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# Dynamic Output Feedback Control for a Class of Stochastic Time-Delay Systems

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***Abstract*— This paper is concerned with the problem of robust control for a class of uncertain stochastic systems with discrete and distributed time-varying delays via dynamic output feedback control. The parameter uncertainties of system matrices are assumed to be norm-bounded. The purpose is to design a dynamic output feedback controller such that the closed-loop system is robustly stochastically stable for all admissible parameter uncertainties. Based on the stability theory of stochastic systems, a new delay-dependent sufficient condition for the existence of such a controller is obtained and the corresponding controller design method is proposed in terms of linear matrix inequality (LMI). Numerical examples are included to illustrate the effectiveness and the benefits of the proposed method.**

## I. INTRODUCTION

Stochastic systems have been widely applied in the fields of communications and radar, dynamic reliability, automation, biological engineering, social sciences due to the fact that, in many engineering systems, the internal parameters, external disturbances and observation noises tend to be stochastic. Over the past few decades, stochastic systems have been extensively investigated in such aspects as system analysis, engineering applications and control synthesis including controllability, robust $H_\infty$ control, stability and stabilization, see, e.g., [1]–[3], and the references therein.

On the other hand, time-delay arises frequently in many practical systems, and can severely degrade closed-loop system performance, in some cases, drive the system to instability. Therefore, the problem of stability analysis and control synthesis for stochastic time-delay systems has received increasing interests. For example, in [4], for a class of stochastic systems with multiple time-delays, a delay-dependent stability criterion was obtained by model transformations and cross term estimation techniques. Another stability criterion was derived by introducing some slack matrices in [5]. Most recently, in order to derive a much less conservative stability condition, in [6], avoid using both model transformation and bounding technique for cross terms, an improved stability criterion for uncertain stochastic time-delay systems was got based on an integral inequality. Furthermore, the problems of robust stochastic stabilization and robust $H_\infty$ control for uncertain neutral stochastic time-delay systems via state feedback control were investigated in [7], while the problem of static output feedback stabilization for stochastic systems with discrete time-delay was considered in [8]. However, to our knowledge, there are few results on dynamic output feedback control for uncertain stochastic systems with both discrete and distributed time-varying delays, which motivates this paper.

In this paper, we consider the problem of dynamic output feedback control for a class of uncertain stochastic systems with both discrete and distributed time-varying delays. By employing Lyapunov Functional theory, a delay-dependent LMI condition is given that can guarantee the stochastic stability of closed-loop system. And the explicit expression of the desired controller gain matrices is also given. Numerical examples show the effectiveness and the benefits of our results as compared to that obtained by other methods.

Notations: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n\times m$ real matrices. The superscripts $-1$ and $T$ indicate the inverse and the transpose of a matrix, respectively. $*$ denotes the symmetry part of a symmetry matrix. $I$ is the identity matrix of appropriate dimension. $diag\{\cdots\}$ denotes a block-diagonal matrix. The symmetry matrix $X>0$ (or $X\geq 0$) means that $X$ is positive definite (or semi-positive definite). $\mathscr{E}\{\cdot\}$ denotes the mathematical expectation operator with respect to some probability measure $\mathscr{P}$. $(\Omega,\mathscr{F},\mathscr{P})$ is a probability space with $\Omega$ the sample space and $\mathscr{F}$ the $\sigma$-algebra of subsets of the sample space.

## II. PROBLEM FORMULATION

Consider the following stochastic system with both discrete and distributed time-delays described by

$$\begin{aligned} dx(t) =\ & [\bar{A}x(t)+\bar{A}_1x(t-d_1(t))+\bar{A}_2\int_{t-d_2(t)}^{t}x(s)ds \\ & +Bu(t)]dt+ \\ & [\bar{S}x(t)+\bar{S}_1x(t-d_1(t))+\bar{S}_2\int_{t-d_2(t)}^{t}x(s)ds]d\omega(t) \\ y(t) =\ & C_1x(t)+C_{1d}x(t-d)+C_{2d}\int_{t-d_2(t)}^{t}x(s)ds \\ x(\theta) =\ & \psi(\theta),\forall\theta\in[-d,0] \end{aligned} \tag{1}$$

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where $x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}^p$ is the control input; $\omega(t)$ is a one-dimension Brownian motion defined on the probability space $(\Omega, \mathscr{F}, \mathscr{P})$ satisfying $\mathscr{E}\{d\omega(t)\} = 0$, $\mathscr{E}\{d\omega^2(t)\} = dt$; $y(t) \in \mathbb{R}^l$ is the measured output; $\psi(\theta)$ is a continuous initial function on $[-d, 0]$. $d_1(t)$ and $d_2(t)$ are the discrete and distributed time-varying delays respectively, satisfying

$$\begin{aligned} &0 < d_1(t) \le d_1,\ \dot{d}_1(t) \le \mu < \infty \\ &0 < d_2(t) \le d_2 \\ &d = \max(d_1, d_2) \end{aligned} \tag{2}$$

