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A New State Observer and Flight Control of Highly Maneuverable Aircraft

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Abstract— **In this paper, a new nonlinear observer (θ - D observer) is proposed to estimate the feedback states for optimal control of a highly maneuverable aircraft. This observer is derived by constructing the dual of a recently developed nonlinear optimal control technique-known as the θ - D technique. The θ - D optimal control approach provides an approximate closed-form solution to the Hamilton-Jacobi-Bellman (HJB) equation. An optimal flight controller using this technique is designed for a highly maneuverable aircraft operating at high angle of attack where the θ - D observer is employed to estimate the states for feedback. The structure of this observer is similar to the State Dependent Riccati Equation Filter (SDREF). However, the new method provides a closed-form observer gain and does not need time-consuming online computations of the algebraic Riccati equation at each instant as the SDREF. The theoretical results about this new observer are given. The simulation shows that the θ - D control and observer exhibit excellent performance for this flight control problem.**

I. Introduction

Observer design for nonlinear systems has been an active field of research over the last few decades. Two primary approaches and their variations have been extensively studied. One well-established method is the exact error linearization [1-3] that is based on a nonlinear state transformation by which the error dynamics of state is linear so that the design of state observer can be performed using linear techniques. However, its integrability condition limits its possible applications and the need to solve a system of partial differential equations complicates the design process. The second approach involves high-gain techniques [4-7]. The system in consideration possesses a structure with a linear and nonlinear part. A linear observer is designed such that the linear part dominates the nonlinearity. Lipschitz condition is assumed to hold for the nonlinear part and the Lipschitz constant is needed to determine the eigenvalues of the error dynamics for stability.

Extended Kalman Filter [8] and its variations are widely used as well in stochastic settings. Since linearization about the current estimate is used to compute observer gain, these methods cannot guarantee the performance for highly nonlinear systems. A relatively new estimation approach called State Dependent Riccati Equation Filter (SDREF) [9] has received much attention during the last decade. SDREF does not include any linearization like the EKF but needs iterative on-line solution to the algebraic Riccati equation at each instant, a process that is computationally intensive. Also stability of the SDREF has not been well established.

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In this paper, a new nonlinear observer, called the $\theta - D$ observer is proposed as state estimation of nonlinear systems. The observer is derived from duality of a recently developed nonlinear optimal control method called the $\theta - D$ technique [10]. This observer takes the same structure as the SDREF. However, the $\theta - D$ observer provides a closed-form observer gain. In this work, the $\theta - D$ technique is used to design a closed-form optimal flight control law for a highly maneuverable aircraft with nonlinear dynamics and the $\theta - D$ observer is utilized to estimate the aircraft states for feedback.

Since the new observer is derived from the $\theta - D$ control technique, this nonlinear control method is reviewed in Section II. The development of the $\theta - D$ observer and its associated theoretical results are presented in Section III. Applications to the flight control and simulation results are shown in Section IV. Some concluding remarks are given in Section V.

II. Review of θ - D Control Technique

The θ - D nonlinear control technique addresses the class of nonlinear time-invariant systems described by

$$\dot{x} = f(x) + Bu \tag{1}$$

with the cost functional: $J = \frac{1}{2}\int_0^\infty [x^T Qx + u^T Ru]dt$ (2)

where $x \in \Omega \subset R^n, f:\Omega \to R^n, B \in R^{n\times m}, u:\Omega \to R^m, Q \in R^{n\times n}, R \in R^{m\times m}$; Q is a positive semi-definite matrix and R is a positive definite matrix; B is a constant matrix and $f(0)=0$; Ω is a compact subset in R^n; Assume that $f(x)$ is continuously differentiable and zero state observable through Q.

The optimal solution to this infinite-horizon nonlinear regulator problem can be obtained by solving the Hamilton-Jacobi-Bellman (HJB) partial differential equation [11]:

$$\frac{\partial V^T}{\partial x} f(x) - \frac{1}{2}\frac{\partial V^T}{\partial x} BR^{-1}B^T \frac{\partial V}{\partial x} + \frac{1}{2}x^T Qx = 0 \tag{3}$$

where $V(x)$ is the optimal cost , i.e.

$$V(x) = \min_u \frac{1}{2}\int_0^\infty (x^T Qx + u^T Ru)dt \tag{4}$$

Optimal control is obtained from the necessary condition as

$$u = -R^{-1}B^T \frac{\partial V}{\partial x} \tag{5}$$

The $\theta - D$ control technique provides an approximate solution to the above HJB equation such that a suboptimal closed-form feedback controller can be obtained.

