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# Analysis and Design of Output Feedback Control Systems in the Presence of State Saturation

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*Abstract*— **This paper studies the problems of stability analysis and dynamic output feedback controller design for continuous-time linear systems under state saturation. In this paper, both full state saturation and partial state saturation are considered. In order to solve the key problem, a new system is constructed. Firstly, a new LMI-based method is presented for estimating the domain of attraction of the origin for a closed-loop system under state saturation. Based on this method, an LMI-based algorithm is developed for constructing dynamic output-feedback controller which guarantees that the domain of attraction of the origin for the closed-loop system is as large as possible. An example is given to illustrate the efficiency of the design method.**

## I. INTRODUCTION

Control systems with saturation are often encountered in practice, and two cases are considered. One is actuator saturation, the other one is state saturation. When actuator saturation occurs, global stability of an otherwise stable linear closed-loop system cannot in general be ensured. And the problem of estimating the domain of attraction for a system with a saturated linear feedback has been studied by many researchers in the last few years and various methods have appeared (see, e.g., [1]-[4], and the references therein). However, in recent years, the number of conclusions about state saturation is much smaller than the one of actuator saturation.

Control systems with state saturation are often encountered in a variety of applications, including signal processing, recurrent neural networks and control systems, and have been studied extensively (see, e.g., [5]-[11], and the references therein). Most of such systems can be modeled by statespace representations with polyhedral or ellipsoidal state constraints, and global stability of an otherwise stable linear closed-loop system can not in general be ensured. A few approaches to the global asymptotic stability of such system were presented in [11]-[14].

Necessary and sufficient conditions for global asymptotic stability were established in [11] and [13], for second order systems with state saturation. For higher order systems, various sufficient conditions for the global asymptotic stability were identified (see, [10], [14], and the references therein). Under the sufficient condition of [15], any system trajectory starting from inside the state saturation region will never reach the boundary of it, i.e., the state never saturates. This saturation avoidance sufficient condition leads to a degree of conservatism.

Extensions of the results of [16] and [17] to the situation involving partial state saturation have been carried out in [18] and [19], respectively. This problem was reconsidered in [20] and a less conservative result was obtained. Using biconvex programming and static output feedback approach, the $H_\infty$ control problem was also considered [21].

This paper not only considers the problem of stability analysis but also considers the problem of dynamic output feedback controller design for continuous time linear systems under state saturation. Both full state saturation and partial state saturation are considered. Firstly, in order to solve the key problem, a new system is constructed. Then, a new LMI-based method is presented for estimating the domain of attraction of the origin for a closed-loop system under state saturation. Based on this method, an LMI-based algorithm is developed for constructing dynamic output-feedback controller which guarantees that the domain of attraction of the origin for the closed-loop system is as large as possible.

The paper is organized as follows. Problem statement is given in Section II. A condition for set invariance is presented for continuous-time LTI systems with state saturation in Section III. The proposed estimation of domain of attraction is presented in Section IV. A controller design method based on LMIs is given in Section V. An illustrative example is presented in Section VI to demonstrate the proposed design methods. Finally, the paper will be concluded in Section VII.

## II. PROBLEM STATEMENT AND PRELIMINARIES

The following definitions and lemma will be used in the sequel.

**Definition 1**: For a vector $x$, define

$$\wp(n,n_p) = \{x \in R^n : \ |x_i| \le x_i^{max}, \ i \in \mathbf{I}[1,n_p], \quad n \ge n_p\},$$

then $\wp(n,n_p)$ is the state saturation region.

In this paper, we will consider two classes of continuous-time linear systems under state constraints. The first class of systems with full state saturation are defined as

$$\begin{aligned}\dot{x}_p(t) &= h[A_p x_p(t) + B_p u(t)] \\ y(t) &= C_p x_p(t)\end{aligned} \tag{1}$$

