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Immersion and Invariance Adaptive Control of Nonlinearly Parameterized Nonlinear Systems

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***Abstract*— A new framework to design adaptive controllers for nonlinearly parameterized systems is proposed in this paper. The key step is the construction of a *monotone* function, which explicitly depends on some of the estimator tuning parameters. Monotonicity—or the related property of convexity—have already been explored by several authors with convexity being an *a priori* assumption that is valid only on some region of state space. In our approach monotonicity is *enforced* by the designer, effectively becoming a synthesis tool. One consequence of this fact is that the controller does not rely on state–dependent switching. In order to dispose of degrees of freedom to render the function monotone we depart from standard adaptive control and adopt instead the recently introduced Immersion and Invariance approach.**

I. INTRODUCTION

It is well known that nonlinear parameterizations are inevitable in any realistic practical problem. On the other hand, designing adaptive (identification or control) algorithms for nonlinearly parameterized systems is a difficult poorly understood problem. A standard procedure to overcome the problem is to introduce extra parameters in order to obtain a linear parametrization, but this suffers from the well known shortcomings of robustness degradation due to the need of a search in a bigger parameter space, the conservativeness introduced when incorporating prior knowledge in restricted parameter estimation and the possibility of generating unexisting pole–zero cancelations [18].

Some results for standard (gradient–like) direct adaptive controllers have been reported in the literature for *convexly* parameterized systems. It was first reported in [9] (see also [22]) that convexity is enough to ensure that the gradient search "goes in the right direction" in a *certain region* in the state space. The idea is then to apply a standard adaptive scheme in this region, while in the "bad" region either the adaptation is frozen and a robust constant parameter controller is switched on [10] or, as proposed in [1], the adaptation is running all the time and stability is ensured with a high–gain mechanism which is suitably adjusted incorporating prior knowledge on the parameters. In [20] *reparametrization* to convexify an otherwise non-convexly parameterized system is proposed. See also [21] and [26] for some recent interesting results along these lines, where the controller and the estimator switch between convex and concave parameterizations or, as in the latter reference, between over/underbounding convex/concave functions.

A new methodology to design direct and indirect adaptive controllers for nonlinear systems, called Immersion and Invariance (I&I), was recently proposed in [3], see also [4]. Although, in principle, the I&I technique is applicable for nonlinearly parameterized systems—besides some basic bounding assumptions of the error terms—no explicit guidelines are given to deal with the nonlinear parameter dependence. In this paper a framework to design I&I adaptive controllers for nonlinearly parameterized systems, which relies on a *monotonicity* property, is developed.[1] It turns out that the way monotonicity can be used in I&I adaptive control differs from the way convexity has been exploited in standard adaptation methods, in at least three fundamental aspects.

- In contrast with standard schemes, where it is *a priori* assumed that the plant is convexly parameterized (or convexifiable), in I&I adaptive control we require monotonicity of a function, which explicitly depends on some of the estimator tuning parameters. Hence, monotonicity is not assumed but can be *enforced*, effectively becoming a design tool.
- In standard adaptation schemes convexity is used to define "good" and "bad" regions in state space, which needs then to be combined with some *switching* strategy. In I&I adaptive control "global" properties may be derived avoiding the need for switching.
- It is shown in the paper that the proposed procedure is applicable to both, direct and indirect I&I adaptive schemes—in contrast to standard adaptation, for which only direct versions have been reported in the literature.

We noticed the interesting work [27]. Similarly to I&I, the authors also consider proportional plus integral adaptation and express the control aim in terms of rendering attractive a given manifold. The main difference with the present work is, however, that the objective manifold in [27] captures the overall systems stabilization, while in our case we are interested only in the estimator part. In the spirit of I&I, whose main motivation is to avoid the intricacies of Lyapunov functions generation, the overall systems is then viewed (and analyzed) as the cascade of a convergent

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[1] Monotonicity has been used in the context of observer design in [2], see also [6] for a recent extension of that result.

