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Robustness Analysis on Constrained Model Predictive Control for Nonholonomic Vehicle Regulation

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Abstract— **A primary concern for nonlinear model predictive control (NMPC) strategies is the evaluation of their control performance, especially robustness. Many researchers show the existence of robustness as a byproduct of stability which is achieved by monotonicity of the cost function. However the design of a control architecture within the MPC frame and the analysis of its robustness to additive uncertainties are far from well solved together as a complete topic. The robust analysis is even more difficult when more than one control values from the optimal control sequence are applied to real systems. In this paper, a general stability condition is proposed to design a NMPC control strategy for a constrained discrete time system. Furthermore, a robustness analysis is also provided for the designed MPC control architecture. Under the proposed stability condition, an admissible invariant set for the nominal system and a terminal constraint set are defined for the MPC regulator. These compact sets make it possible to analyze the bound for additive uncertainties so that the closed-loop system is input-to-state stable with relation to additive uncertainties under this given bound.**

I. INTRODUCTION

Model predictive control or receding horizon control (RHC) is a kind of control algorithm suitable for the case in which pre-computation of a control law is not feasible. In this control strategy, at each sampling instant, the current control law is obtained by solving a finite horizon open-loop optimal control problem. An optimal control sequence is achieved and only the first value in this sequence is applied to the real system. With the current state as the initial state, this on-line optimal control problem will be solved repeatedly. As a process control paradigm, MPC is considered for solving regulation problem of the proposed vehicle dynamic system. This can be attributed to several factors. First, MPC technology considers the vehicle dynamics over a future time horizon and introduces feedforward control so that the measurable disturbances can be anticipated and compensated. Secondly, the vehicle kinematic model used here is a standard nonholonomic system. It could not be asymptotically stabilized by a time invariant feedback control law [1]. Discrete time nonlinear MPC strategy can solve this problem by designing a time-varying controller. Thirdly, MPC can consider input and state constraints in such a simple way by including them in the controller design when solving the optimal control problem. Finally, the improvement of computation speed and the mature of all kinds of numerical optimization algorithms make it possible to apply this control strategy to fast dynamic system like vehicles, robots and so on [2], [3].

Though with the above priorities, MPC have some practically relevant problems such as stability and robustness performance whenever plant-model mismatch, parametric uncertainties or additive disturbance inputs are considered. In MPC robustness analysis, special difficulties result from the open loop nature of the optimal control problem combined with the implicit feedback provided by MPC policy. To tackle the point at stake, extensive research work has been done and many of existing literatures resort to min-max formulations in order to derive an H_∞ MPC controller which accounts for the worst-case disturbance [4], [5]. It is worth to mention that LMI-based MPC is a hot spot since much of existing robust control theory could be recast in LMI framework. Reference [6] is a master piece in summarizing LMI related control problems. Much works appear later in this field such as [7], [8], and so on.

Actually, MPC robustness can be a byproduct of stability ensured by the cost monotonicity condition. That is to say, to satisfy monotonicity condition, MPC controller usually is designed as a conservative solution with some stability margin so that if the disturbance is not very large, stability can be preserved in a certain region. Reference [9] is such a good example where input-to-state stability (ISS) is achieved under uncertainties with a given bound. Nominal predictions are used and parameters of MPC controller are designed to achieve robust feasibility of the closed-loop system. Along the same line, reference [10] established regional ISS for constrained nonlinear systems and provided a good result for robustness analysis of MPC algorithms. A fairly complete discussion of nominal MPC could be found in [11], where design techniques are based on a nominal objective function and plant model uncertainties are compensated to some extent by state measurement feedback at every next sampling time.

A clear analysis on control performance could be found in reference [9] where several assumptions are made to assure the existence of robustness coming from stability margin. However, it doesn't tell us how to design a control strategy so that these assumptions can be satisfied. That is to say, the results are derived assuming monotonicity condition has been satisfied. Our work in this paper focuses on robust analysis for a control architecture we designed according to a general stability condition which we proposed for discrete time nonlinear systems. Based on [9], our research could find the bound of uncertainties under which the optimal problem is feasible. Based on the ISS theorem,

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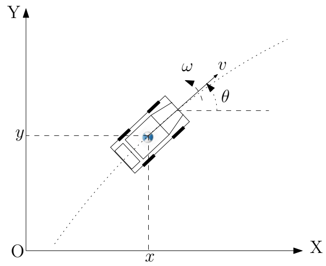

Fig. 1. Coordinate frames for the kinematic model

robustness of the proposed controller is analyzed and stability of disturbed closed-loop system is proved with respect to additive uncertainties. The results hold not only for a general nonlinear regulator but also for the cases where more than one control values from the optimal control sequence are applied to real systems. Since the computation speed of NMPC is always a hindrance for its application, undoubtedly it is meaningful to explore the control performance when increased the length of the applied control inputs in every iteration.