The $\theta - D$ control method can be summarized by the following procedure [10].

Write the original nonlinear state equation as:

$$\dot{x} = f(x) + Bu = F(x)x + Bu = \left[A_0 + \theta \frac{A(x)}{\theta} \right] x + Bu \tag{6}$$

where A_0 is a constant matrix such that (A_0, B) is a stabilizable pair and $[F(x), B]$ is pointwise controllable.

Add a perturbation series $\sum_{i=1}^{\infty} D_i \theta^i$ to the cost function (2),

$$J = \frac{1}{2}\int_0^{\infty} \left[x^T \left(Q + \sum_{i=1}^{\infty} D_i \theta^i \right) x + u^T R u \right] dt \tag{7}$$

where the usage of this series will be elaborated afterward. Assuming a power series expansion $\frac{\partial V}{\partial x} = \sum_{i=0}^{\infty} T_i(x,\theta)\theta^i x$ and solving the perturbed optimal control problem (6) and (7) through the HJB equation yield a suboptimal control

$$u = -R^{-1}B^T \sum_{i=0}^{\infty} T_i(x,\theta)\theta^i x \tag{8}$$

where $T_i(x,\theta)$ ($i = 0,\cdots,n,\cdots$) is a symmetric matrix and is solved recursively by the following algorithm.

$$T_0 A_0 + A_0^T T_0 - T_0 B R^{-1} B^T T_0 + Q = 0 \tag{9a}$$

$$T_1(A_0 - BR^{-1}B^T T_0) + (A_0^T - T_0 BR^{-1}B^T)T_1 = -\frac{T_0 A(x)}{\theta} - \frac{A^T(x)T_0}{\theta} - D_1 \tag{9b}$$

$$\vdots$$

$$T_n(A_0 - BR^{-1}B^T T_0) + (A_0^T - T_0 BR^{-1}B^T)T_n = -\frac{T_{n-1}A(x)}{\theta} - \frac{A^T(x)T_{n-1}}{\theta} + \sum_{j=1}^{n-1} T_j BR^{-1}B^T T_{n-j} - D_n \tag{9c}$$

Equation (9a) is an algebraic Riccati equation. The rest of equations are Lyapunov equations that are *linear* in terms of T_i ($i = 1,2,\cdots,n$). Since all the coefficients of T_i are the same constant matrices, i.e. $A_0 - BR^{-1}B^T T_0$ and $A_0^T - T_0 BR^{-1}B^T$, closed-form solution for $T_i(x,\theta)$ can be easily obtained by solving Eqs. (9a)-(9c) successively [10].

The D_i matrix is constructed in the form of:

$$D_1 = k_1 e^{-l_1 t} \left[-\frac{T_0 A(x)}{\theta} - \frac{A^T(x)T_0}{\theta} \right] \tag{10a}$$

$$\vdots$$

$$D_n = k_n e^{-l_n t} \left[-\frac{T_{n-1}A(x)}{\theta} - \frac{A^T(x)T_{n-1}}{\theta} + \sum_{j=1}^{n-1} T_j BR^{-1}B^T T_{n-j} \right] \tag{10b}$$

where k_i and $l_i > 0$ are design parameters such that

$$-\frac{T_{i-1}A(x)}{\theta} - \frac{A^T(x)T_{i-1}}{\theta} + \sum_{j=1}^{i-1} T_j BR^{-1}B^T T_{i-j} - D_i$$
$$= \varepsilon_i \left[-\frac{T_{i-1}A(x)}{\theta} - \frac{A^T(x)T_{i-1}}{\theta} + \sum_{j=1}^{i-1} T_j BR^{-1}B^T T_{i-j} \right] \tag{11}$$

where $\varepsilon_i = 1 - k_i e^{-l_i t}$ ($i = 1,\cdots n$).