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where $x_p(t) \in \wp(n,n)$ is the plant state, $u = [u_1\ u_2 \dots u_m] \in R^m$ is the control input, $y \in R^p$ is the measured output. $A_p = [a_{ij}]$, $B_p = [b_{iq}]$, $C_p$ are known constant matrices of appropriate dimensions, and

$$h[A_p x_p(t) + B_p u(t)] = \begin{bmatrix} h_1[\sum_{j=1}^{n} a_{1j}x_j + \sum_{q=1}^{m} b_{1q}u_q] \\ h_2[\sum_{j=1}^{n} a_{2j}x_j + \sum_{q=1}^{m} b_{2q}u_q] \\ \vdots \\ h_n[\sum_{j=1}^{n} a_{nj}x_j + \sum_{q=1}^{m} b_{nq}u_q] \end{bmatrix}$$

with for each $i \in [1,n]$

$$h_i[\sum_{j=1}^{n} a_{ij}x_j + \sum_{q=1}^{m} b_{iq}u_q] = \begin{cases} 0, \quad if\ |x_i| = 1\ and\ (\sum_{j=1}^{n} a_{ij}x_j + \sum_{q=1}^{m} b_{iq}u_q)x_i > 0 \\ \sum_{j=1}^{n} a_{ij}x_j + \sum_{q=1}^{m} b_{iq}u_q, \quad otherwise \end{cases}$$

System (1) is defined on a closed hypercube as all state variables are constrained to the hypercube. For this reason, system (1) is sometimes called as continuous-time linear system under state saturation, that is, saturation occurs in the state $x_i$ if $|x_i| > x_i^{max}$.

The other class of systems are continuous-time linear systems with partial state saturation and are formulated as

$$\begin{aligned} \dot{x}_{p1}(t) &= h[A_{p11}x_{p1}(t) + A_{p12}x_{p2}(t) + B_{p1}u(t)] \\ \dot{x}_{p2}(t) &= A_{p21}x_{p1}(t) + A_{p22}x_{p2}(t) + B_{p2}u(t) \\ y(t) &= C_{p1}x_{p1}(t) + C_{p2}x_{p2}(t) \end{aligned} \tag{2}$$

$$h[A_{p11}x_{p1}(t) + A_{p12}x_{p2}(t) + B_{p1}u(t)] = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{n_{p1}} \end{bmatrix}$$

where

$$\begin{aligned} Y_i = & h_i[\sum_{j=1}^{i} a_{p11ij}x_j + \sum_{v=1}^{n_p - n_{p1}} a_{p12iv}x_v \\ & + \sum_{q=1}^{m} b_{p1iq}u_q] \end{aligned}$$

and $x_{p1}(t) \in R^{n_{p1}}$ is the plant state with sate saturation, $x_{p2}(t) \in R^{n_p - n_{p1}}$ is the plant state without state saturation, $u \in R^m$ is the control input, $y \in R^p$ is the measured output. $A_{p11}, A_{p12}, A_{p21}, A_{p22}, B_{p1}, B_{p2}, C_{p1}, C_{p2}$ are known constant matrices of appropriate dimensions. Notes that all the state variables are under state saturation if $n_{p1} = n_p$, in which case, system (2) reduces to system (1).

**Definition 2**: For a matrix $M \in R^{n_{p1} \times (n_p + n_c)}$, denote the $i$th row of $M$ as $M_i$, define

$$\Im(M) = \{\xi \in R^{n_p + n_c} : \quad |M_i \xi| \le x_i^{max}, \quad i \in \mathbf{I}[1, n_{p1}]\},$$

For $x(0) = x_0 \in R^n$, denote the state trajectory of system (1) as $\psi(t, x_0)$. Then the *domain of attraction* of the origin is

$$\ell := \{x_0 \in R^n : \quad lim_{t\to\infty}\psi(t, x_0) = 0\}.$$

Consider the following two systems

$$\begin{cases} \dot{x}_{a1}(t) = h[f_1(x_{a1}(t),\ x_{a2}(t)) + g_1(u_a(t))], \\ \dot{x}_{a2}(t) = f_2(x_{a1}(t),\ x_{a2}(t)) + g_2(u_a(t)), \\ y_a(t) = C_{p1}x_{a1}(t) + C_{p2}x_{a2}(t), \\ \dot{x}_{ca}(t) = f_3(x_{ca}(t)) + g_3(y_a(t)), \\ u_a(t) = f_4(x_{ca}(t)) + g_4(y_a(t)), \end{cases} \tag{3}$$

$$\begin{cases} \dot{x}_{b1}(t) = f_1(\sigma[x_{b1}(t)],\ x_{b2}(t)) + g_1(u_b(t)), \\ \dot{x}_{b2}(t) = f_2(\sigma[x_{b1}(t)],\ x_{b2}(t)) + g_2(u_b(t)), \\ y_b(t) = C_{p1}\sigma[x_{b1}(t)] + C_{p2}x_{b2}(t), \\ \dot{x}_{cb}(t) = f_3(x_{cb}(t)) + g_3(y_b(t)), \\ u_b(t) = f_4(x_{cb}(t)) + g_4(y_b(t)), \end{cases} \tag{4}$$