The remainder of this paper is organized as follows. In Section II, the proposed control architecture and terminal controller are introduced and stability condition is given. Section III presents some preliminary results which are useful for robust analysis. In Section IV, input-to-state stability of the closed-loop system is analyzed with consideration of additive uncertainties in the model. Simulation results are provided in Section V and conclusions are drawn in Section VI.

II. PROPOSED CONTROL ARCHITECTURE

A. Problem Formulation

The kinematic car-like vehicle model (1) is used in this paper as shown in Fig. 1. The state is represented by $\chi = [x, y, \theta]' \in \mathcal{C} = \mathbb{R}^2 \times \mathcal{S}$, where $\mathcal{C}$ denotes the configuration space including vehicle position and orientation. $\mathcal{S} := (-\pi, \pi]$. (x, y, θ) are the Cartesian coordinates of the vehicle and its orientation with respect to an inertial coordinate frame $\{O, X, Y\}$. $\mathrm{u} = [v\ \omega]'$ is the control input, i.e., the linear and the angular velocities, respectively.

$$\left\{ \begin{array}{rcl} \dot{x} & = & v\cos\theta \\ \dot{y} & = & v\sin\theta \\ \dot{\theta} & = & \omega \\ |\frac{v}{\omega}| & \geq & R_{min} \end{array} \right. \tag{1}$$

This model assumed that there is a pure rolling contact between the wheels and the ground. Then the vehicle moves without slipping on a plane, that is to say, the vehicle is subject to a nonholonomic constraint (2).

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0 \tag{2}$$

For this model, the closed-loop control relates to the determination of steering inputs assuring the states of the plant asymptotically converge to the origin (parking target). According to the well known work of [1], Cartesian state space representations of car model is among a class of systems which could not be asymptotically stabilized by a time invariant feedback control law. In addition, the input v and ω have saturations, $\underline{\mathrm{u}}$ and $\overline{\mathrm{u}}$, resulted from physical limitations of the actuators. The minimum turning radius of a vehicle determined the relation between v and ω, that is,

$$|\frac{v}{\omega}| \geq R_{min}. \tag{3}$$

Without loss of generality, we can consider autonomous parking as a regulation problem in which the desired operating point is the origin, that is, reference state $\chi_r = 0$. Since autonomous vehicle is usually driven by control signals from computer, it is necessary to discretize the system model. Considering a step size T, using Euler's approximation, error state system is defined in (4)

$$\left\{ \begin{array}{rcl} x_e(k+1) & = & x_e(k) - v(k)\cos\theta_e(k)T \\ y_e(k+1) & = & y_e(k) - v(k)\sin\theta_e(k)T \\ \theta_e(k+1) & = & \theta_e(k) - \omega(k)T \end{array} \right. \tag{4}$$

where $\chi_e(k) = [x_e(k), y_e(k), \theta_e(k)]' = \chi_r(k) - \chi(k) = -\chi(k)$. Or denote it as (5)

$$\begin{array}{rcl} \chi_e(k+1) & = & f(\chi_e(k), \mathrm{u}(k)) \\ \chi_e(k) & \in & \mathcal{X} \\ \mathrm{u}(k) & \in & U \end{array} \tag{5}$$

where $\mathcal{X}$ is a closed set and U is a compact set. Then, the regulation ("parking") problem can be reformulated as finding a control sequence $\mathrm{u}(k)$ so that the current state $\chi_e(k)$ will converge to the origin when $k \to \infty$. The cost function we used in this paper is as follows (6).

$$\begin{array}{l} J(k, \mathrm{u}^M) = \chi_e(k+M)'\chi_e(k+M) + \\ \sum\limits_{i=0}^{M-1} [\chi_e(k+i)'Q\chi_e(k+i) + \mathrm{u}(k+i)'R\mathrm{u}(k+i)] \end{array} \tag{6}$$

Subject to:

$$\begin{array}{rcl} \chi_e(k+i) & = & f(\chi_e(k+i-1), \mathrm{u}(k+i-1)) \\ & & for\ i = 0, \cdots, M \\ \underline{\mathrm{u}} & \leq & \mathrm{u}(k+i) \ \leq\ \overline{\mathrm{u}} \\ & & for\ i = 0, \cdots, M-1 \\ \chi_e(k+M) & \in & \Omega \end{array} \tag{7}$$

where $\chi_e(k+M)'\chi_e(k+M)$ is the terminal penalty term. Q and R are positive definite weighting matrices. Denote M as the control horizon and P as the prediction horizon. $M \leq P$. $\mathrm{u}_{k+i} = 0$ when $i = M, M+1, \cdots, P$. Here, we choose $M = P$. $\mathrm{u}^M = (\mathrm{u}_k, \cdots, \mathrm{u}_{k+M-1})$ is the control sequence corresponding to each prediction horizon. The first m ($m \leq M$) inputs $(\mathrm{u}_k, \mathrm{u}_{k+1}, \cdots, \mathrm{u}_{k+m-1})$ will be applied to the plant at each time instant. And Ω is the terminal state constraint set.