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estimator and a perturbed controlled system. As witnessed by the developments in this paper, the main advantage of adopting this approach is that it yields *much simpler*, hence more widely applicable, estimation algorithms.

The remaining of the paper is organized as follows. To facilitate the comparison between standard and I&I adaptive controllers, in Section II we briefly describe the way convexity is exploited in the former case—the material presented in this section constitutes a slight extension of the existing literature. Sections III and IV present our results on direct and indirect I&I adaptive controllers, respectively. In these sections only a brief summary of I&I adaptation is given and the interested readers are referred to [3], [4] for further details and motivation. In Section V several examples and simulated results are given. The paper is wrapped up with some concluding remarks and open problems in Section VI.

II. Exploiting Convexity in Standard Adaptive Control

Throughout the paper we consider uncertain nonlinear systems described by

$$\dot{x} \;=\; F(x,u)+\Phi(x,\theta) \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $\theta \in \mathbb{R}^q$ is a constant vector representing the *unknown parameters* and $u \in \mathbb{R}^m$ is the control input. The control objective is to stabilize an assignable equilibrium $x_\star \in \mathbb{R}^n$.

For the adaptive control problem to make sense the following stabilizability assumption is clearly needed.

Assumption 1: There exists a function $\psi : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^m$, such that the system

$$\dot{x} = f_\star(x,\theta) := F(x,\psi(x,\theta))+\Phi(x,\theta) \tag{2}$$

has a globally asymptotically stable equilibrium at $x_\star$ with a proper Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}_{>0}$.

As described in [3] (or Subsection 3.3 of [4]), Lyapunov function matching adaptive control provides one alternative to solve the stabilization problem. In this approach, it is assumed that the function $F(x,\psi(x,\theta))$ depends linearly in the parameters, that is, it can be written in the form

$$F(x,\psi(x,\theta)) = F_0(x)+F_1(x)\theta,$$

and the function $[\frac{\partial V}{\partial x}(x,\theta)]^\top F_1(x)$ is *measurable*, that is, it is independent of θ. Under these conditions, the adaptive controller

$$\begin{array}{rcl} u &=& F_0(x)+F_1(x)\hat{\theta} \\ \dot{\hat{\theta}} &=& -\Gamma F_1^\top(x)\frac{\partial V}{\partial x}(x,\theta), \end{array} \tag{3}$$

with adaptation gain $\Gamma = \Gamma^\top > 0$, ensures boundedness of trajectories and stability of the equilibrium with Lyapunov function

$$W(x,\tilde{\theta}) = V(x,\theta)+\frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta}, \tag{4}$$

where $\tilde{\theta} = \hat{\theta}-\theta$. Indeed, the closed–loop system takes the form

$$\dot{x} = f_\star(x,\theta)+F_1(x)\tilde{\theta},$$

and

$$\dot{W} = [\frac{\partial V}{\partial x}(x,\theta)]^\top f_\star(x,\theta) \le 0, \tag{5}$$

where the inequality stems from Assumption 1.

The assumptions of controller linearity and Lyapunov function matching can be obviated if the Lyapunov function is *independent of* θ and the parameterized function

$$\mathscr{Q}^x(\theta) := [\frac{\partial V}{\partial x}(x)]^\top F(x,\psi(x,\theta)), \tag{6}$$

is *convex in* θ, in some subset of the state space $\Omega \subset \mathbb{R}^n$. It is well known [5] that, due to convexity,

$$\mathscr{Q}^x(\hat{\theta})-\mathscr{Q}^x(\theta) \le \tilde{\theta}^\top\frac{\partial \mathscr{Q}^x}{\partial \hat{\theta}}(\hat{\theta}), \quad \forall x \in \Omega. \tag{7}$$

This allows to prove that the adaptive controller

$$\begin{array}{rcl} u &=& \psi(x,\hat{\theta}) \\ \dot{\hat{\theta}} &=& -\Gamma\frac{\partial \mathscr{Q}^x}{\partial \hat{\theta}}(\hat{\theta}), \end{array} \tag{8}$$

which yields the closed–loop system

$$\dot{x} = f_\star(x,\theta)+F(x,\psi(x,\hat{\theta}))-F(x,\psi(x,\theta)),$$

ensures (5) holds for all $x(t) \in \Omega$.