In system (1), $\bar{A} = A + \Delta A$, $\bar{A}_1 = A_1 + \Delta A_1$, $\bar{A}_2 = A_2 + \Delta A_2$, $\bar{S} = S + \Delta S$, $\bar{S}_1 = S_1 + \Delta S_1$ and $\bar{S}_2 = S_2 + \Delta S_2$. $A$, $A_1$, $A_2$, $B$, $S$, $S_1$, $S_2$, $C_1$, $C_{1d}$ and $C_{2d}$ are known real constant matrices of appropriate dimensions; $\Delta A$, $\Delta A_1$, $\Delta A_2$, $\Delta S$, $\Delta S_1$ and $\Delta S_2$ are unknown time-varying matrices representing the parameter uncertainties, and are assumed to be of the following form

$$\begin{aligned} [\Delta A,\ \Delta A_1,\ \Delta A_2] &= L_1 F(t)[E_a,\ E_{ad1},\ E_{ad2}] \\ [\Delta S,\ \Delta S_1,\ \Delta S_2] &= L_2 F(t)[E_s,\ E_{sd1},\ E_{sd2}] \end{aligned} \tag{3}$$

where $L_1$, $L_2$, $E_a$, $E_{ad1}$, $E_{ad2}$, $E_s$, $E_{sd1}$ and $E_{sd2}$ are known real constant matrices of appropriate dimensions, and $F(t)$ is an unknown time-varying matrix function satisfying

$$F^{\mathrm{T}}(t)F(t) \le I, \quad \forall t \tag{4}$$

It is assumed that all the elements of $F(t)$ are Lebesgue measurable. The parameter uncertainties $\Delta A$, $\Delta A_1$, $\Delta A_2$, $\Delta S$, $\Delta S_1$ and $\Delta S_2$ are said to be admissible if both (3) and (4) hold simultaneously.

Consider the following full order dynamic output feedback controller

$$\begin{aligned} dx_k(t) &= [A_k x_k(t) + B_k y(t)]dt \\ u(t) &= C_k x_k(t) \end{aligned} \tag{5}$$

where $x_k(t) \in \mathbb{R}^n$ is the controller state, $A_k$, $B_k$ and $C_k$ are controller gain matrices to be determined.

Combining system (1) and controller (5) yields the closed-loop system

$$\begin{aligned} d\xi(t) = {} & [\hat{A}\xi(t) + \hat{A}_1\xi(t - d_1(t)) + \hat{A}_2 \int_{t-d_2(t)}^{t} \xi(s)ds]dt + \\ & [\hat{S}\xi(t) + \hat{S}_1\xi(t - d_1(t)) + \hat{S}_2 \int_{t-d_2(t)}^{t} \xi(s)ds]d\omega(t) \end{aligned} \tag{6}$$

where $\xi^{\mathrm{T}}(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & x_k^{\mathrm{T}}(t) \end{bmatrix}$, and

$$\begin{aligned} &\hat{A} = \begin{bmatrix} \bar{A} & BC_k \\ B_k C_1 & A_k \end{bmatrix},\ \hat{A}_1 = \begin{bmatrix} \bar{A}_1 \\ B_k C_{1d} \end{bmatrix} E,\ \hat{A}_2 = \begin{bmatrix} \bar{A}_2 \\ B_k C_{2d} \end{bmatrix} E \\ &\hat{S} = E^{\mathrm{T}}\bar{S}E,\ \hat{S}_1 = E^{\mathrm{T}}\bar{S}_1 E,\ \hat{S}_2 = E^{\mathrm{T}}\bar{S}_2 E,\ E = \begin{bmatrix} I & 0 \end{bmatrix} \end{aligned} \tag{7}$$

Now, we present the following definitions and lemmas which will be used to develop our main results.

*Definition 1:* System (1) is said to be robustly stochastically stable if system (1) with $u(t) = 0$ and $v(t) = 0$ is asymptotically stable in mean square for all admissible parameter uncertainties (3), i.e., for any initial condition,

$$\lim_{t\to\infty} \mathscr{E}\{\|x(t)\|^2\} = 0.$$

*Definition 2:* System (1) is said to be robustly stochastically stabilizable if there exists a controller (5), such that the closed-loop system (6) is robustly stochastically stable.