ε_i is a small number chosen to overcome the large control problem because the state dependent term $A(x)$ on the right-hand side of the equations (9b)-(9c) may cause large magnitude of $T_i(x,\theta)$ if the initial states are large.

ε_i is also required in the proof of convergence and stability of the above algorithm [10]. The exponential term $e^{-l_i t}$ lets the perturbation terms in the cost function (7) diminish as time evolves since we don't want the perturbation terms to change the original cost function too much. k_i and l_i are also used to modulate system transient performance. A systematic method to determine these parameters can be referred to [10,12].

Remark 2.1 θ is just an intermediate variable. The introduction of θ is for the convenience of power series expansion, and it is cancelled when $T_i(x,\theta)$ multiply θ^i in the final control calculations, i.e., equation (8) [10]. The cancellation will be clearly seen in the $\theta - D$ observer development in the next section.

Theoretic results concerning the convergence of the series $\sum_{i=0}^{\infty} T_i(x,\theta)\theta^i$, closed-loop stability, and optimality of truncating the series can be found in Ref. [10].

III. FORMULATION OF $\theta - D$ OBSERVER

Considering duality property between the linear optimal regulator and observer, we can formulate the observer counterpart of the $\theta - D$ controller.

Consider the nonlinear system described by

$$\dot{x} = f(x) \tag{12}$$

$$y(x) = Hx \tag{13}$$

where $f(x)$ is assumed to be of class C^1, and H is a constant matrix. Rewrite (12) as:

$$\dot{x} = f(x) = F(x)x = \left[A_0 + \theta \frac{A(x)}{\theta} \right] x \tag{14}$$

The $\theta - D$ observer is given by

$$\dot{\hat{x}} = \left[A_0 + \theta \frac{A(\hat{x})}{\theta} \right] \hat{x} + K_f(\hat{x})[y(x) - H\hat{x}] \tag{15}$$

where $K_f(\hat{x}) = P(\hat{x})H^T V^{-1}$; $P(\hat{x}) = \sum_{i=0}^{\infty} \hat{T}_i(\hat{x},\theta)\theta^i$ (16)

$\hat{x}$ is the estimate state and $\hat{T}_i(\hat{x},\theta)$ is the solution to the following equations:

$$\hat{T}_0 A_0^T + A_0\hat{T}_0 - \hat{T}_0 H^T V^{-1} H\hat{T}_0 + W = 0 \tag{17a}$$

$$\hat{T}_1(A_0^T - H^T V^{-1} H\hat{T}_0) + (A_0 - \hat{T}_0 H^T V^{-1} H)\hat{T}_1 = -\frac{\hat{T}_0 A(\hat{x})}{\theta} - \frac{A^T(\hat{x})\hat{T}_0}{\theta} - \hat{D}_1 \tag{17b}$$

$$\vdots$$

$$\hat{T}_n(A_0^T - H^T V^{-1} H\hat{T}_0) + (A_0 - \hat{T}_0 H^T V^{-1} H)\hat{T}_n = -\frac{\hat{T}_{n-1} A(\hat{x})}{\theta} - \frac{A^T(\hat{x})\hat{T}_{n-1}}{\theta} + \sum_{j=1}^{n-1}\hat{T}_j H^T V^{-1} H\hat{T}_{n-j} - \hat{D}_n \tag{17c}$$

where $\hat{D}_1, \cdots, \hat{D}_n$ have the similar expressions as (10) with x replaced by $\hat{x}$.

In the above equations, $V > 0$ and $W \geq 0$ are weights to improve the convergence of the observer. They become noise covariance in the stochastic setting.

Eqs. (17a)-(17c) are solved recursively following the same procedure as solving Eqs. (9a)-(9c). Note that solving the first algebraic Riccati equation (17a) can be easily done *offline*. Since Eqs. (17b,c) are linear equations in terms of $\hat{T}_1$ and $\hat{T}_n$ with the same constant coefficient $A_0^T - H^T V^{-1} H\hat{T}_0$ and $A_0 - \hat{T}_0 H^T V^{-1} H$, they can be solved in closed-form *offline*. Hence, we can get the closed-form observer gain $K_f(\hat{x}) = \sum_{i=0}^{\infty}\hat{T}_i(\hat{x},\theta)\theta^i H^T V^{-1}$ if we take a finite number of terms.