B. Control Strategy Design and Stability Condition

The control structure is designed within finite prediction horizon MPC frame considering the speed requirement of real time systems. To clearly describe the predictive control process, the similar notation as that appeared in [12] is adopted here. The evolution of the system will be over time index of the form $t_k^j := t_0 + (j + km)T$, with j varying in

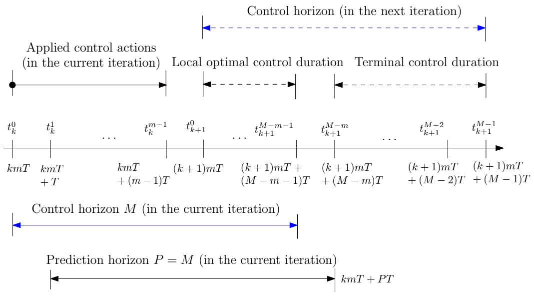

Fig. 2. Time index for optimal problem

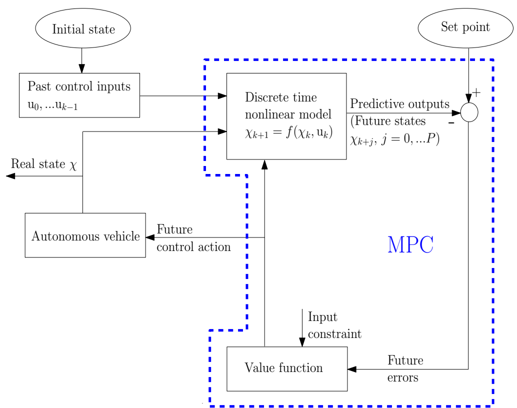

Fig. 3. Control architecture

the interval $j = 0, \ldots, M-1$, while k is kept constant at $k = 0, 1, 2 \ldots$. Here, we choose $t_0 = 0$ and the time index will be as follows.

$$\{\ldots, t_k^0, t_k^1, \ldots, t_k^m = t_{k+1}^0, t_{k+1}^1, \ldots, t_{k+1}^m = t_{k+2}^0, \ldots\} \quad \forall k \in \mathbb{Z}^+$$

As shown in Fig. 2, there are several sets of time duration in which the corresponding optimal problem will be solved. Using the iteration in Fig. 2 as an example, we can get an optimal control sequence $u_{k+1}^0, u_{k+1}^1, \ldots, u_{k+1}^{M-1}$. The first $M-m$ control inputs are called local optimal control, denoted by u_{op} and the rest m control inputs are called terminal control, denoted by u_T. The corresponding time index $t_{k+1}^0, \ldots, t_{k+1}^{M-m-1}$ and $t_{k+1}^{M-m}, \ldots, t_{k+1}^{M-1}$ are called local optimal control duration and terminal control duration respectively. Only the first m control inputs, $u_{k+1}^0, u_{k+1}^1, \ldots, u_{k+1}^{m-1}$ will be the future control action applied to vehicle steering system.

The control architecture is shown in Fig. 3 where the initial state is a current vehicle position and the set point is a goal parking position. Future control actions are decided by MPC module. Motivated by the continuous MPC for tracking control in [13], discrete time MPC is used here for a regulation problem. The main focus will be on control strategy configuration, terminal controller design, and parameters setup so that nonlinear MPC parking controller is stable. Besides, the generated trajectory should meet the minimum curvature requirement from a specific vehicle. The detail control process is as follows.

Step 1. Get the current error state $\chi_e(t_k^0)$.

Step 2. Solve the following optimal control problem on time index $t_k^0, t_k^1, \ldots, t_k^{M-1}$.

$$\begin{aligned} & \min_{\mathrm{u}^M} J(t_k^j, \mathrm{u}^M) \\ = \; & \chi_e(t_k^M)'\chi_e(t_k^M) + \sum_{j=0}^{M-1} [\chi_e(t_k^j)'Q\chi_e(t_k^j) \\ & + \mathrm{u}(t_k^j)'R\mathrm{u}(t_k^j)] \end{aligned} \tag{8}$$

Subject to:

$$\begin{aligned} \chi_e(t_k^{j+1}) &= f(\chi_e(t_k^j), \mathrm{u}(t_k^j)) \\ & \quad for\ j = 0, \cdots, M-1 \\ \underline{\mathrm{u}} &\le \mathrm{u}(t_k^j) \le \bar{\mathrm{u}} \\ \chi_e(t_k^M) &\in \Omega. \end{aligned} \tag{9}$$

Get the optimal control sequence $\hat{\mathrm{u}} = (\mathrm{u}_k^0, \mathrm{u}_k^1, \ldots, \mathrm{u}_k^{M-1})$ and apply $\mathrm{u}_k^0, \mathrm{u}_k^1, \ldots, \mathrm{u}_k^{m-1}$ to the error state system.