*Lemma 1:* [9] For any positive definite matrix $R \in \mathbb{R}^{n\times n}$, positive scalar $h$, and vector function $\dot{x}(t) \in \mathbb{R}^n$, we have

$$-h\int_{t-h}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)ds \le \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -R & R \\ R & -R \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

*Lemma 2:* [10] (Schur Complement Formula) Let $S_{11} = S_{11}^{\mathrm{T}}$, $S_{12}$ and $S_{22} = S_{22}^{\mathrm{T}}$ be matrices of appropriate dimensions. Then, we have

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix} < 0$$

if and only if $S_{11} < 0, S_{22} - S_{12}^{\mathrm{T}} S_{11}^{-1} S_{12} < 0$ or equivalently $S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^{\mathrm{T}} < 0$.

*Lemma 3:* [11] Let $D \in \mathbb{R}^{n\times m}$, $E \in \mathbb{R}^{m\times n}$ be real matrices, and $F(t) \in \mathbb{R}^{m\times m}$ be time-varying matrix with $\|F(t)\| \leqslant 1$. Then, we have

(a) For any scalar $\varepsilon > 0$,

$$DF(t)E + E^{\mathrm{T}}F^{\mathrm{T}}(t)D^{\mathrm{T}} \leqslant \varepsilon^{-1}DD^{\mathrm{T}} + \varepsilon E^{\mathrm{T}}E$$

(b) For any positive definite matrix $R \in \mathbb{R}^{m\times m}$,

$$DE + E^{\mathrm{T}}D^{\mathrm{T}} \leqslant DR^{-1}D^{\mathrm{T}} + E^{\mathrm{T}}RE$$

The objective of this paper is to design a dynamic output feedback controller (5) for system (1). More specially, we are dedicated to finding controller matrices $A_k$, $B_k$ and $C_k$ such that, for all admissible parameter uncertainties (3), the closed-loop system (6) is robustly stochastically stable.

## III. Main Results

Initially, considering system (6), we have the following conclusion.

*Theorem 1:* Given scalars $d_1 > 0$, $d_2 > 0$ and $\mu$, the closed-loop system (6) is stochastically stable, if there exist matrices $P > 0$, $Q > 0$, $R_i > 0$ $(i = 1, 2)$ and $N_j$ $(j = 1, 2, 3, 4)$ such that the following inequality holds:

$$\Gamma + \Gamma_1 + \Gamma_1^{\mathrm{T}} < 0 \tag{8}$$

where

$$\Gamma = \begin{bmatrix} \Gamma_{11} & R_1 + P\hat{A}_1 & P\hat{A}_2 & -N_1 & \hat{S}^{\mathrm{T}}P \\ * & -(1-\mu)Q - R_1 & 0 & -N_2 & \hat{S}_1^{\mathrm{T}}P \\ * & * & -R_2 & -N_3 & \hat{S}_2^{\mathrm{T}}P \\ * & * & * & d_1^2 R_1 - N_4 - N_4^{\mathrm{T}} & 0 \\ * & * & * & * & -P \end{bmatrix}$$

$\Gamma_{11} = Q - R_1 + d_2^2 R_2 + P\hat{A} + \hat{A}^{\mathrm{T}}P$, $\Gamma_1 = \tilde{N}A_{cl}$

$\tilde{N} = \begin{bmatrix} N_1^{\mathrm{T}} & N_2^{\mathrm{T}} & N_3^{\mathrm{T}} & N_4^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}}$, $A_{cl} = \begin{bmatrix} \hat{A} & \hat{A}_1 & \hat{A}_2 & 0 & 0 \end{bmatrix}$

*Proof:* Define the new vectors as

$$\begin{aligned} r(t) &= \hat{A}\xi(t) + \hat{A}_1\xi(t - d_1(t)) + \hat{A}_2 \int_{t-d_2(t)}^{t} \xi(s)ds \\ g(t) &= \hat{S}\xi(t) + \hat{S}_1\xi(t - d_1(t)) + \hat{S}_2 \int_{t-d_2(t)}^{t} \xi(s)ds \end{aligned} \tag{9}$$

Then, the closed-loop system (6) with $v(t) = 0$ can be represented as

$$d\xi(t) = r(t)dt + g(t)d\omega(t) \tag{10}$$

Denote $\eta(t)dt = d\xi(t)$ and consider the following Lyapunov-Krasovskii functional candidate

$$\begin{aligned} V(\xi_t,t) = & \xi^{\mathrm{T}}(t)P\xi(t) + \int_{t-d_1(t)}^{t} \xi^{\mathrm{T}}(s)Q\xi(s)ds \\ & + d_1 \int_{-d_1}^{0}\int_{t+\theta}^{t} \eta^{\mathrm{T}}(s)R_1\eta(s)dsd\theta \\ & + d_2 \int_{-d_2}^{0}\int_{t+\theta}^{t} \xi^{\mathrm{T}}(s)R_2\xi(s)dsd\theta \end{aligned} \tag{11}$$

