2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Variable Structure Extremum Seeking Control Based on Sliding Mode Gradient Estimation for a Class of Nonlinear Systems

Lina Fu and Ümit Özgüner

Abstract— **This paper presents a new approach to extremum seeking control for a class of single-input-single-output nonlinear systems. With the analytic form of the performance function unknown *a priori*, a sliding mode observer is designed to estimate the gradient of the performance map. Based on the estimated gradient, a variable structure controller is proposed to search for the optimal operating point. We establish the conditions for the system states to enter a neighborhood of the optimal operating point, and obtain an ultimate bound on the size of the neighborhood. The robustness of the proposed controller is also discussed with respect to unmodeled fast dynamics and measurement noise.**

I. INTRODUCTION

As a branch of adaptive control, extremum seeking control addresses the control problems where a nonlinear plant is to be regulated to the optimal operating point or to track the optimal trajectory according to a certain performance criterion. The explicit form of the performance function is unknown *a priori*, with only the real time measurements of its value available. The extremum seeking controller steers the system states to the optimal point based on the measurements. There have been accumulating interest in the research of extremum seeking control in the 1950's and 1960's (for example [1], [2]). The topic was revisited in 1980's and was considered one of the most promising field in adaptive control (Astrom and Wittenmark [3], Section 13.3) in 1995.

One of the popular approaches toward extremum seeking control is based on perturbation and averaging. Meerkov ([4]-[6]) presented pioneering work on averaging based analysis in an extremum seeking system in 1967. His study was extended by Krstic and Wang for stability analysis on an extremum seeking feedback scheme with a general nonlinear system in 2000 ([7],[8]).

During the same period, sliding mode based approaches were also proposed to address the difficulty of unknown performance functions. The theoretical development traces back to [9], where static optimization was considered. Drakunov and Özgüner developed a sliding mode extremum seeking control structure for output optimization of dynamic systems [10]. They applied the structure in anti-lock brake system control on automobiles [11]. Haskara *et al.* proposed a two-time scale approach [12]. In the slow time scale, they employed sliding mode optimization to update a free parameter and determine set points for the system corresponding to the parameter. In the fast time scale, they utilized stabilizing control to regulate the system to the set points. Pan *et al.* analyzed in detail the stability and performance improvement of the approach [13]. Rong and Özgüner discussed extremum seeking control via sliding mode with two surfaces [14].

Both approaches require that the dynamics of the nonlinear plant be fast enough and the adaptation gain be relatively small to ensure convergence. There are also works that seek to estimate the gradient and employ gradient-based optimization. For example, Sera *et al.* [15] studied photovoltaic power systems and proposed to estimate the sign of the gradient by perturbation and observation to determine the direction of hill-climbing in order to track maximum power point. For a more general plant that might have slow or time-delayed dynamics, Teel and Popovic ([16]) proposed to introduce a waiting period between consecutive control updates and measurements of performance, allowing the system dynamics to settle before new control is determined. The control updates can explore to utilize the already sophisticated nonlinear programming (NLP) algorithms. For the most popular NLP algorithms that are based on gradient information, it is possible to probe the performance profile for gradient estimate.

This paper presents a new approach for extremum seeking control, which aims to improve the convergence speed of extremum seeking as well as robustness against measurement noise of the performance profile, without the need of explicit external perturbation. A discrete sliding mode observer is designed to estimate the unknown gradient of the performance map. Based on the sign of the gradient estimation, a variable structure controller generates the control input for the nonlinear plant. The gradient estimation error decreases rapidly to the vicinity of zero and stays inside afterwards, and the variable structure controller enforces the system to converge to a neighborhood of the optimal operating point, such that the deviation from the point is ultimately bounded. The stability condition and the ultimate bound is derived. The robustness of the control scheme is discussed in the presence of unmodeled fast dynamics and measurement noise.

The paper is organized as follows. Section 2 states the control problem and main assumptions of the paper. Section 3 elaborates the controller design. Section 4 analyzes the stability and optimization performance, followed by Section 5 with the discussion of the effect of unmodeled actuator dynamics and measurement noise. Simulation results of an example system is given in Section 6. Finally, Section 7 presents the conclusion.

L. Fu is with the Department of Electrical and Computer Engineering, Ohio State University, Columbus, OH 43210, USA fu.83@osu.edu

Ü. Özgüner is with the Faculty of Electrical and Computer Engineering, Ohio State University, Columbus, OH 43210, USA umit@ece.osu.edu

978-1-4244-4524-0/09/$25.00 ©2009 AACC

II. Problem Formulation

Consider a nonlinear single-input-single-output (SISO) plant

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$ are smooth functions.