Step 3. Use $\chi_e(t_{k+1}^0)$ as the initial state and the first $M-m$ local optimal control inputs $\mathrm{u}_{k+1}^0, \mathrm{u}_{k+1}^1, \ldots, \mathrm{u}_{k+1}^{M-m-1}$, which are equal to $\mathrm{u}_k^m, \mathrm{u}_k^{m+1}, \ldots, \mathrm{u}_k^{M-1}$ in Step 2, and m terminal control inputs together as (10) will be the new initial solution for the optimal control problem.

$$\tilde{\mathrm{u}} = (\overbrace{\mathrm{u}_{k+1}^0, \ldots, \mathrm{u}_{k+1}^{M-m-1}}^{\text{local optimal control}}, \overbrace{\mathrm{u}_{k+1}^{M-m}, \ldots, \mathrm{u}_{k+1}^{M-1}}^{\text{terminal control}}) \tag{10}$$

where

$$\begin{aligned} \mathrm{u}_{\mathrm{k+1}}^{\mathrm{j}} &= [v_{k+1}^j \quad \omega_{k+1}^j]' \\ &= [\eta\sqrt{(x_e(t_{k+1}^j))^2 + (y_e(t_{k+1}^j))^2} \quad \xi\theta_e(t_{k+1}^j)]' \\ & \quad for\ j = M-m, \ldots, M-1. \end{aligned} \tag{11}$$

and η, ξ are output feedback gains. This choice of the initial solution is a necessary condition for achieving NMPC stability. The detail proof could be found in [14].

Step 4. Solve (8) and (9) once more. The first m control inputs among the solution sequence will be applied to error state system.

Step 5. $k+1 \rightarrow k$ and continue this procedure until the parking error is small enough.

As shown in (9), actuator saturations and minimum curvature requirements are considered as constraints in the optimal problem. Denote $L = \chi_e'Q\chi_e + \mathrm{u}'R\mathrm{u}$ and $\tilde{L} = \tilde{\chi}_e'Q\tilde{\chi}_e + \tilde{\mathrm{u}}'R\tilde{\mathrm{u}}$ where $\tilde{\chi}_e$ be the error state by applying the control sequence $\tilde{\mathrm{u}}$. To achieve stability of the proposed nonlinear MPC architecture, the condition in Theorem 2.1 should be satisfied and it also serves as the criteria for setting up all design parameters of this controller.

Theorem 2.1: (Stability) The system (4) is asymptotic stable, if terminal control inputs exist so that the following relation holds:

$$\begin{aligned} & \sum_{j=0}^{m-1} [\tilde{\chi}_e(t_k^M + (j+1)T)'\tilde{\chi}_e(t_k^M + (j+1)T) \\ & -\tilde{\chi}_e(t_k^M + jT)'\tilde{\chi}_e(t_k^M + jT) + \tilde{L}(t_k^M + jT)] \le 0. \end{aligned}$$

Remark 2.1: The stability condition has requirements on each term within terminal control duration in $\tilde{\mathrm{u}}$ which is defined in (10). That is to say, the stability condition can be turned into terminal control constraints.

Remark 2.2: The weight matrices Q and R are both diagonal with Q=diag$\{q_{11}, q_{22}, q_{33}\}$ and R=diag$\{r_{11}, r_{22}\}$. After substituting (4) in the stability condition, the following terminal control constraints can be derived.

$$\begin{aligned}(\tfrac{\tilde{x}_e}{\sqrt{\tilde{x}_e^2+\tilde{y}_e^2}}\cos\tilde{\theta}_e + \tfrac{\tilde{y}_e}{\sqrt{\tilde{x}_e^2+\tilde{y}_e^2}}\sin\tilde{\theta}_e) \\ \geq \tfrac{(\eta^2T^2+\max\{q_{11},q_{22}\}+r_{11}\eta^2)}{2\eta T}\end{aligned} \tag{12}$$

$$2\xi T \geq \xi^2 T^2 + q_{33} + r_{22}\xi^2 \tag{13}$$

III. PRELIMINARY ANALYSIS

With predictions from nominal system, the designed MPC architecture did not consider any uncertainties or external disturbances such as model mismatch, measurement noise. Stability might be lost under these conditions. Here, the NMPC stability is achieved by enforcing a quasi-monotonicity condition, that is, choose a proper terminal cost and terminal constraints so as to achieve the monotonicity of cost function. There is some margin for satisfying stability requirements. Consider an uncertain nonlinear discrete time system given by

$$\chi_{k+1} = f(\chi_k, \mathrm{u}_k) + w_k \tag{14}$$

where $w_k \in \mathcal{C} = \mathbb{R}^2 \times S$ is the additive uncertainty and it belongs to a compact set $\mathcal{W}$ that contains the origin. Consider that the admissible set of uncertainties $\mathcal{W}$ is bounded by γ in s-norm. If γ is not too large compared to the margin, stability can be preserved in a suitable region for uncertain systems. Therefore, the analysis of robust stability for the proposed MPC is a feasible and interesting problem. The following observations exist for the designed control architecture. For simplicity, we use χ_1, χ_2 to denote error states at any two different instants and omit the time index and only use k to denote any instant if there is no confusion.