By **Itô** formula, the stochastic differential of $V(\xi_t,t)$ along the trajectories of system (6) is

$$\begin{aligned} & dV(\xi_t,t) = [2\xi^{\mathrm{T}}(t)Pr(t) + d_1^2\eta^{\mathrm{T}}(t)R_1\eta(t) \\ & + \xi^{\mathrm{T}}(t)Q\xi(t) - (1-\dot{d}_1(t))\xi^{\mathrm{T}}(t-d_1(t))Q\xi(t-d_1(t)) \\ & + d_2^2\xi^{\mathrm{T}}(t)R_2\xi(t) - d_1\int_{t-d_1}^{t}\eta^{\mathrm{T}}(s)R_1\eta(s)ds \\ & + g^{\mathrm{T}}(t)Pg(t) - d_2\int_{t-d_2}^{t}\xi^{\mathrm{T}}(s)R_2\xi(s)ds]dt \\ & + 2\xi^{\mathrm{T}}(t)Pg(t)d\omega(t) \end{aligned} \tag{12}$$

Form Lemma 1, we have

$$\begin{aligned} & -d_1\int_{t-d_1}^{t}\eta^{\mathrm{T}}(s)R_1\eta(s)ds \\ & \le -d_1\int_{t-d_1(t)}^{t}\eta^{\mathrm{T}}(s)R_1\eta(s)ds \\ & \le -(\int_{t-d_1(t)}^{t}\eta(s)ds)^{\mathrm{T}}R_1(\int_{t-d_1(t)}^{t}\eta(s)ds) \\ & = \begin{pmatrix}\xi(t)\\ \xi(t-d_1(t))\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}-R_1 & R_1\\ R_1 & -R_1\end{bmatrix}\begin{pmatrix}\xi(t)\\ \xi(t-d_1(t))\end{pmatrix} \end{aligned} \tag{13}$$

and

$$\begin{aligned} & -d_2\int_{t-d_2}^{t}\xi^{\mathrm{T}}(s)R_2\xi(s)ds \\ & \le -d_2\int_{t-d_2(t)}^{t}\xi^{\mathrm{T}}(s)R_2\xi(s)ds \\ & \le -(\int_{t-d_2(t)}^{t}\xi(s)ds)^{\mathrm{T}}R_2(\int_{t-d_2(t)}^{t}\xi(s)ds) \end{aligned} \tag{14}$$

It follows from $\eta(t)dt = d\xi(t)$ and system (10) that, for any matrix $N$ of appropriate dimension, the following equation holds

$$2\zeta^{\mathrm{T}}(t)N\{[-\eta(t)+r(t)]dt + g(t)d\omega(t)\} = 0 \tag{15}$$

where

$$\zeta^{\mathrm{T}}(t) = \begin{bmatrix}\xi^{\mathrm{T}}(t) & \xi^{\mathrm{T}}(t-d_1(t)) & \int_{t-d_2(t)}^{t}\xi^{\mathrm{T}}(s)ds & \eta^{\mathrm{T}}(t)\end{bmatrix}$$

Substituting (13), (14) and (15) into (12), one gets,

$$dV(\xi_t,t) \le \mathscr{L}V(\xi_t,t)dt + [2\xi^{\mathrm{T}}(t)Pg(t) + 2\zeta^{\mathrm{T}}(t)Ng(t)]d\omega(t)$$

where

$$\mathscr{L}V(\xi_t,t) = \zeta^{\mathrm{T}}(t)(\Phi + \Phi_1 + \Phi_1^{\mathrm{T}} + \Phi_2)\zeta(t)$$

$$\Phi = \begin{bmatrix}(1,1) & R_1 + P\hat{A}_1 & P\hat{A}_2 & 0\\ * & -(1-\mu)Q - R_1 & 0 & 0\\ * & * & -R_2 & 0\\ * & * & * & d_1^2R_1\end{bmatrix}$$

$$\begin{aligned} & (1,1) = P\hat{A} + \hat{A}^{\mathrm{T}}P + Q - R_1 + d_2^2R_2 \\ & \Phi_1 = N\tilde{A},,\ \Phi_2 = \tilde{S}^{\mathrm{T}}P\tilde{S},\ N^{\mathrm{T}} = \begin{bmatrix}N_1^{\mathrm{T}} & N_2^{\mathrm{T}} & N_3^{\mathrm{T}} & N_4^{\mathrm{T}}\end{bmatrix} \\ & \tilde{A} = \begin{bmatrix}\hat{A} & \hat{A}_1 & \hat{A}_2 & -I\end{bmatrix},\ \tilde{S} = \begin{bmatrix}\hat{S} & \hat{S}_1 & \hat{S}_2 & 0\end{bmatrix} \end{aligned}$$