The state nonlinearity with the consideration of a piecewise-linear saturation is described as

$$\sigma(x_i) = \begin{cases} x_i, & |x_i| \le x_i^{max}, \\ sign(x_i)x_i^{max}, & |x_i| > x_i^{max}, \end{cases} \tag{5}$$

for $i \in \mathbf{I}[1, n_p]$. Here we have slightly abused the notation by using $\sigma$ to denote both the scalar valued and the vector valued saturation functions.

Denote the state trajectory of system (3) as $\psi_a(t, x_a(0)) = \begin{bmatrix} \psi_{a1}(t, x_a(0)) \\ \psi_{a2}(t, x_a(0)) \\ \psi_{ca}(t, x_a(0)) \end{bmatrix}$, and denote the state trajectory of system (4) as $\psi_b(t, x_b(0)) = \begin{bmatrix} \psi_{b1}(t, x_a(0)) \\ \psi_{b2}(t, x_a(0)) \\ \psi_{cb}(t, x_a(0)) \end{bmatrix}$. Let $x_a(t) = \begin{bmatrix} x_{a1}(t) \\ x_{a2}(t) \\ x_{ca}(t) \end{bmatrix}$, $x_b(t) = \begin{bmatrix} x_{b1}(t) \\ x_{b2}(t) \\ x_{cb}(t) \end{bmatrix}$, then the following lemma is given.

**Lemma 1**: For any initial state $x_a(0) = x_b(0) \in \wp(n, n_{p1})$, the following two statements are equivalent.

(I) $\lim_{t\to\infty} \psi_a(t, x_a(0)) = 0$

(II) $\lim_{t\to\infty} \psi_b(t, x_b(0)) = 0$

**Proof**:

$$\begin{aligned} & x_a(0) = x_b(0) \in \wp(n, n_{p1}) \\ \Rightarrow & \begin{cases} x_{a1}(0) = \sigma[x_{b1}(0)] \\ x_{a2}(0) = x_{b2}(0) \\ x_{ca}(0) = x_{cb}(0) \end{cases} \\ \Rightarrow & \begin{cases} f_1(x_{a1}(0), x_{a2}(0)) = f_1(\sigma[x_{b1}(0)],\ x_{b2}(0)) \\ f_2(x_{a1}(0),\ x_{a2}(0)) = f_2(\sigma[x_{b1}(0)],\ x_{b2}(0)) \\ y_a(0) = y_b(0) \\ u_a(0) = u_b(0) \end{cases} \\ \Rightarrow & \begin{cases} f_1(x_{a1}(0), x_{a2}(0)) + g_1(u_a(0)) \\ \quad = f_1(\sigma[x_{b1}(0)],\ x_{b2}(0)) + g_1(u_b(0)) \\ f_2(x_{a1}(0),\ x_{a2}(0)) + g_2(u_a(0)) \\ \quad = f_2(\sigma[x_{b1}(0)],\ x_{b2}(0)) + g_2(u_b(0)) \\ \psi_{ca}(t, x_a(0)) = \psi_{cb}(t, x_b(0)) \end{cases} \end{aligned}$$

Then, by the definition of function $h[x]$, we have

$$\begin{cases} \psi_{a1}(t, x_a(0)) = \sigma[\psi_{b1}(t, x_b(0))] \\ \psi_{a2}(t, x_a(0)) = \psi_{b2}(t, x_b(0)) \\ \psi_{ca}(t, x_a(0)) = \psi_{cb}(t, x_b(0)) \end{cases}$$

Then, Lemma 1 is proved.

**Problem 1**: The problem under consideration is described as followsFind a controller such that, the domain of asymptotic stability is enlarged as possible for the closed-loop system with state saturation.