As indicated in the introduction, when $x(t) \notin \Omega$ the controller must be switched to ensure that the function $W(x(t),\tilde{\theta}(t))$ is non–increasing and establish stability of the equilibrium. This operation obviously requires some additional (parameter prior knowledge) assumptions. We refer the reader to the aforementioned references for further details.

III. Direct I&I Adaptive Control

I&I adaptive control is a non–certainty equivalent technique where a function $\beta : \mathbb{R}^n \to \mathbb{R}^q$ is added to the estimated parameter vector.In its direct version this is done in the control law, that is,

$$u = \psi(x,\hat{\theta}+\beta(x)). \tag{9}$$

The objective in I&I is to render the manifold

$$\{(x,\theta) \in \mathbb{R}^n \times \mathbb{R}^q \mid \hat{\theta}-\theta+\beta(x) = 0\}$$

invariant and (asymptotically) attractive. This is achieved driving the, so–called, off-the-manifold coordinate

$$z := \hat{\theta}-\theta+\beta(x) \tag{10}$$

to zero. Towards this end, the derivative of z along the trajectories of the closed–loop system

$$\dot{x} = f_\star(x,\theta)+F(x,\psi(x,\hat{\theta}+\beta(x)))-F(x,\psi(x,\theta)), \tag{11}$$

where $f_\star(x,\theta)$ is defined in (2), is calculated as

$$\begin{array}{rcl} \dot{z} &=& \dot{\hat{\theta}}+[\frac{\partial \beta}{\partial x}(x)]^\top \\ && \cdot\left[f_\star(x,\theta)+F(x,\psi(x,\hat{\theta}+\beta(x)))-F(x,\psi(x,\theta))\right]. \end{array} \tag{12}$$

Following the direct I&I adaptive technique [4] we introduce.

Assumption 2: The function $[\frac{\partial \beta}{\partial x}(x)]^\top f_\star(x,\theta)$ is computable, *i.e.*, independent of θ.

As indicated in the Introduction our design exploits a monotonicity–like property, of a suitably designed mapping, that we call *P–monotonicity*. The latter is a variation of the classical property of monotonicity, which is defined as follows.

Definition 1: Given a matrix $P \in \mathbb{R}^{n\times n}$, $P = P^\top > 0$. A mapping $L : \mathbb{R}^q \to \mathbb{R}^q$ is P–monotone [resp. strictly P–monotone] if and only if, for all $a,b \in \mathbb{R}^q$,

$$\begin{aligned} (a-b)^\top P[L(a)-L(b)] &\quad \geq \quad 0 \\ \Big[\text{resp.} \quad (a-b)^\top P[L(a)-L(b)] &\quad > \quad 0, \quad \forall\, a \neq b\Big]. \end{aligned}$$

To present the main result of this section the following assumption is needed.

Assumption 3: There exists a set $\Omega_D \subset \mathbb{R}^n$ such that the parameterized mapping $\mathscr{Q}_{\mathtt{D}} : \mathbb{R}^q \to \mathbb{R}^q$

$$\mathscr{Q}_{\mathtt{D}}(\theta) := -[\frac{\partial \beta}{\partial x}(x)]^\top F(x,\psi(x,\theta)) \tag{13}$$

is P–monotone [resp. strictly P–monotone] for all $x \in \Omega_D$.