The cost criterion is a function of the system output y:

$$z = H(y)$$

The function $H(\cdot)$ or some of its parameters are unknown *a priori*. With only real-time measurement of z, the control objective is to steer the plant to operate at the maximum or minimum of z. Without of loss of generality, we consider minimization here.

We make a few assumptions on the plant and the performance function:
1. There exist a global diffeomorphism

$$y = \gamma(\xi)$$

such that the function $Z(\xi) = H \circ \gamma(\xi) = H(\gamma(\xi)))$ is smooth and has a unique minimum at ξ^*.
2. The second-order derivative of $Z(\xi)$ is bounded. That is, there exist $M > 0$, such that $|\frac{\partial^2 Z}{\partial \xi^2}| < M, \ \forall \xi \in \mathbb{R}$.
3. The system with ξ as the output

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ \xi &= \varphi(x) \end{aligned} \tag{2}$$

where $\varphi(x) = \gamma^{-1} \circ g(x)$, has global relative degree ρ. Moreover, there exists a global diffeomorphism that transforms the system into normal form

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \zeta) \\ \dot{\zeta} &= A_c\zeta + B_c\beta(x)[u - \alpha(x)] \\ \xi &= \zeta_1 \end{aligned} \tag{3}$$

Here $\zeta = [\zeta_1, \zeta_2, \cdots, \zeta_\rho]^T \in \mathbb{R}^\rho$, (A_c, B_c) are in Brunovsky form:

$$A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

4. The internal dynamics $\dot{\eta} = f_0(\eta, \zeta)$ of the system is input-to-state stable with respect to ζ.

III. Controller Design

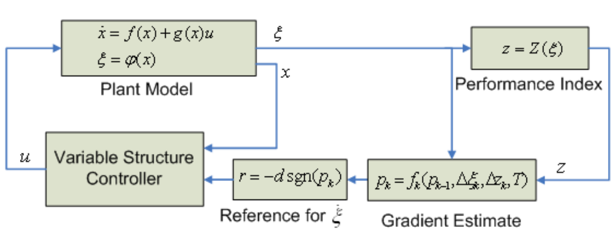

Fig. 1. The block diagram for the controller design.

Fig. 1 illustrates the block diagram for the proposed extremum seeking controller. The design of the controller has two folds. First of all, a sliding mode observer is built to estimate the gradient of the performance map. Then a variable structure controller is constructed to enforce finite-time convergence of ξ to the optimal point ξ^*. Based on the sign of the estimated gradient, the controller determines a reference rate r for the derivative of ξ so as to minimize $Z(\xi)$. The plant is regulated such that $\dot{\xi}$ converges to the reference r asymptotically.

A. Sliding mode observer design for gradient estimation

For the design of the discrete-time sliding mode observer, consider a sampling period T. At the sampling instant kT, z_k is available by measurement, and the value of ξ_k is also attainable through the plant output:

$$\xi_k = \gamma^{-1}(y_k)$$

Due to possible noise in performance measurement, the finite-difference approximation of gradient could amplify the noise drastically. To avoid the problem, sliding mode is utilized in our observer for estimating the gradient of $Z(\xi)$. We choose the sliding surface as

$$s_k = p_{k-1} - \frac{\Delta z_k}{\Delta \xi_k}$$

where $\Delta z_k = z_k - z_{k-1}$, and $\Delta \xi_k = \xi_k - \xi_{k-1}$. The observer for $\frac{\partial Z}{\partial \xi}$ is designed as:

$$p_k = p_{k-1} - V_0 T \text{sgn}(s_k \Delta \xi_k)\text{sgn}(\Delta \xi_k) \tag{4}$$

The $\text{sgn}(\cdot)$ function is defined as

$$\text{sgn}(s) = \begin{cases} 1, & s \geq 0 \\ -1, & s < 0 \end{cases}$$

Note that by definition $\text{sgn}(s) \neq 0$ for all $s \in \mathbb{R}$. This is important for the observer to work properly. If $\text{sgn}(\Delta\xi_k)$ were zero at some point, p_k would stop updating, and in turn $\text{sgn}(\Delta\xi_k)$ would remain at zero thereafter. Therefore, the way we define $\text{sgn}(\cdot)$ is crucial to avoid possible deadlock in the observer.

B. Variable structure controller design for the plant

The objective of the controller is to ensure that ξ converge to the optimal operating point ξ^* with a constant rate specified by the designer. For minimization of $Z(\xi)$, the reference r for the derivative of ξ is determined by the estimated gradient as:

$$r = -d\text{sign}(p_k)$$

where $d > 0$.