Lemma 3.1: The nominal model (5) is such that origin is a steady state and $f(\chi_e(k), \mathrm{u}_k)$ is locally Lipschitz with a Lipschitz constant $0 < L_f < \infty$ so that

$$\begin{array}{rcl}\|f(\chi_1,\mathrm{u}) - f(\chi_2,\mathrm{u})\|_s & \leq & L_f\|\chi_1-\chi_2\|_s \\ \forall \chi_1,\chi_2 \in \mathcal{X} & and & \mathrm{u} \in U\end{array}$$

where s-norm could be any norm and it will influence the value of Lipschitz constant.

The proof for Lemma 3.1 is obvious.

Lemma 3.2: $L(\chi_e(k), \mathrm{u}_k)$ is such that $L(0, 0) = 0$ and there exist a constant $a > 0$ and $\sigma \geq 1$ so that

$$L(\chi_e(k), \mathrm{u}_k) \geq a \cdot \|(\chi_e(k), \mathrm{u}_k)\|_s^\sigma.$$

$L(\chi_e(k), \mathrm{u}_k)$ is Lipschitz with a Lipschitz constant $0 < L_c < \infty$ so that

$$|L(\chi_1,\mathrm{u}) - L(\chi_2,\mathrm{u})| \leq L_c\|\chi_1-\chi_2\|_s$$

$$\forall \chi_1,\chi_2 \in \mathcal{X} \;\; and \;\; \mathrm{u} \in U.$$

Proof: First, $L(\chi_e(k), \mathrm{u}_k) = \chi_e(k)'Q\chi_e(k) + \mathrm{u}_k'R\mathrm{u}_k$. Obviously, $L(0, 0) = 0$. Using ∞-norm, we have

$$\begin{aligned}L(\chi_e(k), \mathrm{u}_k) &= (q_{11}x^2 + q_{22}y^2 + q_{33}\theta^2 + r_{11}v^2 + r_{22}\omega^2) \\ &\geq \min\{q_{11}, q_{22}, q_{33}, r_{11}, r_{22}\}\|(\chi_e(k), \mathrm{u}_k)\|_\infty^2\end{aligned}$$

Secondly, for $|L(\chi_1, \mathrm{u}) - L(\chi_2, \mathrm{u})|$, we have

$$\begin{array}{rl} & |L(\chi_1,\mathrm{u}) - L(\chi_2,\mathrm{u})| \\ = & |\chi_1'Q\chi_1 - \chi_2'Q\chi_2| \\ = & |\chi_1'Q\chi_1 - \chi_1'Q\chi_2 + \chi_1'Q\chi_2 - \chi_2'Q\chi_2| \\ \leq & |\chi_1'Q(\chi_1-\chi_2)| + |(\chi_1'-\chi_2')Q\chi_2| \\ \leq & \|\chi_1\|_s\lambda_Q^M\|\chi_1-\chi_2\|_s + \|\chi_1-\chi_2\|_s\lambda_Q^M\|\chi_2\|_s \\ \leq & (\|\chi_1\|_s + \|\chi_2\|_s)\lambda_Q^M\|\chi_1-\chi_2\|_s \end{array}$$

where λ_Q^M is the maximum eigenvalue of matrix Q. Also, since the algorithm is asymptotically stable [14], by definition, there exists an upper bound $\mathcal{B} = \mathcal{B}(\chi_0, \gamma)$ for $\chi(t)$ such that $\|\chi(t)\|_s \leq \mathcal{B}$ for all $t \geq 0$. Here, $\mathcal{B}$ is a positive number and its value depends on initial condition χ_0 and the bound of uncertainties γ. Thus, we have

$$|L(\chi_1,\mathrm{u}) - L(\chi_2,\mathrm{u})| \leq 2\mathcal{B}\lambda_Q^M\|\chi_1-\chi_2\|_s = L_c\|\chi_1-\chi_2\|_s.$$

■

Lemma 3.3: Denote $V_T = \chi_e'\chi_e$, the terminal control duration as u_T. There exists a set Φ which is defined as $\Phi = \{\chi_e \in \mathcal{X} : V_T(\chi_e) \leq \alpha\}$ satisfying $\Phi \subseteq \mathcal{A} = \{\chi_e \in \mathcal{X} : \mathrm{u}_T(\chi_e) \in U\}$. And

$$\left\{\begin{array}{rcl} V_T(f(\chi_e(k), \mathrm{u}_T(\chi_e(k)))) & - & V_T(\chi_e(k)) \\ & \leq & -L(\chi_e(k), \mathrm{u}_T(\chi_e(k))) \\ \forall \chi_e(k) & \in & \Phi \end{array}\right. \tag{15}$$

where k denotes any time instant of $t_k^M + (j + 1)T$, for $j = 0, \ldots, m - 1$.

