(t) + u_2(t) + v_2(t) + w_2(t) \\ x_3(t+1) &= 0.6x_3(t) + u_3(t) + v_3(t) + w_3(t) \\ x_4(t+1) &= 0.6x_4(t) + u_4(t) + v_4(t) + w_4(t) \end{aligned}$$

with the constraints

$$\begin{aligned} v_1(t) &= 0.1x_2(t) \\ v_2(t) &= 0.3x_1(t) + 0.1x_3(t) \\ v_3(t) &= 0.3x_2(t) + 0.1x_4(t) \\ v_4(t) &= 0.3x_3(t) \end{aligned}$$

Consider the optimal control problem

$$\min_{\mu} \mathbf{E} \sum_{i=1}^{4} \left(|x_i|^2 + |u_i|^2\right) = \max_{p} \min_{\mu,\eta} \sum_i J_i(x_i, u_i, v_i, p)$$

where

$$\begin{aligned} J_1(x_1,u_1,v_1,p) &= \mathbf{E}\left(|x_1|^2 + |u_1|^2 + 2p_1v_1 - 0.6p_2x_1\right) \\ J_2(x_2,u_2,v_2,p) &= \mathbf{E}\left(|x_2|^2 + |u_2|^2 + 2p_2v_2 - (0.2p_1 + 0.6p_3)x_2\right) \\ J_3(x_3,u_3,v_3,p) &= \mathbf{E}\left(|x_3|^2 + |u_3|^2 + 2p_2v_2 - (0.2p_2 + 0.6p_4)x_3\right) \\ J_4(x_4,u_4,v_4,p) &= \mathbf{E}\left(|x_4|^2 + |u_4|^2 + 2p_4v_4 - 0.2p_3x_4\right) \end{aligned}$$

is obtained by

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = -\underbrace{\begin{bmatrix} 0.3420 & 0.0737 & 0.0046 & 0.0002 \\ 0.1839 & 0.3448 & 0.0736 & 0.0047 \\ 0.0103 & 0.1840 & 0.3447 & 0.0726 \\ 0.0008 & 0.0104 & 0.1808 & 0.3296 \end{bmatrix}}_{L} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

$$\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{bmatrix} = -\underbrace{\begin{bmatrix} 0.3420 & 0.0737 & 0.0046 & 0.0002 \\ 0.1839 & 0.3448 & 0.0736 & 0.0047 \\ 0.0103 & 0.1840 & 0.3447 & 0.0726 \\ 0.0008 & 0.0104 & 0.1808 & 0.3296 \end{bmatrix}}_{M} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

(Notice that $L = M$ when $R = B^T$.) The corresponding stationary node costs are

$$\begin{aligned} J_1^* &= 1.3145 \\ J_2^* &= 1.2705 \\ J_3^* &= 1.2674 \\ J_4^* &= 1.1510 \end{aligned}$$

with the total cost

$$J^* = \mathbf{E} \sum_{i=1}^{4} \left(|x_i|^2 + |u_i|^2\right) = 5.0033$$

The diagonal dominance suggests that L could be approximated by

$$\bar{L} = \begin{bmatrix} 0.3420 & 0 & 0 & 0 \\ 0 & 0.3448 & 0 & 0 \\ 0 & 0 & 0.3447 & 0 \\ 0 & 0 & 0 & 0.3296 \end{bmatrix}$$

without too much deviation from optimality. The accuracy of this approximation will now be evaluated using the distributed test of Theorem 1 with prices generated by the same approximation $p = \bar{M}x$, $\bar{M} = \bar{L}$.

Running (19) with the control law $u = -\bar{L}x$ gives the solution $(\bar{x}, \bar{u})$ with the total cost

$$\mathbf{E} \sum_{i=1}^{4} \left(|\bar{x}_i(t)|^2 + |\bar{u}_i(t)|^2\right) = 5.3349$$

Define $\bar{p} = -\bar{M}\bar{x} = \bar{u}$. The costs in the individual nodes then become

$$\begin{aligned} J_1(\bar{x}_1, \bar{u}_1, 0.1\bar{x}_2, \bar{p}) &= 1.2375 \\ J_2(\bar{x}_2, \bar{u}_2, 0.3\bar{x}_1(t) + 0.2\bar{x}_3(t), \bar{p}) &= 1.3710 \\ J_3(\bar{x}_3, \bar{u}_3, 0.2\bar{x}_2(t) + 0.3\bar{x}_4(t), \bar{p}) &= 1.3862 \\ J_4(\bar{x}_4, \bar{u}_4, 0.1\bar{x}_3, \bar{p}) &= 1.3403 \end{aligned}$$

It should be noted that $\bar{M}$ only affects the individual costs, not the total. Moreover, the total cost always grows with deviations from the optimal control law, but this is not necessarily the case with individual costs.