Proposition 1: Consider system (1), verifying Assumption 1 with $x_\star \in \Omega_D$, in closed–loop with the direct I&I adaptive controller (9) and the estimator

$$\dot{\hat{\theta}} = -[\frac{\partial \beta}{\partial x}(x)]^\top f_\star(x,\theta), \tag{14}$$

where the function $\beta(x)$ is such that Assumptions 2 and the strict[2] version of 3 are satisfied. Assume, furthermore, that system (11) verifies

$$\lim_{t\to\infty} z(t) = 0 \;\Rightarrow\; x(t) \in \mathscr{L}_{\infty e}\,. \tag{15}$$

Then, for all $(x(0),\hat{\theta}(0)) \in \Omega_D \times \mathbb{R}^q$ such that $x(t) \in \Omega_D$ for all $t \geq 0$, the trajectories are bounded and

$$\lim_{t\to\infty} x(t) = x_\star.$$

Proof: The first part of the proof consists in establishing that $z^\top P\dot{z} < 0$, hence $\lim_{t\to\infty} z(t) = 0$. Replacing (14) in (12) and using the definition (13) one gets

$$\dot{z} = \mathscr{Q}_{\mathtt{D}}(\theta) - \mathscr{Q}_{\mathtt{D}}(\hat{\theta} + \beta(x)). \tag{16}$$

Consequently,

$$z^\top P\dot{z} = [\hat{\theta} + \beta(x) - \theta]^\top P[\mathscr{Q}_{\mathtt{D}}(\theta) - \mathscr{Q}_{\mathtt{D}}(\hat{\theta}+\beta(x))], \tag{17}$$

where (10) has been used. From (17) and Assumption 3 the bound

$$z^\top(t)P\dot{z}(t) < 0, \quad \forall x(t) \in \Omega_D,$$

is obtained. Convergence to zero of $z(t)$ follows then from invariance of the set Ω_D.

The proof is completed writing the closed–loop system (11) in the perturbed form

$$\dot{x} = f_\star(x,\theta) + \varepsilon(x,t)$$

where, in the light of (15),

$$\varepsilon(x,t) := F(x,\psi(x,z(t)+\theta)) - F(x,\psi(x,\theta))$$

verifies $\lim_{t\to\infty}\varepsilon(x(t),t) = 0$ and invoking Assumption 1 and standard arguments of stability of perturbed systems [25], [14]. ■

Monotonicity is related to convexity via Kachurovskii's Theorem that establishes the equivalence between convexity of a function and monotonicity of its gradient, (cf. Theorem 4.1.4 in Chapter IV of [11]). For the purpose of this work the following interesting result of Demidovich [8] provides a simple way to verify P–monotonicity. The proof of the claim, which is obtained invoking the mean value theorem and some simple analysis arguments, may be found in [23].

Lemma 1: A sufficient condition for a differentiable mapping $L : \mathbb{R}^q \to \mathbb{R}^q$ to be P–monotone [resp. strictly P–monotone] is

$$P\frac{\partial L}{\partial \theta}(\theta) + [\frac{\partial L}{\partial \theta}(\theta)]^\top P \geq 0 \quad [\text{resp.} > 0]\,.$$

The following remarks are in order.

(R1) Assuming some parameter prior knowledge (*e.g.*, knowledge of a set $\Theta \subset \mathbb{R}^q$ such that $\theta \in \Theta$) and incorporating standard estimate projection schemes [9], [24], monotonicity may be imposed only on Θ.

(R2) Although the assumption of invariance of Ω_D seems quite restrictive it should be underscored that, in contrast to standard adaptive schemes where the only "choice" is the Lyapunov function $V(x)$ that should, furthermore, be independent of θ, in I&I adaptive control the function $\beta(x)$ is at the designers disposal—hence opening the possibility of attaining $\Omega_D = \mathbb{R}^n$. This important feature is illustrated in the examples of Section V.

(R3) It is shown in [4] that Assumption 2 is weaker than the Lyapunov function matching assumption of Section II.[3] For affine systems of the form $\dot{x} = f(x,\theta) + g(x)u$, it is satisfied if the uncertain parameters are in the image of $g(x)$. This is the case for the class of systems considered in [1], [20], [21], where it is furthermore assumed that g is constant.

(R4) For the affine systems described in (R3) the mapping (13) takes the simpler form $\mathscr{Q}_{\mathtt{D}}(\theta) = -[\frac{\partial \beta}{\partial x}(x)]^\top g(x)\psi(x,\theta)$.