$$\left\{\begin{array}{rcl} |V_T(\chi_1) - V_T(\chi_2)| & \leq & L_v\|\chi_1-\chi_2\|_s \\ \forall \chi_1,\chi_2 & \in & \Phi \end{array}\right. \tag{16}$$

Proof: The local controller (11) in the designed MPC frame is used as terminal control inputs in each iteration. And stability condition in Theorem 2.1 is also the requirement on terminal control inputs. Then, for each item from the terminal control duration u_T, we have

$$\begin{array}{rcl}\mathrm{u}_T(k) & = & [v(k) \quad \omega(k)]' \\ & = & [\eta\sqrt{(x_e((k)))^2 + (y_e(k))^2} \quad \xi\theta_e(k)]'\end{array} \tag{17}$$

and

$$\chi_e(k+1)'\chi_e(k+1) - \chi_e(k)'\chi_e(k) + L(k) \leq 0. \tag{18}$$

Recall that $\underline{\mathrm{u}} \leq \mathrm{u}(k) \leq \overline{\mathrm{u}}$ and in our case $|\underline{\mathrm{u}}| = |\overline{\mathrm{u}}|$. Choose

$$\alpha = \min\{\frac{\overline{v}^2}{\eta^2}, \frac{\overline{\omega}^2}{\xi^2}\},$$

then we have

$$V_T(\chi_e) = x_e^2 + y_e^2 + \theta_e^2 \leq \alpha.$$

Thus, again for each item in terminal control duration u_T, omitting time index, we have

$$\begin{array}{rcl}\mathrm{u}_T(k) & = & h(\chi_e(k)) = [v(k) \quad \omega(k)]' \\ & = & [\eta\sqrt{x_e(k)^2 + y_e(k)^2} \quad \xi\theta_e(k)]',\end{array}$$

$$|v(k)| = |\eta\sqrt{x_e(k)^2 + y_e(k)^2}| \leq |\eta\sqrt{\min\{\frac{\overline{v}^2}{\eta^2}, \frac{\overline{\omega}^2}{\xi^2}\}}| \leq |\overline{v}|,$$

$$|\omega(k)| = |\xi\theta_e(k)| = \sqrt{\xi^2\theta_e(k)^2} \le |\xi\sqrt{\min\{\frac{\overline{v}^2}{\eta^2}, \frac{\overline{\omega}^2}{\xi^2}\}}| \le |\overline{\omega}|.$$

Thus, $\Phi \subseteq \mathcal{A} = \{\chi_e \in \mathcal{X} : \mathrm{u}_T(k) \in U\}$ and (15) is an immediate consequence of (18). The proof for (16) is the same as the second step in Lemma 3.2. ■

Lemma 3.4: The terminal constraint set Ω is given by $\Omega = \{\chi_e \in \mathcal{X} : V_T(\chi_e) \le \alpha_v\}$ so that $\forall \chi_e \in \Phi, f(\chi_e, \mathrm{u}_T(\chi_e)) \in \Omega$.

Proof: For system (4), we have the following relations.

$$\begin{aligned} & V_T(f(\chi_e, \mathrm{u}_T(\chi_e))) \\ = \; & (x_e - \eta T \cos\theta_e \sqrt{x_e^2 + y_e^2})^2 \\ & + (y_e - \eta T \sin\theta_e \sqrt{x_e^2 + y_e^2})^2 + (\theta_e - \xi\theta_e T)^2 \\ = \; & x_e^2 + y_e^2 + \theta_e^2 + T^2\eta^2(x_e^2 + y_e^2) + T^2\xi^2\theta_e^2 - 2\xi\theta_e^2 T \\ & -2T\eta x_e \cos\theta_e \sqrt{x_e^2 + y_e^2} - 2T\eta y_e \sin\theta_e \sqrt{x_e^2 + y_e^2} \end{aligned}$$

From terminal constraints (12), we have

$$\begin{aligned} & 2Tx_e \cos\theta_e \sqrt{x_e^2 + y_e^2} + 2Ty_e \sin\theta_e \sqrt{x_e^2 + y_e^2} \\ \ge \; & (\eta T^2 + q_{11}/\eta + r_{11}\eta)(x_e^2 + y_e^2). \end{aligned}$$

Thus,

$$\begin{aligned} & V_T(f(\chi_e, \mathrm{u}_T(\chi_e))) \\ \le \; & (1 + T^2 \max\{\eta^2, \xi^2\})(x_e^2 + y_e^2 + \theta_e^2) \\ & -(\eta T^2 + q_{11}/\eta + r_{11}\eta)(x_e^2 + y_e^2) - 2\xi T\theta_e^2 \\ \le \; & (1 + T^2 \max\{\eta^2, \xi^2\} \\ & - \max\{(\eta T^2 + q_{11}/\eta + r_{11}\eta), 2\xi T\})(x_e^2 + y_e^2 + \theta_e^2) \end{aligned}$$

where

$$\begin{aligned} c \; = \; & (1 + T^2 \max\{\eta^2, \xi^2\} \\ & - \max\{(\eta T^2 + q_{11}/\eta + r_{11}\eta), 2\xi T\}) < 1. \end{aligned} \quad (19)$$