By Lemma 2, it is easy to see that (8) can ensure $\Phi + \Phi_1 + \Phi_1^{\mathrm{T}} + \Phi_2 < 0$, that is to say, $\mathscr{L}V(\xi_t,t) < 0$. Then, system (6) is stochastically stable. Thus, it completes the proof. ■

*Remark 1:* By employing Lyapunov-Krasovskii functional and free-weighting matrix approach, Theorem 1 gives a delay-dependent and rate-dependent sufficient condition for the existence of a dynamic feedback controller (5) such that the closed-loop system (6) is stochastically stable. The result of Theorem 1 can be easily extended to the uncertain stochastic systems with multiple discrete and distributed delays. For simplicity, here we only consider the uncertain stochastic time-delay system (1).

*Remark 2:* It follows from the proof of Theorem 1 that system (1) is stochastically stabilizable via a dynamic feedback controller (5) for any delays $d_1(t)$ and $d_2(t)$ satisfying (2). If the derivative of the discrete delay $d_1(t)$ is unknown, we can set $Q = 0$ in (11) and derive a new stability condition which is delay-dependent and rate-independent.

Neglecting the parameter uncertainties, we consider the following deterministic system

$$\dot{x}(t) = Ax(t) + A_1x(t-d_1(t)) + A_2\int_{t-d_2(t)}^{t}x(s)ds \tag{16}$$

Similarly to the proof of Theorem 1, the following corollary is immediate.

*Corollary 1:* Given scalars $d_1 > 0$, $d_2 > 0$ and $\mu$, the deterministic system (16) is asymptotically stable for any time-delays $d_1(t)$ and $d_2(t)$ satisfying (2), if there exist matrices $P > 0$, $Q > 0$, $R_i > 0$ $(i = 1,2)$, and $N_j$ $(j = 1,2,3,4)$ such that the following inequality holds:

$$\begin{aligned} & \begin{bmatrix}\Theta & R_1 + PA_1 & PA_2 & -N_1\\ * & -(1-\mu)Q - R_1 & 0 & -N_2\\ * & * & -R_2 & -N_3\\ * & * & * & d_1^2R_1 - N_4 - N_4^{\mathrm{T}}\end{bmatrix} \\ & +2\begin{bmatrix}N_1^{\mathrm{T}} & N_2^{\mathrm{T}} & N_3^{\mathrm{T}} & N_4^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}A & A_1 & A_2 & 0\end{bmatrix} < 0 \end{aligned} \tag{17}$$

where

$$\Theta = PA + A^{\mathrm{T}}P + Q - R_1 + d_2^2R_2.$$

Next, from (7), it is obvious that (8) is a nonlinear matrix inequality with respect to the parameter matrices $P$, $A_k$, $B_k$, and $C_k$ since some crosses of these matrices are appearing in (8) in nonlinear fashion. In order to facilitate solving the controller, in the following, we are interested to transform (8) into a LMI by some manipulations. From (8) and Lemma 3, we have

$$\Gamma_1 + \Gamma_1^{\mathrm{T}} \le \tilde{N}P^{-1}\tilde{N}^{\mathrm{T}} + A_{cl}^{\mathrm{T}}PA_{cl} \tag{18}$$

By Lemma 2, (8) is equivalent to

$$\begin{bmatrix} \Gamma_{11} & R_1+P\hat{A}_1 & P\hat{A}_2 & -N_1 & \hat{S}^{\mathrm{T}}P & \hat{A}^{\mathrm{T}}P & N_1 \\ * & \Gamma_{22} & 0 & -N_2 & \hat{S}_1^{\mathrm{T}}P & \hat{A}_1^{\mathrm{T}}P & N_2 \\ * & * & -R_2 & -N_3 & \hat{S}_2^{\mathrm{T}}P & \hat{A}_2^{\mathrm{T}}P & N_3 \\ * & * & * & \Gamma_{44} & 0 & 0 & N_4 \\ * & * & * & * & -P & 0 & 0 \\ * & * & * & * & * & -P & 0 \\ * & * & * & * & * & * & -P \end{bmatrix} < 0 \quad (19)$$

where $\Gamma_{11}$ is the same as that defined in (8), and

$$\Gamma_{22} = -(1-\mu)Q - R_1,\ \Gamma_{44} = d_1^2 R_1 - N_4 - N_4^{\mathrm{T}}.$$