IV. Indirect I&I Adaptive Control

In indirect I&I adaptive control the aim is to estimate the parameters of the plant—an objective that is achieved, as in its direct version, by driving to zero the off-the-manifold coordinate (10). Towards this end, the dynamics of z is expressed in the form

$$\begin{aligned} \dot{z} &= \dot{\hat{\theta}} + [\frac{\partial \beta}{\partial x}(x)]^\top [F(x,u) + \Phi(x,\theta)] \\ &= \mathscr{Q}_{\mathtt{I}}(\theta) - \mathscr{Q}_{\mathtt{I}}(\hat{\theta} + \beta(x)), \end{aligned} \tag{18}$$

where the estimator has been chosen as

$$\dot{\hat{\theta}} = -[\frac{\partial \beta}{\partial x}(x)]^\top \left[F(x,u) + \Phi(x,\hat{\theta}+\beta(x))\right], \tag{19}$$

[2]For ease of presentation *strict* P–monotonicity is required. This can be relaxed to P–monotonicity only—see Remark R5 in Section IV and Section V.

[3]The observation regarding the "given" function $V(x)$ and the designer–chosen function $\beta(x)$ of the previous remark are also pertinent in this respect.

and the parameterized mapping $\mathscr{Q}_{\mathtt{I}} : \mathbb{R}^q \to \mathbb{R}^q$

$$\mathscr{Q}_{\mathtt{I}}(\theta) := [\frac{\partial \beta}{\partial x}(x)]^\top \Phi(x,\theta) \qquad (20)$$

has been defined. Comparing (16) with (18) motivates the next.

Assumption 4: There exists a set $\Omega_I \subset \mathbb{R}^n$ such that the mapping $\mathscr{Q}_{\mathtt{I}}(\theta)$ is P–monotone [resp. strictly P–monotone] for all $x \in \Omega_I$.

Mimicking the proof of Proposition 1 the result below is easily established.

Proposition 2: Consider system (1) and the estimator (19), where the function $\beta(x)$ is such that the strict version of Assumption 4 is satisfied.

(P1) For all $(x(0), \hat{\theta}(0)) \in \Omega_I \times \mathbb{R}^q$ and $u(t)$ such that $x(t) \in \Omega_I$ for all $t \geq 0$ and $\mathscr{Q}_{\mathtt{I}}(\theta)$ is bounded we have

$$\lim_{t\to\infty} z(t) = 0.$$

(P2) If Assumption 1 holds with $x_\star \in \Omega_I$, then the I&I controller (9) guarantees that, for all $(x(0), \hat{\theta}(0)) \in \Omega_I \times \mathbb{R}^q$ such that $x(t) \in \Omega_I$, the trajectories are bounded and $\lim_{t\to\infty} x(t) = x_\star$.

The following remarks are in order.

(R5) Assumptions 3 and 4 may be relaxed to P–monotonicity, which ensures $\dot{z}^\top P z \leq 0$—*i.e.*, $z(t)$ is non–increasing. This, more reasonable condition, may be sufficient to establish, via some signal chasing, the desired stability properties—see [4] for several illustrative examples. Of course, the issue here is to relate "growth rate" of the perturbation term in (11) with the properties of the Lyapunov function of the matched system $\dot{x} = f_\star(x)$—which we recall is assumed globally asymptotically stable.

(R6) One fundamental difference between direct and indirect I&I adaptive control is that Assumption 2 is conspicuous by its absence in the latter. Although both use the same control law, they differ in the estimators, *e.g.*, (14) and (19), and in the mappings, *e.g.*, (13) and (20), that must be rendered P–monotone with a suitable selection of $\beta(x)$. Since in its indirect version, this is the only task of $\beta(x)$ it is reasonable to expect that its realm of applicability is wider—this, of course, is strongly dependent on form of the functions $F(x, \psi(x,\theta))$ and $\Phi(x,\theta)$.