Choose $\alpha_v = c\alpha$, then $\forall \chi_e \in \Phi, f(\chi_e, \mathrm{u}_T(\chi_e)) \in \Omega$. This condition ensures at the end of each iteration the states enter the terminal region set. ■

Corollary 3.5: Since Lemma 3.1 ∼ 3.4 exist for system 5, by Theorem 1 in [9], we can define a set of states of the system, D_N, where the MPC optimization problem is feasible. And the closed loop is stable in D_N if the uncertainties are bounded by:

$$\gamma \le \frac{\alpha - \alpha_v}{L_v - L_f^{M-1}}.$$

Corollary 4.2 provides a theoretical estimation of disturbance bound under which stability can be achieved. In the following section, we will proof of ISS for MPC algorithm under this bounded uncertainties.

IV. ROBUST MPC BASED ON INPUT-TO-STATE STABILITY THEOREM

Consider the cost function (6), we have

Lemma 4.1: The cost function in the NMPC problem is an ISS-Lyapunov candidate.

Proof: Define V and $\tilde{V}$ as the cost function for applying MPC controller and $\tilde{\mathrm{u}}$ respectively. Denote u^* as the optimal control sequence from each iteration and $\hat{\chi}$ as the predicted nominal state of the system by applying the whole solution sequence from Step 4. m is the number of applied control inputs which is solved from the NMPC problem. Then we have

$$\begin{aligned} & V(t_k^m, \chi(t_k^m)) - V(t_k^0, \chi(t_k^0)) \\ \le \; & \tilde{V}(t_k^m, \tilde{\chi}(t_k^m)) - V(t_k^0, \chi(t_k^0)) \\ = \; & V_T(\tilde{\chi}(t_k^{m+M})) - V_T(\hat{\chi}(t_k^M)) \\ & + \sum_{i=0}^{M-1} \{L(\tilde{\chi}(t_k^{m+i}), \tilde{\mathrm{u}}(t_k^{m+i})) - L(\hat{\chi}(t_k^i), \mathrm{u}^*(t_k^i))\} \\ = \; & \sum_{i=0}^{M-m-1} \{L(\tilde{\chi}(t_k^{m+i}), \tilde{\mathrm{u}}(t_k^{m+i})) \\ & -L(\hat{\chi}(t_k^{m+i}), \mathrm{u}^*(t_k^{m+i}))\} \\ & +V_T(\tilde{\chi}(t_k^{m+M})) - V_T(\tilde{\chi}(t_k^{m+M-1})) \\ & +V_T(\tilde{\chi}(t_k^{m+M-1})) - V_T(\tilde{\chi}(t_k^{m+M-2})) + \ldots \\ & +V_T(\tilde{\chi}(t_k^M)) - V_T(\hat{\chi}(t_k^M)) \\ & +L(\tilde{\chi}(t_k^{m+M-1}), \tilde{\mathrm{u}}(t_k^{m+M-1})) \\ & +L(\tilde{\chi}(t_k^{m+M-2}), \tilde{\mathrm{u}}(t_k^{m+M-2})) + \ldots \\ & +L(\tilde{\chi}(t_k^M), \tilde{\mathrm{u}}(t_k^M)) \\ & -L(\chi(t_k^0), \mathrm{u}^{MPC}(t_k^0)) - L(\chi(t_k^1), \mathrm{u}^{MPC}(t_k^1)) - \ldots \\ & -L(\chi(t_k^{m-1}), \mathrm{u}^{MPC}(t_k^{m-1})). \end{aligned}$$

From Lemma 3.3, we have

$$\begin{aligned} \sum_{i=1}^{m}\{V_T(\tilde{\chi}(t_k^{M+i})) \; & - \; V_T(\tilde{\chi}(t_k^{M+i-1})) \\ & + \; L(\tilde{\chi}(t_k^{M+i-1}), \tilde{\mathrm{u}}(t_k^{M+i-1}))\} \le 0 \\ V_T(\tilde{\chi}(t_k^M)) \; & - \; V_T(\hat{\chi}(t_k^M)) \le \gamma L_v L_f^{M-1}. \end{aligned}$$

From Lemma 3.2, we have

$$\begin{aligned} \sum_{i=0}^{M-m-1}\{L(\tilde{\chi}(t_k^{m+i}), \tilde{\mathrm{u}}(t_k^{m+i})) \; & - \; L(\hat{\chi}(t_k^{m+i}), \mathrm{u}^*(t_k^{m+i}))\} \\ & \le \; \gamma L_c \tfrac{L_f^{M-m}-1}{L_f-1} \\ L(\chi(t_k^0), \mathrm{u}^{MPC}(t_k^0)) \; & \ge \; a\|\chi(t_k^{m-1})\|_s^\sigma. \end{aligned}$$