Consider system (3). Define $\tilde{\zeta}_2 = \zeta_2 - r$. Consider the subsystem

$$\begin{aligned} \dot{\tilde{\zeta}}_2 &= \zeta_3 \\ &\vdots \\ \dot{\zeta}_\rho &= \beta(x)[u - \alpha(x)] \end{aligned} \tag{5}$$

Let (A'_c, B'_c) be in Brunovsky form of $(\rho-1)$-th order, and K be such that $A'_c + B'_c K$ is Hurwitz. To achieve asymptotic stability of the origin of the subsystem, the control is designed as

$$u = \frac{1}{\beta(x)} K\tilde{\zeta} + \alpha(x) \tag{6}$$

where $\tilde{\zeta} = [\tilde{\zeta}_2, \zeta_3, \cdots, \zeta_\rho]^T$.

IV. Stability Analysis

A. Gradient Estimation

It follows from (4) that

$$\begin{aligned} s_{k+1} - s_k &= p_k - \frac{z_{k+1} - z_k}{\xi_{k+1} - \xi_k} - (p_{k-1} - \frac{z_k - z_{k-1}}{\xi_k - \xi_{k-1}}) \\ &= -\frac{z_{k+1} - z_k}{\xi_{k+1} - \xi_k} + \frac{z_k - z_{k-1}}{\xi_k - \xi_{k-1}} - V_0 T \text{sgn}(s_k) \end{aligned}$$

From Assumption 1, the cost function $Z(\xi)$ is smooth. By Taylor expansion,

$$Z(\xi) = Z(\xi_0) + \frac{\partial Z}{\partial \xi}(\xi_0)(\xi - \xi_0) + \frac{\partial^2 Z}{\partial \xi^2}(\xi_0)(\xi - \xi_0)^2 + O((\xi - \xi_0)^3)$$

Then,

$$z_{k+1} = z_k + \frac{\partial Z}{\partial \xi}(\xi_k)\Delta\xi_{k+1} + \frac{\partial^2 Z}{\partial \xi^2}(\xi_k)\Delta\xi_{k+1}^2 + O(\Delta\xi_{k+1}^3)$$

Consequently,

$$\begin{aligned} s_{k+1} - s_k = &-\frac{\partial Z}{\partial \xi}(\xi_k) + \frac{\partial Z}{\partial \xi}(\xi_{k-1}) - \frac{\partial^2 Z}{\partial \xi^2}(\xi_k)\Delta\xi_{k+1} \\ &+ \frac{\partial^2 Z}{\partial \xi^2}(\xi_{k-1})\Delta\xi_k - O(\Delta\xi_{k+1}^2) + O(\Delta\xi_k^2) \\ &- V_0 T \text{sgn}(s_k) \end{aligned}$$

By Assumption 2, $\left|\frac{\partial^2 Z}{\partial \xi^2}\right| < M$. On the other hand, there exists $\hat{\xi} \in \mathbb{R}$ such that $\frac{\partial Z}{\partial \xi}(\xi_1) - \frac{\partial Z}{\partial \xi}(\xi_2) = \frac{\partial^2 Z}{\partial \xi^2}(\hat{\xi})(\xi_1 - \xi_2)$. Therefore,

$$\left|\frac{\partial Z}{\partial \xi}(\xi_1) - \frac{\partial Z}{\partial \xi}(\xi_2)\right| < M|\xi_1 - \xi_2|, \ \forall \xi_1, \xi_2 \in \mathbb{R}$$

As a result,

$$\begin{aligned} &\left|-\tfrac{\partial Z}{\partial \xi}(\xi_k) + \tfrac{\partial Z}{\partial \xi}(\xi_{k-1}) - \tfrac{\partial^2 Z}{\partial \xi^2}(\xi_k)\Delta\xi_{k+1} + \tfrac{\partial^2 Z}{\partial \xi^2}(\xi_{k-1})\Delta\xi_k\right| \\ &< 2M|\Delta\xi_k| + M|\Delta\xi_{k+1}| \end{aligned}$$

To derive the bound on $\Delta\xi_k$, let $\bar{\zeta} = [\zeta_2, \zeta_3, \ldots, \zeta_\rho]^T$. Consider the $\bar{\zeta}$-subsystem in the normal form (3) with control (6):

$$\begin{aligned} \dot{\bar{\zeta}} &= A'_c\bar{\zeta} + B'_c\beta(x)[u - \alpha(x)] \\ &= A'_c\bar{\zeta} + B'_c K\tilde{\zeta} \\ &= (A'_c + B'_c K)\bar{\zeta} - B'_c K_1 r \end{aligned}$$

where K_1 is the first column of the feedback matrix K. Because $A'_c + B'_c K$ is Hurwitz, there exist positive definite matrices P and Q, such that