Rewriting the matrices in (7) to be of the following form

$$\begin{aligned} \hat{A} &= E^{\mathrm{T}}\bar{A}E + E_0\begin{bmatrix}A_k & B_k\end{bmatrix}E_{c1} + E^{\mathrm{T}}BC_kE_0^{\mathrm{T}} \\ \hat{A}_1 &= E^{\mathrm{T}}\bar{A}_1E + E_0\begin{bmatrix}A_k & B_k^{\mathrm{T}}\end{bmatrix}E_{c1d} \\ \hat{A}_2 &= E^{\mathrm{T}}\bar{A}_2E + E_0\begin{bmatrix}A_k & B_k\end{bmatrix}E_{c2d} \\ E_0 &= \begin{bmatrix}0\\I\end{bmatrix},\ E^{\mathrm{T}} = \begin{bmatrix}I\\0\end{bmatrix},\ E_{c1} = \begin{bmatrix}0 & I\\ C_1 & 0\end{bmatrix} \\ E_{c1d} &= \begin{bmatrix}0 & 0\\ C_{1d} & 0\end{bmatrix},\ E_{c2d} = \begin{bmatrix}0 & 0\\ C_{2d} & 0\end{bmatrix} \end{aligned}$$

Then, we have

$$\begin{aligned} P\hat{A} &= PE^{\mathrm{T}}AE + PE^{\mathrm{T}}\Delta AE + PE_0\begin{bmatrix}A_k & B_k\end{bmatrix}E_{c1} + PE^{\mathrm{T}}BC_kE_0^{\mathrm{T}} \\ P\hat{A}_1 &= PE^{\mathrm{T}}A_1E + PE^{\mathrm{T}}\Delta A_1E + PE_0\begin{bmatrix}A_k & B_k\end{bmatrix}E_{c1d} \\ P\hat{A}_2 &= PE^{\mathrm{T}}A_2E + PE^{\mathrm{T}}\Delta A_2E + PE_0\begin{bmatrix}A_k & B_k\end{bmatrix}E_{c2d} \end{aligned}$$

Setting

$$V_1 = PE_0\begin{bmatrix}A_k & B_k\end{bmatrix},\ V_2 = PE^{\mathrm{T}}BC_k.$$

We suppose that the matrix $B$ is of full column rank. And then the following conclusion is immediate.

*Theorem 2:* Given scalars $d_1 > 0$, $d_2 > 0$ and $\mu$, for system (1), there exist a dynamic output feedback controller (5) such that the closed-loop system (6) is robustly stochastically stable for all admissible parameter uncertainties (3), if there exist matrices $P > 0$, $Q > 0$, $R_i > 0$ $(i = 1,2)$, $V_j$ $(j = 1,2)$ and $N_l$ $(l = 1,2,3,4)$, as well as a scalar $\varepsilon > 0$ such that LMI (20) holds, which is shown at the top of next page, where

$$\begin{aligned} \Phi_{11} &= Q - R_1 + d_2^2R_2 + P\check{A} + V_1E_{c1} + V_2E_0^{\mathrm{T}} + \check{A}^{\mathrm{T}}P \\ &\quad + E_{c1}^{\mathrm{T}}V^{\mathrm{T}} + E_0V_2^{\mathrm{T}} + \varepsilon\check{E}_a^{\mathrm{T}}\check{E}_a + \varepsilon\check{E}_s^{\mathrm{T}}\check{E}_s \\ \Phi_{12} &= R_1 + P\check{A}_1 + V_1E_{c1d} + \varepsilon\check{E}_a^{\mathrm{T}}\check{E}_{ad1} + \varepsilon\check{E}_s^{\mathrm{T}}\check{E}_{sd1} \\ \Phi_{13} &= P\check{A}_2 + V_1E_{c2d} + \varepsilon\check{E}_a^{\mathrm{T}}\check{E}_{ad2} + \varepsilon\check{E}_s^{\mathrm{T}}\check{E}_{sd2} \\ \Phi_{22} &= -(1-\mu)Q - R_1 + \varepsilon\check{E}_{ad1}^{\mathrm{T}}\check{E}_{ad1} + \varepsilon\check{E}_{sd1}^{\mathrm{T}}\check{E}_{sd1} \\ \Phi_{23} &= +\varepsilon\check{E}_{ad1}^{\mathrm{T}}\check{E}_{ad2} + \varepsilon\check{E}_{sd1}^{\mathrm{T}}\check{E}_{sd2} \\ \Phi_{33} &= -R_2 + \varepsilon\check{E}_{ad2}^{T}\check{E}_{ad2} + \varepsilon\check{E}_{sd2}^{\mathrm{T}}\check{E}_{sd2} \\ &\begin{bmatrix}\check{A} & \check{A}_1 & \check{A}_2 & \check{S} & \check{S}_1 & \check{S}_2\end{bmatrix} \\ &\quad = E^{\mathrm{T}}\begin{bmatrix}A & A_1 & A_2 & S & S_1 & S_2\end{bmatrix}E \\ &\begin{bmatrix}\check{E}_a & \check{E}_{ad1} & \check{E}_{ad2} & \check{E}_s & \check{E}_{sd1} & \check{E}_{sd2}\end{bmatrix} \\ &\quad = E^{\mathrm{T}}\begin{bmatrix}E_a & E_{ad1} & E_{ad2} & E_s & E_{sd1} & E_{sd2}\end{bmatrix}E \end{aligned}$$