Thus,

$$\begin{aligned} & V(t_k^m, \chi(t_k^m)) - V(t_k^0, \chi(t_k^0)) \\ \le \; & (L_c \tfrac{L_f^{M-m}-1}{L_f-1} + L_v L_f^{M-1})\gamma - a\|\chi(t_k^{m-1})\|_s^\sigma. \end{aligned}$$

From the definition 3.2 in [15], V is an ISS-Lyapunov candidate. ■

Corollary 4.2: From the input-to-state stability theorem for discrete time nonlinear systems in [15], the system (5) is ISS with the proposed MPC control architecture.

TABLE I

PARAMETER VALUES

Parameters	$T(s)$	q_{11}	q_{22}	q_{33}	r_{11}	r_{22}	η	ξ
Value	0.2	0.1	0.1	0.1	0.1	0.1	1	1

V. SIMULATION RESULTS

In this section, simulation results are given to illustrate the performance of NMPC algorithm for parking maneuver under bounded additive uncertainties. To satisfy conditions (12), (13) and (19), the parameters could be chosen as in Table I as one example. In Fig. 4 the 6 different initial postures (including position and orientation) are chosen as follows.

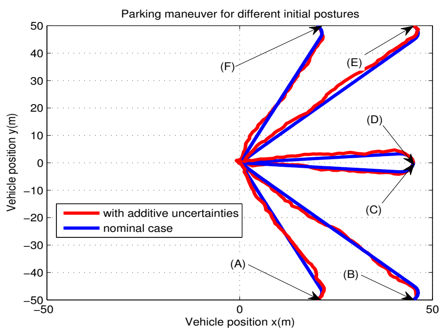

Fig. 4. Parking maneuver from different initial postures

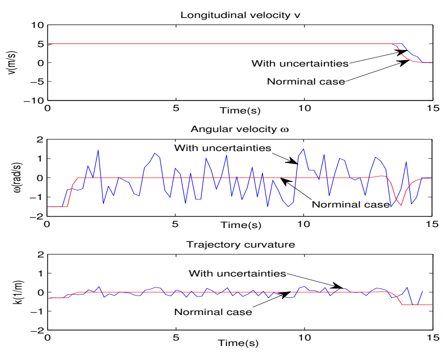

Fig. 5. Control variables and trajectory curvature

$(A)[20, -50, \pi/4]$ $(B)[45, -50, \pi/4]$ $(C)[45, 0, -\pi/2]$
$(D)[45, 0, \pi/2]$ $(E)[45, 50, -\pi/4]$ $(F)[20, 50, -\pi/4]$

For all these initial configurations, under additive uncertainties of bound γ=0.2 m, the vehicle could arrive the final parking configuration, (0, 0, π or -π), via a reasonable path. The bound for linear velocity v is set as -5 and 5 m/s and angular velocity ω is between -1.5 and 1.5 rad/s. The simulation is ended whenever parking error is small enough. As shown in Fig. 5, the simulation results from initial posture (E) for both v and ω are within the required range even under the uncertainties. Besides, the curvature of the trajectory is also bounded since a vehicle has minimum turning radius (1.5 m in our case). By adding the proposed constraints in our algorithm, the curvature k of the trajectory is small enough. In Fig. 6, the simulation is running for a longer time

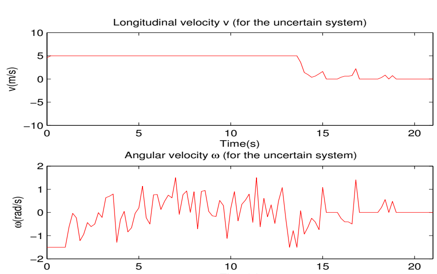

Fig. 6. Control variables for the uncertain system

than necessary to show the stability of the proposed NMPC strategy.

It is worth mentioning that the estimation of bound for additive uncertainties, $\frac{\alpha-\alpha_v}{L_v-L_f^{M-1}}$, depends on α, α_v, L_v and L_f. Therefore, how to set these parameters has influence on the estimation of this bound.

VI. CONCLUSIONS

In this paper, a model predictive control strategy is proposed for solving nonholonomic vehicle regulation problem. Based on nonlinear discrete time system model, a stable terminal controller is designed for each prediction horizon. The stability condition provides not only a guidance for parameter setup but also a margin for rejecting additive uncertainties. ISS of the proposed MPC module is assured by combining terminal controller design and terminal region constraints. Based on the ISS theorem and MPC feasibility theorem for discrete time system, robustness of MPC regulator with respect to bounded additive uncertainties is analyzed so that property of the given algorithm becomes more clear.

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