$$(A'_c + B'_c K)^T P + P(A'_c + B'_c K) \le -Q$$

Let $V_{\bar{\zeta}} = \bar{\zeta}^T P \bar{\zeta}$. Its derivative satisfies

$$\begin{aligned} \dot{V}_{\bar{\zeta}} &\le -\bar{\zeta}^T Q\bar{\zeta} + 2|\bar{\zeta}^T PBK_1 r| \\ &\le -\lambda_{min}|\bar{\zeta}|^2 + 2d|PBK_1||\bar{\zeta}| \\ &= -\lambda_{min}|\bar{\zeta}|(|\bar{\zeta}| - \varepsilon) \end{aligned}$$

where λ_{min} is the smallest eigenvalue of Q, and $\varepsilon = \frac{2d}{\lambda_{min}}|PBK_1|$. Thus whenever $|\bar{\zeta}| > \varepsilon$, we have $\dot{V}_{\bar{\zeta}} < 0$. Let λ_1 and λ_2 be the smallest and the largest eigenvalue of P, respectively, then $\lambda_1|\bar{\zeta}|^2 \le \bar{\zeta}^T P\bar{\zeta} \le \lambda_2|\bar{\zeta}|^2$. Therefore if the initial condition of the system satisfies $\left|\bar{\zeta}_0\right| \le \frac{\lambda_2}{\lambda_1}\varepsilon$,

$$|\bar{\zeta}| \le \frac{\lambda_2}{\lambda_1}\varepsilon, \ \forall t \ge 0$$

Obviously $|\zeta_2| \le |\bar{\zeta}|$, and

$$|\zeta_2| \le \frac{\lambda_2}{\lambda_1}\varepsilon, \ \forall t \ge 0$$

It follows

$$|\Delta\xi_k| = \left|\int_{(k-1)T}^{kT} \zeta_2 dt\right| \le \frac{\lambda_2}{\lambda_1}\varepsilon T$$

Hence,

$$|\Delta\xi_k| \le \frac{\lambda_2}{\lambda_1}\varepsilon T, \ \forall \rho \ge 1, \forall t \ge 0$$

Note that we can choose related parameters to ensure $|\frac{\lambda_2}{\lambda_1}\varepsilon T| \ll 1$. As a result, the term

$$-O(\Delta\xi_{k+1}^2) + O(\Delta\xi_k^2)$$

is negligible because both $|\Delta\xi_k| \ll 1$ and $|\Delta\xi_{k+1}| \ll 1$.

Consider the Lyapunov function $V_k = s_k^2$. Note that

$$\Delta V_k = V_{k+1} - V_k = s_{k+1}^2 - s_k^2 = (s_{k+1} + s_k)(s_{k+1} - s_k)$$

Choose $V_0 > 3M\frac{\lambda_2}{\lambda_1}\varepsilon$. Let $\delta = 2V_0$. When $|s_k| > \delta$, it is true that $\text{sgn}(s_{k+1}) = \text{sgn}(s_k)$ and thereby $\text{sgn}(s_{k+1} + s_k) = \text{sgn}(s_k)$ because $|s_{k+1} - s_k| < \delta$. On the other hand, $\text{sgn}(s_{k+1} - s_k) = -\text{sgn}(s_k)$. Therefore,

$$\Delta V_k < 0, \ \forall |s_k| > \delta$$

In summary, the conditions for the gradient estimation to enter the boundary layer $|s_k| < \delta$ of the sliding surface $s_k = 0$ and stay inside it afterwards are:

$$\begin{aligned} &V_0 > 3M\tfrac{\lambda_2}{\lambda_1}\varepsilon, \text{ and} \\ &\tfrac{\lambda_2}{\lambda_1}\varepsilon T \ll 1 \end{aligned} \tag{7}$$

B. Optimization Accuracy

After the sliding mode gradient observer reaches the boundary layer $|s_k| < \delta$, the estimation error of the gradient is bounded by δ, that is

$$\left|p_k - \frac{\partial Z}{\partial \xi}(\xi_k)\right| = |s_k| < \delta$$

As a result, when $\left|\frac{\partial Z}{\partial \xi}(\xi_k)\right| > \delta$, we have $\text{sgn}(p_k) = \text{sgn}(\frac{\partial Z}{\partial \xi}(\xi_k))$. Without loss of generality, assume $\frac{\partial Z}{\partial \xi}(\xi_k) < -\delta$, then the reference rate $r = d > 0$. With the asymptotically stabilizing control (6), the states of the $\tilde{\zeta}$-subsystem

(5) enter the invariant manifold $\mathcal{S}_\zeta = \{\tilde{\zeta} \in \mathbb{R}^{\rho-1} \Big| |\tilde{\zeta}| < d\}$ in finite time. Inside the manifold $\mathcal{S}_\zeta$, it is guaranteed that $|\tilde{\zeta}_2| < d$, and equivalently $\dot{\xi} = \zeta_2 > 0$. As a result, ξ increases. Therefore, ξ eventually enters the manifold $\mathcal{S}_\xi = \{\xi \in \mathbb{R} \Big| |\frac{\partial Z}{\partial \xi}| < \delta\}$.