## III. A CONDITION FOR SET INVARIANT

In this section, we will establish new sufficient conditions for global asymptotic stability for both systems (1) and (2). All the state variables are under state saturation if $n_{p1} = n_p$, in which case, system (2) reduces to system (1), so in the following, we need only to consider system (2). To this end, we first establish a new system as follows.

$$\begin{aligned} \dot{x}_{p1}(t) &= A_{p11}\sigma[x_{p1}(t)] + A_{p12}x_{p2}(t) + B_{p1}u(t) \\ \dot{x}_{p2}(t) &= A_{p21}\sigma[x_{p1}(t)] + A_{p22}x_{p2}(t) + B_{p2}u(t) \\ y(t) &= C_{p1}\sigma[x_{p1}(t)] + C_{p2}x_{p2}(t) \end{aligned} \quad (6)$$

which can be rewritten as

$$\begin{aligned} \dot{x}_p(t) &= A_p \begin{bmatrix} \sigma(x_{p1}(t)) \\ x_{p2}(t) \end{bmatrix} + B_p u(t) \\ y(t) &= C_p \begin{bmatrix} \sigma(x_{p1}(t)) \\ x_{p2}(t) \end{bmatrix} \end{aligned}$$

where

$$A_p = \begin{bmatrix} A_{p11} & A_{p12} \\ A_{p21} & A_{p22} \end{bmatrix},\ B_p = \begin{bmatrix} B_{p1} \\ B_{p2} \end{bmatrix},\quad C_p = \begin{bmatrix} C_{p1} & C_{p2} \end{bmatrix},$$

and $x_p(t) \in R^{n_p}$ is the plant state, $x_{p1}(t) \in R^{n_{p1}}$ is the plant state with sate saturation, $x_{p2}(t) \in R^{n_p - n_{p1}}$ is the plant state without state saturation, $u \in R^m$ is the control input, $y \in R^p$ is the measured output. $A_p$, $B_p$, $C_p$ are known constant matrices of appropriate dimensions.

To formulate suitably the corresponding LMIs, we need to introduce additional notation which corresponds to representing the closed-loop system in a compact way. Then, the following equation is given

$$q(t) = x_{p1}(t) - \sigma(x_{p1}(t)) \quad (7)$$

Then, we have

$$\begin{aligned} \dot{x}_p(t) &= A_p x_p(t) - A_{p1}q(t) + B_p u(t) \\ y(t) &= C_p x_p(t) - C_{p1}q(t) \end{aligned}$$

where $A_{p1} = \begin{bmatrix} A_{p11} \\ A_{p21} \end{bmatrix}$.

The controller structure is chosen as

$$\begin{aligned} \dot{x}_c(t) &= A_k x_c(t) + B_k y(t) \\ u(t) &= C_k x_c(t) + D_k y(t) \end{aligned} \quad (8)$$

Next, define the overall state variable $x \in \mathbf{R}^n$, where $n = n_p + n_c$, as

$$x = [x_p^T \quad x_c^T]^T$$

which allows the linear dynamics of the plant and controller to be combined and written as

$$\dot{x}(t) = A_e x(t) + B_e q(t) \quad (9)$$

where

$$A_e = \begin{bmatrix} A_p + B_p D_k C_p & B_p C_k \\ B_k C_p & A_k \end{bmatrix}$$
$$B_e = \begin{bmatrix} -A_{p1} - B_p D_k C_{p1} \\ -B_k C_{p1} \end{bmatrix}$$

**Remark 1**: When full state saturation is considered, system (9) can be replaced by the following system

$$\dot{x}(t) = A_e x(t) + B_e q(t) \quad (10)$$

where

$$A_e = \begin{bmatrix} A_p + B_p D_k C_p & B_p C_k \\ B_k C_p & A_k \end{bmatrix}$$
$$B_e = \begin{bmatrix} -A_p - B_p D_k C_p \\ -B_k C_p \end{bmatrix}$$

**Definition 4**: Let $P \in R^{n\times n}$ be a positive-define matrix. Denote

$$\varepsilon(P,\delta) = \{x \in R^n : \quad x^T P x \le \delta\}.$$

Assume that the standard dynamic output feedback controller has been designed. Then, for system (6) controlled by the designed controller (8), the following lemma is presented to estimate the domain of attraction of the origin.

**Lemma 2:** For system (6) given an ellipsoid $\varepsilon(P,1)$, $P \in R^{n\times n}$, if there exist matrices $Q > 0$, $U > 0$, $G$ such that

$$\begin{bmatrix} A_e Q + Q A_e^T & B_e U + Q \begin{bmatrix} I \\ 0 \end{bmatrix} - G^T \\ * & -2U \end{bmatrix} < 0 \quad (11)$$

for $\varepsilon(P,1) \subset \Im(M), i.e., |M_i x| \le x_i^{max}$ for all $x \in \varepsilon(P,1), i \in \mathbf{I}[1, n_{p1}]$, then $\varepsilon(P,1)$ is a contractively invariant set.