The following theorem shows the convergence of the series $\sum_{i=0}^{\infty}\hat{T}_i(\hat{x},\theta)\theta^i$.

Theorem 3.1: If the following conditions are satisfied:

(i) $\hat{x} \in \Omega$, where $\Omega \subset R^n$ is a compact set;

(ii) (A_0, H) and (A_0^T, G) are observable, where $W = GG^T$;

(iii) $A(\hat{x})$ is continuous on Ω and $\|A(\hat{x})\|_2 \neq 0, \forall \hat{x} \in \Omega$;

(iv) $\lambda_{\max}[(A_0 - \hat{T}_0 H^T V^{-1} H) + (A_0^T - H^T V^{-1} H\hat{T}_0)] < 0$,

where $\lambda_{\max}$ denotes the largest eigenvalue,

the series $\sum_{i=0}^{\infty}\hat{T}_i(\hat{x},\theta)\theta^i$ given by the algorithm in Eqs. (17a)-(17c) is a pointwise convergent series.

Proof. Considering (17b) and the selection of $\hat{D}_1$, Eq. (17b) can be written as:

$$\hat{T}_1(A_0^T - H^T V^{-1} H\hat{T}_0) + (A_0 - \hat{T}_0 H^T V^{-1} H)\hat{T}_1 = -\varepsilon_1(\hat{T}_0 A + A^T \hat{T}_0)\frac{1}{\theta} \tag{18}$$

with
$$\varepsilon_1 = 1 - k_1 e^{-l_1 t} \tag{19}$$

Assume that the solution to the equation

$$\overline{T}_1(A_0^T - H^T V^{-1} H\hat{T}_0) + (A_0 - \hat{T}_0 H^T V^{-1} H)\overline{T}_1 = -\varepsilon_1(\hat{T}_0 A + A^T \hat{T}_0) \tag{20}$$

is $\overline{T}_1$ with
$$\overline{T}_0 = \hat{T}_0 \tag{21}$$

By using the linearity of the Lyapunov equation (20), the solution to (18) becomes
$$\hat{T}_1 = \frac{1}{\theta}\overline{T}_1 \tag{22}$$

In the same manner for (17c),
$$\hat{T}_n = \frac{1}{\theta^n}\overline{T}_n \tag{23}$$

where $\overline{T}_n$ is the solution to

$$\overline{T}_n(A_0^T - H^T V^{-1} H\hat{T}_0) + (A_0 - \hat{T}_0 H^T V^{-1} H)\overline{T}_n = -\varepsilon_n\left[\overline{T}_{n-1}A + A^T\overline{T}_{n-1} - \sum_{j=1}^{n-1}\overline{T}_j H^T V^{-1} H\overline{T}_{n-j}\right] \tag{24}$$

Therefore, proving the convergence of $\sum_{i=0}^{\infty}\hat{T}_i(\hat{x},\theta)\theta^i$ is equivalent to proving the convergence of $\sum_{i=0}^{\infty}\overline{T}_i(\hat{x})$ because θ^i gets cancelled as seen from (22) and (23). Note that this proof substantiates the Remark 2.1.

The objective now is to find a norm bound for each $\overline{T}_i$ in order to prove the convergence of the series $\sum_{i=0}^{\infty}\overline{T}_i(\hat{x})$.

Given a Lyapunov equation
$$\overline{A}^T\overline{P} + \overline{P}\overline{A} = -\overline{Q} \tag{25}$$

if $\overline{A}$ is a stable matrix, the norm bound for $\overline{P}$ exists [14]

$$\|\overline{P}\|_\bullet \leq \frac{\|\overline{Q}\|_\bullet}{-\mu_\bullet(\overline{A}^T) - \mu_\bullet(\overline{A})} \tag{26}$$

where $\mu_\bullet(\overline{A})$ is a matrix measure of $\overline{A}$ induced from $\|\cdot\|_\bullet$.

In the case of 2-norm,
$$\mu_2(\overline{A}) \triangleq \frac{1}{2}\lambda_{\max}(\overline{A} + \overline{A}^T) \tag{27}$$

In the following, $\|\cdot\|$ is defined as a 2-norm and denotes $\mu(\cdot) = \mu_2(\cdot)$.