Once ξ is inside $\mathcal{S}_\xi$, the distance between ξ and $\mathcal{S}_\xi$, defined as

$$dist(\xi, \mathcal{S}_\xi) = \min_{\xi_1 \in \mathcal{S}_\xi} |\xi - \xi_1|$$

will be bounded by a constant value thereafter. The reasoning is as follows. When ξ leaves $\mathcal{S}_\xi$, the reference rate is $r = -d\mathrm{sign}(\frac{\partial Z}{\partial \xi})$ because $\mathrm{sgn}(p_k) = \mathrm{sgn}(\frac{\partial Z}{\partial \xi})$ at this time. Consider the time t_s needed for $\dot{\xi}$ to enter the neighborhood $B_d = \{|\dot{\xi} - r| \le d\}$ of r so that $\mathrm{sgn}(\dot{\xi}) = -\mathrm{sgn}(\frac{\partial Z}{\partial \xi})$. Define $V_{\tilde{\zeta}} = \tilde{\zeta}^T P \tilde{\zeta}$. Then $\lambda_1 |\tilde{\zeta}|^2 \le V_{\tilde{\zeta}} \le \lambda_2 |\tilde{\zeta}|^2$, and

$$\dot{V}_{\tilde{\zeta}} = -\tilde{\zeta}^T Q \tilde{\zeta} \le -\lambda_{\min} |\tilde{\zeta}|^2$$

It has been proved earlier that $|\bar{\zeta}| < \frac{\lambda_2}{\lambda_1}\varepsilon$, $\forall t \ge 0$. Also, $\dot{\xi} = \zeta_2 \in B_d$ if $|\bar{\zeta}| < d$. It follows

$$t_s \le \frac{\lambda_2 (\frac{\lambda_2}{\lambda_1}\varepsilon)^2 - \lambda_1 d^2}{\lambda_{\min} d^2}$$

Let $l = t_s \max(\frac{\lambda_2}{\lambda_1}\varepsilon, d)$, then

$$dist(\xi, \mathcal{S}_\xi) \le l$$

Let ξ^l and ξ^r be the closest points to ξ^* that satisfy $\frac{\partial Z}{\partial \xi}(\xi^l) = -\delta$ and $\frac{\partial Z}{\partial \xi}(\xi^r) = \delta$, respectively. The optimization accuracy, in terms of the error between ξ and ξ^*, is ultimately bounded by

$$|\xi - \xi^*| < \max(|\xi^l - \xi^*|,\ |\xi^r - \xi^*|) + l$$

We conclude this section with the following theorem:

Theorem 4.1: Assume that the plant (1) satisfies Assumptions 1-3 in Section 2, and that there exist a global diffeomorphism that transforms the plant with new output (2) into normal form (3). Assume also the initial condition of the system satisfies that $|\bar{\zeta}_0| \le \frac{\lambda_2}{\lambda_1}\varepsilon$. Then the sliding mode estimator (4) with (7) and the variable structure controller (6) in Section 3 solve the extremum seeking control problem. In particular:

1. the sliding mode gradient observer converges to a δ-vicinity of the real value in finite time, that is, there exists $T_1 > 0$, such that

$$\left| p_k - \frac{\partial Z}{\partial \xi} \right| < \delta,\ \forall t > T_1$$

2. the optimizing variable ξ converges to a neighborhood of the optimal value ξ^* in finite time, and the error $|\xi - \xi^*|$ is ultimately bounded; that is, there exist $T_2 > 0$ and $l' > 0$, such that

$$|\xi - \xi^*| < l',\ \forall t > T_2$$

where $l' = \max(|\xi^l - \xi^*|,\ |\xi^r - \xi^*|) + l$.

Remark 1: The assumption on the initial condition $|\bar{\zeta}_0| \le \frac{\lambda_2}{\lambda_1}\varepsilon$ is not restrictive. When it is not satisfied, set $r = 0$, and the system will achieve the condition in finite time. After that, we can apply the designed controller to solve the extremum seeking problem as described above.

V. The Effect of Unmodeled Actuator Dynamics and Measurement Noise

A. Unmodeled Actuator Dynamics

In variable structure control, unmodeled actuator dynamics is often a potential source of deteriorated performance or even instability. In previous sections, we have neglected the actuator dynamics for simplicity of design and stability analysis. However, it is important that we examine the effect of unmodeled actuator dynamics on the stability and optimization performance of the controlled system.