**Proof**: Choose the following Lyapunov function

$$V(k) = x(k)^T P x(k)$$

Let $M = GQ^{-1}$, $W = U^{-1}$, $P = Q^{-1}$. We have that inequality (11) is equivalent to

$$\begin{bmatrix} PA + A^T P & PBW^{-1} + \begin{bmatrix} I \\ 0 \end{bmatrix} - M^T \\ * & -2W^{-1} \end{bmatrix} < 0$$

$$\Leftrightarrow \begin{bmatrix} PA + A^T P & PB + \begin{bmatrix} I \\ 0 \end{bmatrix} W - M^T W \\ * & -2W \end{bmatrix} < 0$$

$$\Rightarrow \begin{bmatrix} x \\ q \end{bmatrix}^T \begin{bmatrix} PA + A^T P & PB + \begin{bmatrix} I \\ 0 \end{bmatrix} W - M^T W \\ * & -2W \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} < 0$$

$$\begin{aligned} &\Leftrightarrow x^T(PA + A^T P)x + x^T PBq + q^T B^T P x \\ &\quad + q^T W([I\,0]x - Mx - q) + ([I\,0]x - Mx - q)^T W q < 0 \end{aligned}$$

By equation (7) we have that, if $|M_i x| \le x_i^{max} \quad i \in \mathbf{I}[1,\ n_p]$,

$$q^T W([I\,0]x - Mx - q) + ([I\,0]x - Mx - q)^T W q \ge 0$$

By employing the $S$-procedure, it is shown that given any symmetric positive definite matrix $W$, if

$$\begin{aligned} &x^T(PA+A^TP)x+x^TPBq+q^TB^TPx \\ &+q^TW([I\ 0]x-Mx-q)+([I\ 0]x-Mx-q)^TWq<0 \\ &|M_ix|\le x_i^{max} \qquad i\in \mathbf{I}[1,\ n_p] \end{aligned}$$

then

$$x^T(PA+A^TP)x+x^TPBq+q^TB^TPx<0$$

**Theorem 1:** For system (3), given an ellipsoid $\varepsilon(P,1)$, $P\in R^{n\times n}$, if there exist matrices $Q>0$, $U>0$, $G$ such that inequality (11) holds for $\varepsilon(P,1)\subset\Im(M), i.e., |M_ix|\le x_i^{max}$ for all $x\in\varepsilon(P,1), i\in\mathbf{I}[1,n_{p1}]$, then $\wp(n,n_{p1})\bigcap\varepsilon(P,1)$ is a contractively invariant set.

**Proof:** By Lemma 2, we have that for any initial state $x(0)\in(\wp(n,n_{p1})\bigcap\varepsilon(P,1))$ the state of system (6) can be attracted to zero. Obviously, $(\wp(n,n_{p1})\bigcap\varepsilon(P,1))\subset\wp(n,n_{p1})$. Then, by Lemma 1 we have that for any initial state $x(0)\in(\wp(n,n_{p1})\bigcap\varepsilon(P,1))$ the state of system (3) can be attracted to zero.

## IV. Estimation of the domain of attraction

From Theorem 1, we can obtain various sets satisfying the set invariance condition. So, how to choose the largest one of them becomes an interesting problem. Because set $\wp(n,n_{p1})$ is given, we need only to enlarge the domain $\varepsilon(P,1)$ as possible for estimating the largest domain of attraction. In this section, we will give a method to find the largest set.

The following definition will be used in the sequel.

**Definition 5:** Define $X_R$ as a prescribed bounded convex set. $X_R=\varepsilon(R,1)=\{x\in R^{n\times n}:\ \ x^TRx\le 1\},\ \ R>0$ or $X_R=co\{x_1,x_2,...,x_l\}$. For a set $S\in R^n$, $\alpha_R(S)=sup\{\alpha>0:\ \ \alpha X_R\subset S\}$.