(R7) In the case of *linearly parameterized* plants Assumption 4 is necessary and sufficient to ensure $\dot{z}^\top P z < 0$, uniformly in x. Indeed, in this case $\Phi(x,\theta) = \Phi_0(x)\theta$ and $\mathscr{Q}_{\mathtt{I}}(\theta) := [\frac{\partial \beta}{\partial x}(x)]^\top \Phi_0(x)\theta$. This mapping is strictly P–monotone if and only if

$$P[\frac{\partial \beta}{\partial x}(x)]^\top \Phi_0(x) + [[\frac{\partial \beta}{\partial x}(x)]^\top \Phi_0(x)]^\top P > 0.$$

An obvious candidate is $\frac{\partial \beta}{\partial x}(x) = \Phi_0(x)$, but this equation may not have a solution, *i.e.*, $\Phi_0(x)$ may not be a Jacobian matrix. In the recent interesting paper [13] this difficulty is removed introducing a dynamic scaling and a filter. Unfortunately, the construction critically relies on linearity—to create a suitable negative quadratic term in z— and is not applicable in our case.

(R8) It is clear from the analysis of the linear case above that P–monotonicity can be made strict *only if* $n \geq q$. A similar constraint applies to the test suggested in Lemma 1.

V. Examples

A. Analysis of separable functions with $q = 1$

Assume a single unknown parameter and, furthermore, that the function $\Phi(x,\theta)$ is *separable*, that is, it can be written in the form

$$\Phi(x,\theta) = \phi(x)\mu(\theta) \qquad (21)$$

where, $\phi : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ and $\mu : \mathbb{R} \to \mathbb{R}^p$. In this case, verifying Assumption 4 reduces to the solution of a standard linear partial differential equation (PDE).

Proposition 3: Let $q = 1$, assume (21) holds and that, at least one of the functions $\mu_i(\theta)$ has a non–zero derivative. All solutions $\beta(x)$ of the linear PDE

$$\phi^\top(x) \frac{\partial \beta}{\partial x}(x) = \ell \qquad (22)$$

where $\ell \in \mathbb{R}^p$ with elements

$$\ell_i = \begin{cases} \mathrm{sign}(\frac{d\mu_i}{d\theta}(\theta)), & \text{if} \quad \mu_i(\theta) \text{ is monotone} \\ \\ 0, & \text{if} \quad \mu_i(\theta) \text{ is arbitrary,} \end{cases}$$

satisfy the strict version of Assumption 4 with $P = 1$.

Necessary and sufficient conditions for solvability of the PDE (22)—even in the case where the right hand side ℓ is a function of x—are given in [7]. Of course, from Frobenius' Theorem [12], a necessary condition is the involutivity of the distribution spanned by the columns of $\phi(x)$.

For the sake of illustration we give in the fact below one particular—admittedly, contrived—example that does not require the solution of the PDE (22) but satisfies Assumption 4. The proof follows via direct substitution and is omitted for brevity.

Fact 1: Let $q = 1$ $n = 2$, assume (21) holds and $\phi(x) = \{\phi_{ij}(x)\}$ verifies:

(i) The second column is function of x_1 *only*, that is, $\phi_{12}(x_1)$ and $\phi_{22}(x_1)$.
(ii) $\phi_{12}(x_1) < 0$.
(iii) $\det \phi(x) > 0$.

Then, for all functions $\mu_1(\theta)$ with *strictly positive* derivative and *arbitrary* functions $\mu_2(\theta)$,

$$\beta(x) = x_2 - \int_0^{x_1} \frac{\phi_{22}(s)}{\phi_{12}(s)} ds - \frac{\phi_{22}(0)}{\phi_{12}(0)} \qquad (23)$$

verifies the strict version of Assumption 4.