It remains to find α such that (8) holds for all i. In particular, for $i = 1$ the value

$$J_1(\bar{x}_1, \bar{u}_1, 0.1\bar{x}_2, \bar{p}) = 1.2375$$

should be compared with

$$\min_{\mu_1,\eta_1} J_1(x_1, u_1, v_1, \bar{p})$$

when minimizing over $u_1 = \mu_1(x)$, $v_1 = \eta_1(x)$ and stationary solutions to

$$x_1(t+1) = A_{11}x_1(t) + u_1(t) + v_1(t) + w_1(t)$$

For each i, a standard LQG optimization gives the appropriate number for comparison. Writing (8) for all four nodes gives

$$\begin{aligned} 1.2375 &\leq 1.2116\,\alpha \\ 1.3710 &\leq 1.1886\,\alpha \\ 1.3862 &\leq 1.1853\,\alpha \\ 1.3403 &\leq 1.1723\,\alpha \end{aligned}$$

Hence the distributed condition (8) of Theorem 1 holds with $\alpha = \frac{1.3862}{1.1853} = 1.17$ the theorem concludes that the control performance of the decentralized controller

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = -\bar{L}x(t) = \begin{bmatrix} -0.3420x_1(t) \\ -0.3448x_2(t) \\ -0.3447x_3(t) \\ -0.3296x_4(t) \end{bmatrix}$$

is at most 17% worse than optimal. Not surprisingly, the bound is conservative. Comparing the actual total costs $\frac{5.3349}{5.0033} = 1.0663$ shows that the actual deviation from optimality is only 6.6%.

Repeating the same calculations for the control law

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = -\begin{bmatrix} 0.3420 & 0.0737 & 0 & 0 \\ 0.1839 & 0.3448 & 0.0736 & 0 \\ 0 & 0.1840 & 0.3447 & 0.0726 \\ 0 & 0 & 0.1808 & 0.3296 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

and the corresponding price generator $\bar{M}$ verifies that the deviation from optimality is less than 1%. □

IV. Distributed Gradient Iterations for Synthesis

Given the successful application of dual decomposition for analysis of optimal control problems, it is natural to consider also control synthesis. Below, we will use inspiration from Proposition 1 and Takahara's algorithm [18], [6] to sketch how distributed synthesis of feedback controllers can be done in analogy with the classical algorithms for distributed optimization.

In section III, the stochastic linear quadratic control problem was rewritten as

$$\max_p \sum_i \min_{\mu_i, \eta_i} \mathbf{E}\left[\ell_i(x_i, u_i, v_i) + 2p_i^T v_i - 2\left(\sum_{j\neq i} p_j^T A_{ji}\right) x_i\right]$$

where the optimal v_i is given by (7). By Pontryagin's maximum principle, optimal prices p_i are generated by the adjoint equation (9) and the optimal control law $u_i = -\sum_j L_{ij}x_j$ must minimize the Hamiltonian

$$\sum_i \mathbf{E}\left[\ell_i(x_i, -\textstyle\sum_j L_{ij}x_j, v_i) - 2p_i^T\left(\sum_j A_{ij}x_j - B_i L_{ij}x_j\right)\right]$$

Differentiating with respect to L_{ij} gives the gradient

$$\nabla_{ij} = -2R_i\mathbf{E}(u_i x_j^T) + 2B_i^T\mathbf{E}(p_i x_j) \qquad (20)$$

Hence a distributed gradient algorithm can be constructed as follows:

Algorithm 1.

1) Run the system with $u = -L^k x$ for $t = 1, \dots, N$ to let each node i compute $\sum_{t=1}^N u_i(t)x_j(t)^T/N$ as an estimate for $\mathbf{E}(u_i x_j^T)$.
2) Using data for $t = 1, \dots, N$ compute $p_i(t)$ backwords in time using the adjoint equation (9), then $\sum_{t=1}^N p_i(t)x_j(t)^T/N$ to estimate $\mathbf{E}(p_i x_j^T)$.
3) Estimate the gradient ∇_{ij} using (20) and let $L_{ij}^{k+1} = L_{ij}^k + \gamma\nabla_{ij}$ for some appropriate step length γ.
4) If the gradient is smaller than some treshhold, then stop, else restart from 1).