With the above shape reference sets, we can choose from all the $\varepsilon(P,1)'s$ that satisfy the condition of Theorem 1 such that the quantity $\alpha_R(\varepsilon(P,1))$ is maximized. The problem can be formulated as follows

$$\begin{aligned} sup\quad & \alpha \\ s.t.\quad & (a)\ \ (11), \\ & (b)\ \ \varepsilon(P,1)\subset\Im(M), \\ & (c)\ \ \alpha X_R\subset\varepsilon(P,1) \end{aligned} \tag{12}$$

Condition (b) is equivalent to

$$\frac{1}{x_i^{max}}M_iP^{-1}(\frac{1}{x_i^{max}}M_i)^T\le 1\Leftrightarrow\begin{bmatrix}1 & \frac{1}{x_i^{max}}M_iP^{-1}\\ * & P^{-1}\end{bmatrix}\ge 0 \tag{13}$$

for all $i\in\mathbf{I}[1,n_{p1}]$, where $M_i$ is the $j$th row of $M$.

If the given shape reference set $X_R$ is a polyhedron as defined in Definition 5, then Constraint (c) is equivalent to

$$\begin{bmatrix}\frac{1}{\alpha^2} & x_j^T\\ * & P^{-1}\end{bmatrix}\ge 0,\quad j\in\mathbf{I}[1,l] \tag{14}$$

If $X_R$ is an ellipsoid $\varepsilon(R,1)$, then (c) is equivalent to

$$\frac{R}{\alpha^2}\ge P\Leftrightarrow\begin{bmatrix}(1/\alpha^2)R & I\\ I & P^{-1}\end{bmatrix}\ge 0. \tag{15}$$

If $X_R$ is a polyhedron, then from (13) and (14), the optimization problem (12) can be described as the following algorithm

**Algorithm 1:**

$$\begin{aligned} \inf_{Q>0,G}\quad & \gamma \\ s.t.\quad & (a1)\ (11) \\ & (b1)\ \begin{bmatrix}1 & \frac{1}{x_i^{max}}g_i\\ * & Q\end{bmatrix}\ge 0, i\in\mathbf{I}[1,n_{p1}] \\ & (c1)\ \begin{bmatrix}\gamma & x_j^T\\ x_j & Q\end{bmatrix}\ge 0,\ j\in\mathbf{I}[1,\ l] \end{aligned}$$

where $\gamma=1/\alpha^2$, $Q=(\frac{P}{\rho})^{-1}$ and $G=MQ$. Let $g_i$ be the $i$th row of $G$. It is easy to see that all constraints are given in LMIs. If $X_R$ is an ellipsoid, we need only to replace (c1) with

$$(c2)\qquad\begin{bmatrix}\gamma R & I\\ I & Q\end{bmatrix}\ge 0.$$

## V. Controller design

In this section we will design a dynamic output feedback controller (8) such that the estimated domain of attraction is maximized with respect to $X_R$.

**Lemma 3:** For matrix variables $Q>0$, $U>0$, $G$, $K$, constraint (11) is equivalent to constraint (16) as follows

$$\begin{bmatrix}T_1 & T_2 & \bar{B}K+Q\bar{C}^T\\ * & T_3 & U\bar{D}^T\\ * & * & -I\end{bmatrix}<0 \tag{16}$$

where

$$\begin{aligned} T_1 &= AQ+QA^T-\bar{B}KK_0^T\bar{B}^T-\bar{B}K_0K^T\bar{B}^T \\ &\quad+\bar{B}K_0K_0^T\bar{B}^T-Q\bar{C}^T\bar{C}Q_0-Q_0\bar{C}^T\bar{C}Q+Q_0\bar{C}^T\bar{C}Q_0 \\ T_2 &= BU+Q[I\ 0]^T-QM^T-Q\bar{C}^T\bar{D}U_0 \\ &\quad-Q_0\bar{C}^T\bar{D}U+Q_0\bar{C}^T\bar{D}U_0 \\ T_3 &= -2U-U\bar{D}^T\bar{D}U_0-U_0\bar{D}^T\bar{D}U+U_0\bar{D}^T\bar{D}U_0 \end{aligned}$$

$$A=\begin{bmatrix}A_p & 0\\ 0 & 0\end{bmatrix}, B=\begin{bmatrix}-A_{p1}\\ 0\end{bmatrix}, K=\begin{bmatrix}A_k & B_k\\ C_k & D_k\end{bmatrix},$$
$$\bar{B}=\begin{bmatrix}0 & B_p\\ I & 0\end{bmatrix}, \bar{C}=\begin{bmatrix}0 & I\\ C_p & 0\end{bmatrix}, \bar{D}=\begin{bmatrix}0\\ -C_{p1}\end{bmatrix}$$