Consider the controlled subsystem (5) with the following actuator dynamics:

$$\begin{aligned} \dot{\tilde{\zeta}} &= A_c \tilde{\zeta} + B_c \beta(x)[v - \alpha(x)] \\ \epsilon \dot{\omega} &= A\omega + B[\frac{1}{\beta(x)} K \tilde{\zeta} + \alpha(x)] \\ v &= C\omega \end{aligned}$$

where A is a Hurwitz matrix, and $-CA^{-1}B = I$. Because actuator dynamics are typically much faster than the plant dynamics, we can assume that ϵ is small, and apply singular perturbation analysis on the overall system. With the controller design in Section 3, the origin of the reduced system (5) is exponentially stable. Define $\tilde{\omega} = \omega + A^{-1}B[\frac{1}{\beta(x)} K \tilde{\zeta} + \alpha(x)]$. Let $\tau = t/\epsilon$. The boundary-layer model is given by

$$\frac{d\tilde{\omega}}{d\tau} = A\tilde{\omega}$$

whose origin is also exponential stable since A is Hurwitz. By the Singular Perturbation Theorem (Theorem 11.4, [17]), the origin of the actual closed-loop system is exponentially stable for sufficiently small ϵ. Therefore, $|\xi - \xi^*|$ is ultimately bounded. However, we should be aware that the ultimate bound could be larger now because of possible increase in t_s.

B. Noise in Performance Measurement

In realistic applications, measurement noise is almost inevitable. A robust controller needs to reduce the disturbance from noise as much as possible. In the case of gradient estimation this is especially important, because noise is often greatly magnified due to the differential calculation.

Zero-mean white noise in performance measurement is considered in the problem. The sliding mode observer has greatly improved the accuracy of gradient estimation by limiting the rate of change of the estimate. Moreover, the fact that the variable structure controller depends only on the sign of the estimate, not the magnitude also adds to the robustness of the overall system.

Note that increasing the sampling period is a good way to reduce the noise level in gradient estimate.

VI. Simulation Examples

Consider the following system

$$\begin{aligned} \dot{x}_1 &= -x_1 + \tfrac{1}{1+x_3^2} u \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 x_3 + u \\ \xi &= x_2 \end{aligned} \tag{8}$$

The objective of the control was to minimize the performance criterion

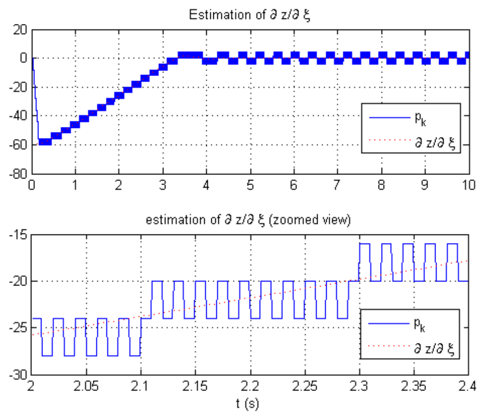

Fig. 2. Gradient Estimate of the performance map.

$$z = 10(\xi - 5)^2 - 10$$

The change of coordinates $[\eta, \zeta_1, \zeta_2] = [x_1 - \tan^{-1} x_3, x_2, x_3]$ transforms the system into normal form

$$\begin{aligned}\dot{\eta} &= -\left(1 + \frac{\zeta_2}{1+\zeta_2^2}\right)\eta - \tan^{-1}\zeta_2\left(1 + \frac{\zeta_2}{1+\zeta_2^2}\right)\\ \dot{\zeta}_1 &= \zeta_2\\ \dot{\zeta}_2 &= x_1 x_3 + u\\ \xi &= \zeta_1\end{aligned}$$

Let $V_\eta = \frac{1}{2}\eta^2$, then

$$\begin{aligned}\frac{dV}{d\eta}\dot{\eta} &= -\left(1 + \frac{\zeta_2}{1+\zeta_2^2}\right)\eta^2 - \tan^{-1}\zeta_2\left(1 + \frac{\zeta_2}{1+\zeta_2^2}\right)\eta\\ &\le -\frac{1}{2}\eta^2 - \tan^{-1}\zeta_2\left(1 + \frac{\zeta_2}{1+\zeta_2^2}\right)\eta\\ &\le -\frac{1}{4}\eta^2 + |\eta|\left(-\frac{1}{4}|\eta| + \frac{3}{2}|\tan^{-1}\zeta_2|\right)\\ &\le -\frac{1}{4}\eta^2 \quad \text{when}|\eta| > 6|\tan^{-1}\zeta_2|\end{aligned}$$

where the fact $\left|\frac{\zeta_2}{1+\zeta_2^2}\right| \le \frac{1}{2}$ is used. Therefore, the η-dynamics satisfies the input-to-state stable condition with respect to ζ.