Since (A_0, H) is an observable pair, (A_0^T, H^T) is controllable. So the condition (ii) implies that the Riccati equation (17a) has a positive definite solution $\hat{T}_0$ and $A_0 - \hat{T}_0 H^T V^{-1} H$ is a stable matrix.

Eq. (20) has the form of Eq. (25) and we have from (26):

$$\|\overline{T}_1\| \leq \frac{\left\|\varepsilon_1\left[\hat{T}_0 A + A^T\hat{T}_0\right]\right\|}{-\mu\left(A_0 - \hat{T}_0 H^T V^{-1} H\right) - \mu\left(A_0^T - H^T V^{-1} H\hat{T}_0\right)} \tag{28}$$

Let
$$C = \frac{1}{-\mu\left(A_0 - \hat{T}_0 H^T V^{-1} H\right) - \mu\left(A_0^T - H^T V^{-1} H\hat{T}_0\right)} \tag{29}$$

From condition (iv), we have $C > 0$.

Then
$$\|\overline{T}_1\| \leq C\varepsilon_1\left\|\hat{T}_0 A + A^T\hat{T}_0\right\| \leq C\varepsilon_1\left[\|\hat{T}_0\|\left(\|A\| + \|A^T\|\right)\right] \tag{30}$$

Since $A(\hat{x})$ is continuous on a compact set Ω, it is bounded on Ω.

Let
$$C_1 = \max_{\hat{x}\in\Omega}\left(\|A(\hat{x})\| + \|A^T(\hat{x})\|\right) \tag{31}$$

Then
$$\|\overline{T}_1\| \leq \varepsilon_1 C C_1\|\hat{T}_0\|. \tag{32}$$

Condition (iii) also implies that $C_1 \neq 0$. If it is zero, the nonlinear system will reduce to the linear system for which the solution is $\hat{T}_0$, the solution to the Riccati equation (17a).

For later use, define $\qquad S_0 = \|\bar{T}_0\| = \|\hat{T}_0\| \qquad (33)$

and $\qquad S_1 = \varepsilon_1 \cdot C \cdot C_1 \cdot \|\hat{T}_0\| \qquad (34)$

Therefore, by choosing sufficiently small ε_1, we can always make $S_1 / S_0 = \varepsilon_1 C C_1 < 1$.

From equation (24), a norm-bounded inequality for $\bar{T}_2$ becomes $\quad \|\bar{T}_2\| \le C\varepsilon_2 \|\bar{T}_1 A + A^T \bar{T}_1 - \bar{T}_1 H^T V^{-1} H \bar{T}_1\| \qquad (35)$

Let $C_H \triangleq \|H^T V^{-1} H\|$, a constant. Then

$$\|\bar{T}_2\| \le C \cdot \varepsilon_2 \left(C_1 \|\bar{T}_1\| + \|\bar{T}_1\|^2 C_H \right) = C^2 \varepsilon_1 \varepsilon_2 C_1^2 \|\hat{T}_0\| \left(1 + \varepsilon_1 C \|\hat{T}_0\| C_H\right) \qquad (36)$$

Let $\qquad C_2 = \sup\left(1 + \varepsilon_1 C \|\hat{T}_0\| C_H\right)$. $\qquad (37)$

Then we get $\qquad \|\bar{T}_2\| \le \varepsilon_1 \varepsilon_2 C^2 \cdot C_1^2 \cdot C_2 \|\hat{T}_0\| \qquad (38)$

Let $\qquad S_2 = \varepsilon_1 \varepsilon_2 C^2 \cdot C_1^2 \cdot C_2 \|\hat{T}_0\|$. $\qquad (39)$

Then we have $\qquad S_2 / S_1 = \varepsilon_2 C C_1 C_2 = O(\varepsilon_2) \qquad (40)$

Therefore, if ε_2 is picked sufficiently small, we can make

$$S_2 / S_1 < 1 \qquad (41)$$

In a similar manner we can derive for $\bar{T}_n$:

$$\|\bar{T}_n\| \le (\varepsilon_1 \cdots \varepsilon_n) \cdot C^n C_1^n C_2 \dots C_n \|\hat{T}_0\| \qquad (42)$$

Once the bound for each $\bar{T}_i$ is determined, the convergence of the series $\sum_{i=0}^{\infty} \bar{T}_i$ can be proved.