**Proof:** Obviously, constraint (11) is equivalent to

$$\begin{aligned}
&\begin{bmatrix} AQ+QA^T & BU+Q[I\ 0]^T-QM^T \\ * & -2U \end{bmatrix} \\
&+\begin{bmatrix} \bar{B}K \\ 0 \end{bmatrix}\begin{bmatrix} \bar{C}Q & \bar{D}U \end{bmatrix} \\
&+\begin{bmatrix} \bar{C}Q & \bar{D}U \end{bmatrix}^T\begin{bmatrix} \bar{B}K \\ 0 \end{bmatrix}^T<0 \\
&\Leftrightarrow \begin{bmatrix} AQ+QA^T & BU+Q[I\ 0]^T-QM^T \\ * & -2U \end{bmatrix} \\
&+(\begin{bmatrix} \bar{B}K \\ 0 \end{bmatrix}+\begin{bmatrix} Q\bar{C}^T \\ U\bar{D}^T \end{bmatrix})(\begin{bmatrix} \bar{B}K \\ 0 \end{bmatrix}+\begin{bmatrix} Q\bar{C}^T \\ U\bar{D}^T \end{bmatrix})^T \\
&-\begin{bmatrix} \bar{B}K \\ 0 \end{bmatrix}\begin{bmatrix} K^T\bar{B}^T & 0 \end{bmatrix} \\
&-\begin{bmatrix} Q\bar{C}^T \\ U\bar{D}^T \end{bmatrix}\begin{bmatrix} \bar{C}Q & \bar{D}U \end{bmatrix}<0
\end{aligned} \tag{17}$$

In addition, as is known to all that for any matrix $V$, there always exists a matrix $V_0$ such that the following inequality holds

$$(V-V_0)(V-V_0)^T \geq 0 \tag{18}$$

Thus, there exist matrix variables $Q_0$, $K_0$ and $U_0$ such that (16) is equivalent to (17). On one hand, according to (18), if (17) holds, then (16) holds. On the other hand, when $Q_0=Q$, $K_0=K$ and $U_0=U$, (17) holds if (16) holds. Thus, the proof for Lemma 3 is complete.

**Remark 2**: When full state saturation is considered, only two matrices defined in Lemma 3 should be replaced with

$$B=\begin{bmatrix} -A_p \\ 0 \end{bmatrix},\ \bar{D}=\begin{bmatrix} 0 \\ -C_p \end{bmatrix}.$$

By solving inequalities (11), (b1), (c1), we can solve Problem 1. But the constraints (11) is not an LMI, we can not solve them directly. To overcome this difficulty, we will give the following algorithm by Lemma 3.

**Algorithm 2:**

**Step 1** For system (1) design a standard dynamic output feedback controller such that the system (1) is asymptotic stable without state saturation.

**Step 2** Based on the controller gain $K^*$ obtained in Step 1, find a feasible set $(Q^*,U^*,G^*,\gamma^*)$ by solving Algorithm 1. Let $\eta=0$.

**Step 3** If $\gamma^*<1$ or $\eta>N$, where $N$ is the maximum number of iterations allowed, exit.

**Step 4** Let $Q_0=Q^*$, $U_0=U^*$, $K_0=K^*$. Solve the following LMI problem

$$\begin{aligned}
&\min\ \gamma \\
&s.t.\quad (16),\quad (b1), \\
&\qquad (c2)\quad \begin{bmatrix} -\gamma & \begin{bmatrix} -x_j^T & 0 \end{bmatrix} \\ * & -Q \end{bmatrix}<0,\quad j\in \mathbf{I}[1,2^{n_p}]
\end{aligned}$$

where $\gamma=\frac{1}{\alpha^2}$

**Step 5** Let $Q^*=Q$, $U^*=U$, $K^*=K$, $\eta=\eta+1$. Return to Step 3

## VI. Examples

**Example 1.** Consider the system of form (1) with

$$A=\begin{bmatrix} -9.9 & 0.8 \\ 1 & 0.5 \end{bmatrix},\quad B=\begin{bmatrix} 1 \\ -9 \end{bmatrix},\quad C=\begin{bmatrix} 1 & 2 \end{bmatrix}$$

and $x_1^{max}=x_2^{max}=1$. Suppose that a controller is given as follows without considering state saturation.