The corresponding controller gain can be taken as

$$A_k = E_0^{\mathrm{T}}P^{-1}V_1E^{\mathrm{T}}, B_k = E_0^{\mathrm{T}}P^{-1}V_1E_0, C_k = B_{\mathrm{R}}EP^{-1}V_2 \quad (21)$$

Here, $B_{\mathrm{R}}$ only needs to satisfy

$$BB_{\mathrm{R}} = \begin{bmatrix}I_p & \star\\ \star & \star\end{bmatrix}$$

where the symbol $\star$ denotes the omitted part of the matrix. Then, we have

$$\begin{aligned} PE^{\mathrm{T}}BC_k &= PE^{\mathrm{T}}BB_{\mathrm{R}}EP^{-1}V_2 \\ &= P\begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}I_p & \star\\ \star & \star\end{bmatrix}\begin{bmatrix}I\\0\end{bmatrix}P^{-1}V_2 = V_2 \end{aligned}$$

Specifically, suppose $B$ is of the form $B^T = \begin{bmatrix}B_1^{\mathrm{T}} & B_2^{\mathrm{T}}\end{bmatrix}$ with $B_1 \in \mathbb{R}^{p\times p}$ and $B_2 \in \mathbb{R}^{(n-p)\times p}$. Then, we can select $B_{\mathrm{R}} = \begin{bmatrix}B_1^{-1} & \star\end{bmatrix}$. It should be pointed out that the existence of the matrix $B_1^{-1}$ is reasonable. Because for any matrix $B$ with full column rank, through the appropriate row transformation of matrix $B$, the invertibility of matrix $B_1$ can be easily guaranteed to be invertible.

*Remark 3:* For system (1), given scalars $d_2$ and $\mu$, we can find the maximum allowable bound of the discrete time-delay $d_1$ by solving the following optimal problem

$$\max_{\substack{P,Q,R_i,V_i,N_i,\varepsilon \\ subject\ to\ (20)}} d_1$$

and vice versa.

*Remark 4:* Theorem 2 proposes a design method of dynamic output feedback controller for uncertain stochastic time-delay system (1). This is the problem of robust stabilization. Furthermore, Theorem 2 can also be generalized to solve problems of robust $H_\infty$ or $L_2$-$L_\infty$ stabilization.

## IV. Numerical Example

*Example 1:* Consider uncertain stochastic system with discrete time-delay in [12]

$$\begin{aligned} dx(t) = &\ [(A+\Delta A)x(t) + (A_1+\Delta A_1)x(t-d_1)]dt \\ &+ [\Delta Sx(t) + \Delta S_1x(t-d_1)]d\omega(t) \end{aligned} \quad (22)$$

with

$$\begin{aligned} A &= \begin{bmatrix}-2 & 0\\ 0 & -0.9\end{bmatrix},\ A_1 = \begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix} \\ L_1 &= L_2 = diag\{0.2,\ 0.2\} \\ E_a &= E_{ad1} = E_s = E_{sd1} = diag\{1,\ 1\} \end{aligned}$$

Taking no account of distributed time-delay, and only considering the discrete constant delay, system (1) reduces to the system (22). With the method presented in [12], it has been obtained that the allowable maximum delay is 0.3555. In this example, using Theorem 1 in this paper, we can find that system (22) is robustly stochastically stable for any $d_1$ satisfying $0 < d_1 \leq 4.3623$. It is obvious that the result in this paper is less conservatism.

*Example 2:* [13], [14] Consider system (16) with

$$\begin{aligned} A &= \begin{bmatrix}-0.9 & 0\\ 0 & -0.9\end{bmatrix},\ A_1 = \begin{bmatrix}-1 & -0.12\\ 0.12 & -1\end{bmatrix} \\ A_2 &= \begin{bmatrix}-0.2 & 0\\ -0.2 & -0.1\end{bmatrix},\ d_2 = 1,\ \mu = 0 \end{aligned}$$