Simulations were carried out for the system $\dot{x} = u + \Phi(x,\theta)$ with

$$\Phi(x,\theta) = \begin{bmatrix} 1+\sin(x_1) & -(1+x_1^2) \\ 1+x_2^2 & (1+x_1^2)^2 \end{bmatrix} \begin{bmatrix} \theta - e^{-\theta} \\ \cos(\theta) \end{bmatrix}$$

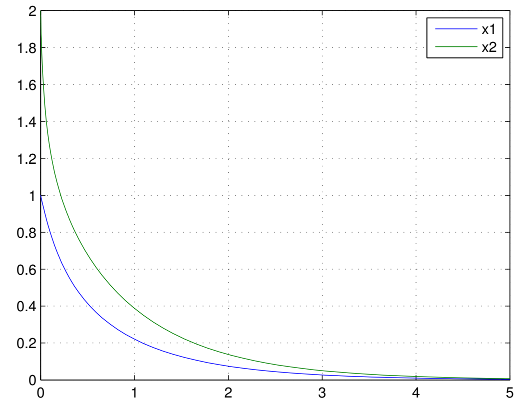

Fig. 1. Transient response of the states

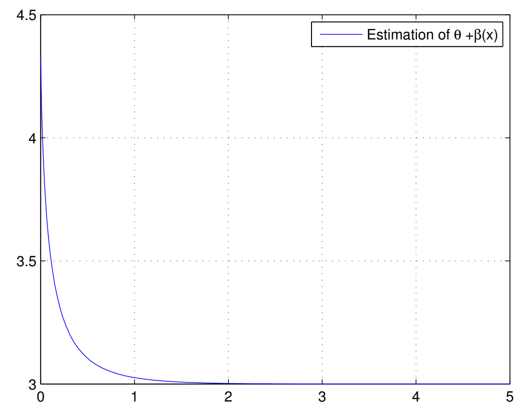

Fig. 2. Estimated parameter

which satisfies the conditions of Fact 1, in closed–loop with the feedback linearizing controller

$$u = -x - \Phi(x, \hat{\theta} + \beta(x)).$$

From (23) we obtain

$$\beta(x) = x_1 + \frac{x_1^3}{3} + x_2 + 1,$$

while the indirect I&I estimator

$$\begin{aligned} \dot{\hat{\theta}} &= [\frac{\partial \beta}{\partial x}(x)]^\top x \\ &= x_1^3 + x_1 + x_2 \end{aligned}$$

is obtained from (19).

The unknown parameter was taken as $\theta = 3$, with the initial conditions $x(0) = [1,2]^\top$, $\hat{\theta}(0) = 0$. Plots in Figure 1 show the good performance of the response of the system states. The estimated parameter $\hat{\theta} + \beta(x)$ is given in Figure 2, which illustrates that $\hat{\theta} + \beta(x)$ converges to the true value of θ.

We wrap up this example with the following interesting observation. Given that in this example $f_\star(x) = -x$, the direct I&I estimator (14) also takes the form above, and both adaptive controllers coincide. Moreover, even though

$$\mathscr{Q}_{\mathtt{D}}(\theta) = \mathscr{Q}_{\mathtt{I}}(\theta) + [\frac{\partial \beta}{\partial x}(x)]^\top x,$$

the monotonicity properties of both functions, obviously, coincide—hence both stability analysis apply.

B. Single reaction systems with Monod's growth laws

Single reaction systems can be described by models of the form (1) with

$$\Phi(x, \theta) = N\eta(x, \theta), \tag{24}$$

where $N \in \mathbb{R}^n$ is the stoichiometric vector and $\eta : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}_{>0}$ is the reaction kinetics—see [28] for additional details on the model and the various control problem formulations. A classical model for the reaction kinetics is Monod's growth law that is given by

$$\eta(x, \theta) = \frac{\lambda_0(x)}{\lambda^\top(x)\theta} \tag{25}$$

where $\lambda_0 : \mathbb{R}^n \to \mathbb{R}_{>0}$ and $\lambda : \mathbb{R}^n \to \mathbb{R}^q_{>0}$. The proposition below characterizes, in terms of a linear PDE, a set of functions $\beta(x)$ to satisfy Assumption 4.

Proposition 4: Consider the function (24) with (25). All solutions $\beta(x)$ of the linear PDE

$$[\frac{\partial \beta}{\partial x}(x)]^\top N = -\lambda(x) \tag{26}$$

satisfy Assumption 4 with $P = 1$.