Example 2 The previous example is reconsidered to iteratively update control laws of the form

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & 0 & 0 \\ l_{21} & l_{22} & l_{23} & 0 \\ 0 & l_{32} & l_{33} & l_{34} \\ 0 & 0 & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

Starting from $L^0 = 0$ with $\gamma = 0.02$ gives

$$L^1 = \begin{bmatrix} 0.24 & 0.32 & 0 & 0 \\ 0.27 & 0.46 & 0.43 & 0 \\ 0 & 0.39 & 0.46 & 0.32 \\ 0 & 0 & 0.27 & 0.24 \end{bmatrix} \quad L^2 = \begin{bmatrix} 0.26 & 0.29 & 0 & 0 \\ 0.26 & 0.45 & 0.39 & 0 \\ 0 & 0.36 & 0.45 & 0.29 \\ 0 & 0 & 0.26 & 0.25 \end{bmatrix}$$

$$L^3 = \begin{bmatrix} 0.26 & 0.27 & 0 & 0 \\ 0.25 & 0.44 & 0.36 & 0 \\ 0 & 0.34 & 0.44 & 0.27 \\ 0 & 0 & 0.25 & 0.26 \end{bmatrix} \quad L^4 = \begin{bmatrix} 0.27 & 0.25 & 0 & 0 \\ 0.25 & 0.43 & 0.33 & 0 \\ 0 & 0.33 & 0.43 & 0.25 \\ 0 & 0 & 0.24 & 0.27 \end{bmatrix}$$

which asymptotically approaches a tridiagonal approximation of the LQ optimal control law. Further analysis of such iterations will be given in a future publication.

Matlab scripts for the examples of this paper are available from the web site of this paper at `http://www.control.lth.se/publications`. □

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References

[1] K.J. Arrow, L. Hurwicz, and H. Uzawa. Studies in linear and nonlinear programming. Stanford University Press, 1958.

[2] Bassam Bamieh, Fernando Paganini, and Munther A. Dahleh. Distributed control of spatially invariant systems. *IEEE Transactions on Automatic Control*, 47(7), July 2002.

[3] Bassam Bamieh and Petros G. Voulgaris. A convex characterization of distributed control problems in spatially invariant systems with communication constraints. *Systems & Control Letters*, 54(6):575–583, June 2005.

[4] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[5] Guy Cohen. Auxiliary problem principle and decomposition of optimization problems. *Journal of Optimization Theory and Applications*, 32(3), November 1980.

[6] Guy Cohen and Bernadotte Miara. Optimization with an auxiliary constraint and decomposition. *SIAM Journal on Control and Optimiztion*, 28(1):137–157, January 1990.

[7] G.B. Danzig and P. Wolfe. The decomposition algorithm for linear programming. *Econometrica*, 4, 1961.

[8] W. Findeisen. *Control and coordination in hierarchical systems*. International series on applied systems analysis. Wiley, 1980. ISBN 0-471-27742-8.

[9] Yu-Chi Ho. Team decision theory and information structures. *Proceedings of the IEEE*, 68(6):644–654, June 1980.

[10] Cédric Langbort, Ramu Sharat Chandra, and Raffaello D´Andrea. Distributed control design for systems interconnected over an arbitrary graph. *IEEE Transactions on Automatic Control*, 49(9):1502–1519, September 2004.

[11] J. Marshak. Elements for a theory of teams. *Management Sci.*, 1:127–137, 1955.

[12] M. D. Mesarovic, D. Macko, and Y. Takahara. *Theory of Hierarchical Multilevel Systems*. Academic Press, New York, 1970.

[13] R. Radner. Team decision problems. *Ann. Math. Statist.*, 33(3):857–881, 1962.

[14] Anders Rantzer. A separation principle for distributed control. In *Proceedings of the 45th IEEE Conference on Decision and Control*, San Diego, CA, December 2006.

[15] Anders Rantzer. On prize mechanisms in linear quadratic team theory. In *Proc. 46th IEEE Conference on Decision and Control*, New Orleans, LA, December 2007.

[16] M. Rotkowitz and S. Lall. A characterization of convex problems in decentralized control. *IEEE Trans. on Automatic Control*, 51(2):274–286, Feb 2006.

[17] Nils R. Sandell and Michael Athans. Solution of some nonclassical LQG stochastic decision problems. *IEEE Transactions on Automatic Control*, 19(2):108–116, April 1974.

[18] Y. Takahara. Multilevel approach to dynamic optimization. Technical Report SRC-50-C-64-18, Case Western Reserve, Cleveland, Ohio, 1964.

[19] H.S. Witsenhausen. A counterexample in stochastic optimum control. *SIAM Journal on Control*, 6(1):138–147, 1968.