Define a series $\sum_{n=0}^{\infty} S_n$ with S_0 and S_1 defined in (33) and (34) and $\quad S_n = (\varepsilon_1 \cdots \varepsilon_n) \cdot C^n C_1^n C_2 \dots C_n \|\hat{T}_0\| \qquad (43)$

Then $\qquad S_n / S_{n-1} = \varepsilon_n \cdot C C_1 C_n = O(\varepsilon_n) \qquad (44)$

By choosing a sufficiently small ε_n such that $\lim_{n \to \infty} \varepsilon_n \cdot C C_1 C_n < 1$, $\sum_{i=0}^{\infty} S_i$ is a convergent series. Since each $\|\bar{T}_i\| \le S_i$, $\sum_{i=0}^{\infty} \bar{T}_i$ is also a convergent series. Thus $\sum_{i=0}^{\infty} \hat{T}_i(\hat{x}, \theta) \theta^i$ is convergent. □

The following lemma shows the asymptotic stability of the $\theta - D$ observer.

Lemma 3.1: Suppose that $f_1(x) = A(x)x$ is Lipschitz continuous on a compact set $\Omega \subset R^n$ with $x, \hat{x} \in \Omega$ and the conditions in Theorem 3.1 are satisfied, then the error dynamics defined by $e = \hat{x} - x$ is asymptotically stable.

Proof. Rewrite Eqs. (14) and (15)

$$\dot{x} = A_0 x + f_1(x) \qquad (45)$$

$$\dot{\hat{x}} = A_0 \hat{x} + f_1(\hat{x}) + K_f(\hat{x})\left[y(x) - H\hat{x}\right] \qquad (46)$$

Define $\qquad e = \hat{x} - x \qquad (47)$

Then $\dot{e} = \dot{\hat{x}} - \dot{x} = \left[A_0 - K_f(\hat{x}) H\right] e + \left[f_1(\hat{x}) - f_1(x)\right] \qquad (48)$

Using (16) in $\dot{e}$ yields

$$\dot{e} = \left\{ \left[A_0 - \hat{T}_0 H^T V^{-1} H \right] e - \sum_{i=1}^{\infty} \bar{T}_i H^T V^{-1} H e + \left[f_1(\hat{x}) - f_1(x) \right] \right\} \qquad (49)$$

From the condition (ii) in Theorem 3.1, we can always make $A_0 - \hat{T}_0 H^T V^{-1} H$ a Hurwitz matrix.

Let $\qquad F_0 = A_0 - \hat{T}_0 H^T V^{-1} H \qquad (50)$

Thus for any given positive-definite matrix $\hat{Q} \in R^{nxn}$, there exists a unique positive-definite $\hat{P} \in R^{nxn}$ such that

$$F_0^T \hat{P} + \hat{P} F_0 = -2\hat{Q} \qquad (51)$$

Now consider the following positive definite Lyapunov function $V(e) = e^T \hat{P} e \qquad (52)$

$$\dot{V}(e) = \left\{ \left[e^T F_0^T - e^T H^T V^{-1} H \sum_{i=1}^{\infty} \bar{T}_i + \left[f_1^T(\hat{x}) - f_1^T(x) \right] \right] \right\} \hat{P} e$$

$$+ e^T \hat{P} \left\{ \left[F_0 e - \sum_{i=1}^{\infty} \bar{T} H^T V^{-1} H e \right] + \left[f_1(\hat{x}) - f_1(x) \right] \right\}$$

$$= -2 e^T \hat{Q} e + 2 e^T \hat{P} \left\{ \left[f_1(\hat{x}) - f_1(x) \right] - \sum_{i=1}^{\infty} \bar{T}_i H^T V^{-1} H e \right\} \qquad (53)$$