It is easy to see that the optimal output here is $\xi^* = 5$. The controller parameters in our simulation are selected as $V_0 = 400$, $T = 0.01s$, $d = 1$, $K = -4$, and the initial condition is $x_{10} = 2, x_{20} = 2, x_{30} = 0$. The sign$(\cdot)$ function in $r = -d\text{sign}(p_k)$ is approximated by a continuous function

$$\Phi(\pi) = \begin{cases}\pi/\mu, & |\pi| < \mu\\ \pi/|\pi|, & |\pi| \ge \mu\end{cases}$$

with $\mu = 0.1$. Fig. 2 demonstrates the gradient estimate. After a short period of reaching phase, the estimate stays close to the real value. Fig. 3 shows the real-time measurement of the performance cost function, the trajectory of ξ, and the corresponding control input. At the rising stage of ξ, the rate of increase follows closely to the reference rate $r = d = 1$. It is straightforward to change the convergence rate of ξ by adjusting the reference rate d. As we can see, the steady-state oscillation of ξ is very small.

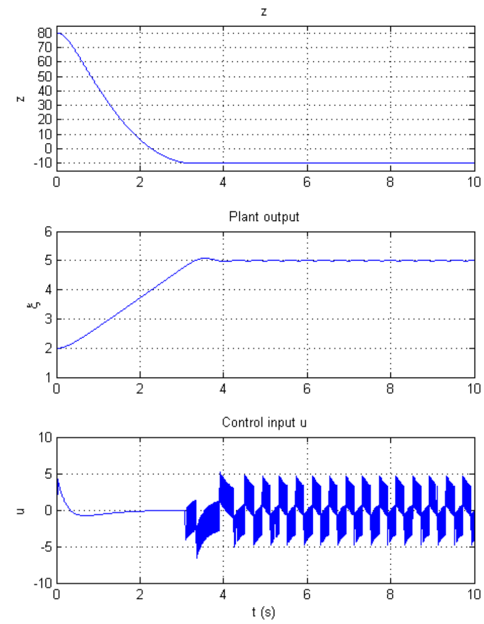

Fig. 3. Performance index z, optimizing variable ξ and control input u.

To simulate the effect of actuator dynamics, we include an unknown second-order fast dynamics $\omega_1 = \frac{1}{(1+\epsilon s)^2}u$ in the previous example. The system of interest is

$$\begin{aligned}\dot{x}_1 &= -x_1 + \frac{1}{1+x_3^2}\omega_1\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= x_1 x_3 + \omega_1\\ \dot{\omega_1} &= \omega_2\\ \epsilon^2\dot{\omega_2} &= u - \omega_1 - 2\epsilon\omega_2\\ \xi &= x_2\end{aligned}$$

Fig. 4 shows the simulation results for this system with $\epsilon = 0.02$, using the same controller designed for the nominal system (8) above. We still observe a steady convergence of ξ to ξ^*. While the ultimate boundedness remains valid, the steady-state oscillation of z and ξ is slightly larger than without actuator dynamics.

Fig. 5 demonstrates the case where we have white noise in the measurement z. The standard variance of the noise in the simulation is $\sigma = 0.1$. Again, ξ converges towards ξ^* steadily, and then oscillates within a neighborhood of the optimal point.

VII. Conclusion

For a class of nonlinear systems, this paper proposes a variable structure extremum seeking scheme based on gradient estimation. The sliding mode observer for performance gradient is proved to converge to the vicinity of the real value. The variable structure controller is shown to be able to enforce the system states to be enter a neighborhood of the equilibrium corresponding to the optimal performance value. The deviation from the optimal operating point is ultimately bounded. For systems with unmodeled fast dynamics, it is

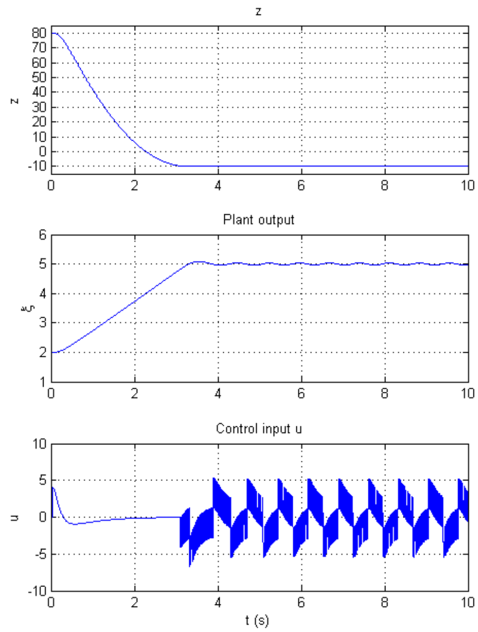

Fig. 4. Performance index z, optimizing variable ξ and control input u for the system with fast dynamics.