$$
\begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} & -N_1 & \check{S}^{\mathrm{T}}P & \check{A}^{\mathrm{T}}P+E_{c1}^{\mathrm{T}}V_1^{\mathrm{T}}+EV_2^{\mathrm{T}} & N_1 & PL_1 & 0 \\
* & \Phi_{22} & \Phi_{22} & -N_2 & \check{S}_1^{\mathrm{T}}P & \check{A}_1^{\mathrm{T}}P+E_{c1d}^{\mathrm{T}}V_1^{\mathrm{T}} & N_2 & 0 & 0 \\
* & * & \Phi_{33} & -N_3 & \check{S}_2^{\mathrm{T}}P & \check{A}_2^{\mathrm{T}}P+E_{c2d}^{\mathrm{T}}V_1^{\mathrm{T}} & N_3 & 0 & 0 \\
* & * & * & d_1^2R_2-N_4-N_4^{\mathrm{T}} & 0 & 0 & N_4 & 0 & 0 \\
* & * & * & * & -P & 0 & 0 & PL_1 & 0 \\
* & * & * & * & * & -P & 0 & 0 & 0 \\
* & * & * & * & * & * & -P & 0 & PL_2 \\
* & * & * & * & * & * & * & -\varepsilon I & 0 \\
* & * & * & * & * & * & * & * & -\varepsilon I
\end{bmatrix} < 0 \qquad (20)
$$

From Corollary 1, the maximum value of the discrete time-delay $d_1(t)$ is $d_1 \leq 4.9999$. Using the results in [13] and [14], the system (16) is asymptotically stable for any $d_1$ satisfying $d_1 \leq 2.8011$ and $d_1 \leq 3.1668$, respectively. Therefore, the delay-dependent stability condition in Corollary 1 in this paper is less conservative than that in [13] and [14].

The following numerical example is given to demonstrate the effectiveness of the proposed dynamic output feedback controller design method.

*Example 3:* Consider the uncertain stochastic time-delay system (1) with

$$
A=\begin{bmatrix}-1 & 2\\ 0 & 0.5\end{bmatrix},\ A_1=\begin{bmatrix}-1 & 0\\ -0.5 & -1\end{bmatrix},\ A_2=\begin{bmatrix}0 & 0.2\\ 0.1 & -0.1\end{bmatrix}
$$

$$
S=\begin{bmatrix}-0.1 & -0.05\\ 0.2 & 0.15\end{bmatrix},\ S_1=\begin{bmatrix}-0.2 & 0.4\\ -0.5 & 0.1\end{bmatrix},\ S_2=\begin{bmatrix}0.1 & -0.2\\ 0 & 0.1\end{bmatrix}
$$

$$
C_1=\begin{bmatrix}2 & -0.5\\ -1.5 & 0.5\end{bmatrix},\ C_{1d}=\begin{bmatrix}0.15 & 0.1\\ -0.1 & 0.1\end{bmatrix},\ C_{2d}=\begin{bmatrix}0.5 & -0.2\\ 0.6 & 0\end{bmatrix}
$$

$$
B=\begin{bmatrix}0.5 & -0.5\\ 0 & 0.6\end{bmatrix},\ L_1=L_2=diag\{0.2,\ 0.2\}
$$

$$
E_a=E_{ad1}=E_{ad2}=E_s=E_{sd1}=S_{sd2}=diag\{0.1,\ 0.1\}
$$

In this case, given $d_1=0.8$, $d_2=0.5$ and $\mu=2$. By solving LMI (20), we can obtain the solutions as follows:

$$
P=\begin{bmatrix}0.1547 & -0.2183 & 0 & 0\\ -0.2183 & 0.5495 & 0 & 0\\ 0 & 0 & 7.5550 & 0.0470\\ 0 & 0 & 0.0470 & 7.0842\end{bmatrix}
$$

$$
V_1=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -4.1684 & 0.9230 & -0.0144 & -0.0129\\ -0.7401 & -4.2588 & -0.0195 & -0.0373\end{bmatrix}
$$

$$
V_2^{\mathrm{T}}=\begin{bmatrix}-0.0186 & 0.0032 & 0 & 0\\ 0.0204 & -0.0632 & 0 & 0\end{bmatrix}
$$

Then, from (21), the gain matrices of stabilizing dynamic output-feedback controller (5) can be obtained as follows:

$$
A_k=\begin{bmatrix}-0.5511 & 0.1259\\ -0.1008 & -0.6020\end{bmatrix},\ B_k=\begin{bmatrix}-0.0019 & -0.0017\\ -0.0027 & -0.0052\end{bmatrix}
$$

$$
C_k=\begin{bmatrix}-0.6690 & -0.3753\\ -0.1592 & -0.2374\end{bmatrix}
$$

## V. Conclusions

In this paper, we have investigated the problem of dynamic output feedback control for a class of uncertain stochastic systems with discrete and distributed time-varying delays and norm-bounded parameter uncertainties. Sufficient delay-dependent condition for the existence of the desired controller is derived. Meanwhile, the output feedback controller design method is also given in terms of LMI. Numerical examples have been provided to illustrate the effectiveness and the benefits of the proposed results.

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