Consider as an example the classical baker's yeast fed–batch fermentation process studied in [19]. In this case $q = 2, n = 3$

$$N = \begin{bmatrix} 1 \\ -c_1 \\ c_2 \end{bmatrix}, \quad \eta(x, \theta) = \frac{x_2}{1 + x_3^2} \frac{1}{\theta_1 + \theta_2 x_2},$$

with $c_1, c_2 > 0$. Hence

$$\lambda_0(x) = \frac{x_2}{1 + x_3^2}, \quad \lambda(x) = \begin{bmatrix} 1 \\ x_2 \end{bmatrix}.$$

The solutions of the PDE (26) are given by

$$\beta(x) = \begin{bmatrix} -x_1 \\ \frac{x_2^2}{2c_1} \end{bmatrix} + \rho(x_2 + c_1 x_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with $\rho : \mathbb{R} \to \mathbb{R}$ an arbitrary function.

C. Temperature control of exothermic CSTRs

Consider the exothermic CSTR studied in [28], see also [10], whose dynamics is described by

$$\begin{aligned} \dot{x}_1 &= \frac{1}{c_1}\psi(x, \theta) + c_2(x_1^{in} - x_1) \\ \dot{x}_2 &= -\frac{1}{c_1}\psi(x, \theta) - c_2 x_2 \\ \dot{x}_3 &= -c_3(x_3 - x_3^\star) - \psi(x, \theta) + u, \end{aligned} \tag{27}$$

where x_1, x_2 are concentrations and x_3 is the temperature, all of them restricted to the positive orthant, $x_1^{in}, c_1, c_2, c_3 > 0$,

$x_3^\star > 0$ is the desired value for x_3, u is the manipulated heat and $-\psi(x,\theta)$ is the reaction kinetics that is assumed verifies Arrhenius' law

$$-\psi(x,\theta) = c_1 x_1 e^{\theta_1} e^{-\frac{\theta_2}{x_3}}.$$

A re-parametrization, $c_4 = e^{\theta_1}$, of the so–called pre–exponential coefficient $c_4 > 0$ in the standard Arrhenius' law has been introduced, and the preliminary feedback suggested in [10] has been applied to the CSTR model.

The following proposition enables the implementation of a direct I&I adaptive controller for this system with control (9).

Proposition 5: Consider the system (27) in closed loop with the control (9), where

$$\beta_1(x) = \frac{\kappa}{2} x_3^2, \quad \beta_2(x) = -\kappa x_3,$$

with $\kappa > 0$. Assumptions 2 and 3, with $P = I$, are satisfied.

VI. Concluding Remarks and Future Work

A new framework to design I&I adaptive controllers for nonlinearly parameterized systems has been proposed. Instrumental for our developments is the *construction* of a $P-$monotone mapping via the selection of the free function $\beta(x)$, which is a degree of freedom that is available for I&I designs. In spite of the simplicity of the arguments—with the exception of [27], where it is used under different assumptions—this is the first time, to the best of our knowledge, that monotonicity has been exploited for adaptive control systems design.

The formulation of the problem is quite general and additional studies are required to define classes of systems for which the technique is effectively applicable. For systems with only one uncertain parameter and separable nonlinearities a rather good understanding of the problem is available. In particular, it is shown that it can be recast in terms of solutions of standard linear PDEs. Unfortunately, for more than one parameter, besides Lemma 1, the authors are not aware of suitable characterizations of monotone maps, rendering quite difficult the task of developing general results. However, it is clear from the examples, that on a case–by–case basis it is possible to exploit the particular structure of the system to verify this condition.

One shortcoming of the general I&I methodology is that the new degree of freedom provided by the design parameter $\beta(x)$ appears in the form of a gradient—see [4] for a detailed discussion—and the same stumbling block arises in the present work. This shortcoming has been ingeniously removed in [13] but, as discussed in Remark R7 of Section IV, the technique is not applicable for nonlinear parameterizations. Further research is needed to extend the ideas of [13] in this direction.

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