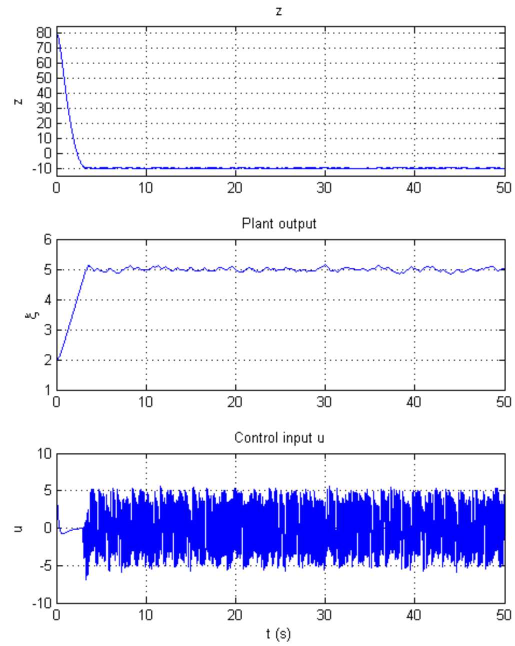

Fig. 5. Performance index z, optimizing variable ξ and control input u for the system with measurement noise.

shown that the ultimate boundedness is still valid when additional conditions are satisfied. The controller is also robust to measurement noise. In both situations, the ultimate bound is subject to a possible increase in size. Simulation results with a prototype problem illustrates the foregoing conclusions.

REFERENCES

[1] I. I. Ostrovskii, Extremum regulation, *Automatic & Remote Control*, vol. 18, 1957, pp. 900-907.

[2] O. L. R. Jacobs and G. C. Shering, Design of a single-input sinusoidal-perturbation extremum-control system, *Proceedings IEE*, vol. 115, 1968, pp. 212-217.

[3] K. J. Astrom and B. Wittenmark, *Adaptive Control*, second edition, Addison-Wesley, Boston, MI; 1995.

[4] S.M. Meerkov, Asymptotic methods for investigating quasistationary states in continuous systems of automatic optimization, *Automation Remote Control*, vol. 11, 1967, pp. 1726-1743.

[5] S.M. Meerkov, Asymptotic methods for investigating a class of forced states in extremal systems, *Automation Remote Control*, vol. 12, 1967, pp. 1916-1920.

[6] S.M. Meerkov, Asymptotic methods for investigating stability of continuous systems of automatic optimization subjected to disturbance action, *Avtomatika i Telemekhanika*, vol. 12, 1967, pp. 14-24 (in Russian).

[7] M. Krstic and H.H. Wang, Stability of Extremum Seeking Feedback for General Nonlinear Dynamic Systems, *Automatica*, vol. 36, 2000, pp. 595-601.

[8] M. Krstic, Performance Improvement and Limitations in Extremum Seeking Control, *Syst. & Cntr. Letters*, vol. 39, 2000, pp. 313-326.

[9] S.K. Korovin and V.I. Utkin, Using Sliding Modes in Static Optimization and Nonliear Programming, *Automatica*, vol. 10, 1974, pp. 525-532.

[10] S. Drakunov and Ü. Özgüner, Optimization of Nonlinear System Output via Sliding Mode Approach, *IEEE International Workshop on Variable Structure and Lyapunov Control of Uncertain Dyanmical System*, UK, 1992, pp. 61-62.

[11] S. Drakunov, Ü. Özguner, P. Dix and B. Ashrafi, ABS Control Using Optimum Search via Sliding Modes, *IEEE Trans. on Control Systems Technology*, vol. 3, issue 1, 1995, pp. 79-85.

[12] İ. Haskara, Ü. Özgüner, and J. Winkelman, Extremum Control for Optimal Operating Point Determination and Set Point Optimization via Sliding Modes, *Journal of Dynamic Systems, Measurement and Control*, vol. 122, 2000, pp. 719-724.

[13] Y. Pan, Ü. Özgüner, and T. Acarman, Stability and performance improvement of extremum seeking control with sliding mode, *Int.J.Control*, vol. 76, no. 9-10, 2003, pp. 968-985.

[14] R. Xu and Ü. Özgüner, Extremum Seeking Control via Sliding Mode with Two Surfaces, *Proceedings of the 16th IFAC World Congress*, Prague, Czech, 2005, pp. 2212-2217.

[15] D. Sera, R. Teodorescu, J. Hantschel and M. Knoll, Optimized Maximum Power Point Tracker for Fast-Changing Environmental Conditions, *IEEE Trans. on Industrial Electronics*, vol. 55, 2008, pp. 2629-2637.

[16] A. R. Teel and D. Popovic, Solving smooth and nonsmooth multivariable extremum seekingproblems by the methods of nonlinear programming, *Proceedings of the American Control Conference*, vol. 3, 2001, pp. 2394-2399.

[17] H.K. Khalil, *Nonlinear Systems*, Third Edition, Prentice Hall, NY; 2002.