$$\begin{aligned}
&A_k=\begin{bmatrix} -1 & 2.5 \\ 30 & -9 \end{bmatrix},\quad B_k=\begin{bmatrix} -0.9 \\ -0.5 \end{bmatrix}, \\
&C_k=\begin{bmatrix} 0.1 & -2 \end{bmatrix},\quad D_k=1
\end{aligned} \tag{19}$$

Let $X_R=co\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\}$. By using Algorithm 1 we draw the following conclusion

$$\begin{aligned}
&\gamma^*=1.4248 \\
&Q^*=\begin{bmatrix} 64.8424 & -32.2461 & 0.2668 & -0.8962 \\ -32.2461 & 19.7468 & 0.4159 & 0.4528 \\ 0.2668 & 0.4159 & 0.1461 & 0.0276 \\ -0.8962 & 0.4528 & 0.0276 & 1.4202 \end{bmatrix} \\
&U^*=\begin{bmatrix} 4.6289 & -2.3545 \\ * & 1.2171 \end{bmatrix}
\end{aligned}$$

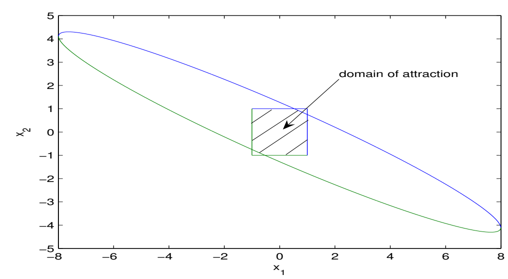


Fig. 1. Cross-section of $\varepsilon(P^*,\ 1)$ at $x_c=0$, under controller (19)

By using Algorithm 2 we have

$$\begin{aligned}
&\gamma^*=0.2067 \\
&Q^*=\begin{bmatrix} 426.5840 & -219.2509 & 0.1454 & 2.2724 \\ -219.2509 & 155.4780 & 7.3361 & -23.4550 \\ 0.1454 & 7.3361 & 1.7842 & -4.3642 \\ 2.2724 & -23.4550 & -4.3642 & 32.6553 \end{bmatrix} \\
&U^*=\begin{bmatrix} 30.4043 & -15.0892 \\ * & 8.0269 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&A_k^*=\begin{bmatrix} -1.8431 & 6.5377 \\ 29.6656 & -40.0248 \end{bmatrix},\quad B_k^*=\begin{bmatrix} -7.2394 \\ 19.6846 \end{bmatrix}, \\
&C_k^*=\begin{bmatrix} 0.4412 & -4.4417 \end{bmatrix},\quad D_k^*=5.3281
\end{aligned} \tag{20}$$

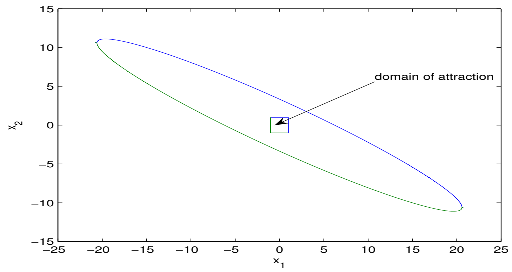


Fig. 2. Cross-section of $\varepsilon(P^*,\ 1)$ at $x_c = 0$, under controller (20)

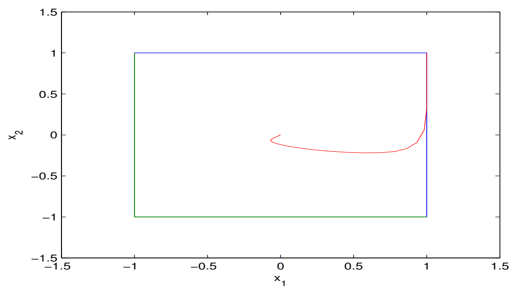


Fig. 3. trajectories of closed-loop systems

## VII. Conclusions

In this paper, for continuous time systems both full state saturation and partial state saturation were considered. In order to solve the problem of this paper a new system was constructed. Then, LMI-based algorithm was proposed for determining if a given ellipsoid is contractively invariant, and an LMI-based algorithm was developed for constructing dynamic output-feedback controllers which guarantee that the domain of attraction of the origin for the closed-loop system is enlarged as possible. An example was given to illustrate the efficiency of the design method.

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