Since $f_1(x)$ is Lipschitz continuous on a compact set Ω, we have $\|f_1(\hat{x}) - f_1(x)\| \le L_f \|\hat{x} - x\| = L_f \|e\|$, where L_f is the Lipschitz constant. Then

$$\dot{V}(e) \le -2\lambda_{\min}(\hat{Q}) \cdot \|e\|^2 + 2 \cdot L_f \cdot \|\hat{P}\| \cdot \|e\|^2 + 2 \cdot \left\| \sum_{i=1}^{\infty} \bar{T}_i \right\| \cdot \|H^T V^{-1} H\| \cdot \|\hat{P}\| \cdot \|e\|^2$$

$$= -2 \left\{ \lambda_{\min}(\hat{Q}) - \|\hat{P}\| \cdot \left[L_f + \left\| \sum_{i=1}^{\infty} \bar{T}_i \right\| \cdot C_H \right] \right\} \|e\|^2 \qquad (54)$$

From (42), as long as we choose proper $\varepsilon_1 \cdots \varepsilon_i$ for $\left\| \sum_{i=0}^{\infty} \bar{T}_i \right\|$ and large enough $\lambda_{\min}(\hat{Q})$ such that

$$\|\hat{P}\| \cdot \left[L_f + \left\| \sum_{i=1}^{\infty} \bar{T}_i \right\| C_H \right] < \lambda_{\min}(\hat{Q}), \qquad (55)$$

then we have $\dot{V}(e) < 0$. Therefore, $e = 0$ is an asymptotically stable equilibrium point. □

Remark 3.1 The selection of k_i and l_i parameters in D_i and $\hat{D}_i$ can be done systematically by utilizing the least-square curve fitting of the maximum singular value of the $\theta - D$ solution with the state dependent Riccati equation solution [10,12].

IV. APPLICATION TO FLIGHT CONTROL OF AIRCRAFT

In this section, the $\theta - D$ observer and control are applied to the flight control of a high performance aircraft operating at high angles of attack. The mathematical model used in this study is similar to the X-31 research aircraft [15] and only the longitudinal mode is considered. The longitudinal model is highly nonlinear at high angle of attack and the conventional linear time invariant framework is not applicable. The state vector that describes the longitudinal motion is

$$x^T = \begin{bmatrix} \Delta V & \Delta\alpha & q & \gamma & \Delta\delta \end{bmatrix}^T \quad (56)$$

where ΔV is the deviation in the velocity from the level flight trim value of 100m/s; $\Delta\alpha$ is the deviation of angle of attack from its trim value of 4.2^0; q is the pitch rate in rad/s; γ is the flight path angle in radians; $\Delta\delta$ is the canard deflection in degrees from its trim value. The scalar control u is the input to the canard actuator. The longitudinal equations of motion are given by

$$\dot{x} = (A_L + x_2 A_{NL})x + Bu \quad (57)$$

where the constant matrices A_L and A_{NL} were obtained by a best least square fit to flight conditions at eight angles of attack ranging from 4.2^0 to 43^0 and are given by [15]; B matrix is $B = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 20.0 \end{bmatrix}^T$.

In this aircraft application, we design an optimal flight controller and observer to control the angle of attack $\Delta\alpha$ and flight path angle γ.

The measurements for the observer are the velocity and the canard deflection, i.e. $y(x) = \begin{bmatrix} x_1 & x_5 \end{bmatrix}^T$.

The optimal controller is designed using the $\theta - D$ technique

$$u(\hat{x}) = -R^{-1}B^T \sum_{i=0}^{\infty} T_i(\hat{x},\theta)\theta^i \hat{x} \quad (58)$$

where $T_i(\hat{x},\theta)$ is obtained by following the algorithm (9) and the feedback state vector $\hat{x}$ is estimated by using the $\theta - D$ observer (15).

Note that both the optimal control law $u(\hat{x})$ and the observer gain $K_f(\hat{x})$ can be solved in closed-form that offers a great computational advantage.

The first simulation scenario is to regulate the states to zero (their respective trim values) from a large initial angle of attack given by $x_0 = \begin{bmatrix} 0 & 30^0 & 0 & 0 & 0 \end{bmatrix}^T$. The initial estimated states are assumed to be: $x_e(0) = \begin{bmatrix} 0 & 25^0 & 0 & 0 & 0 \end{bmatrix}^T$.

After some numerical experiments, the control weights and the observer weights are chosen to be:

Q=diag([80,1,1,300,1]) , R=300; W=I_5, V=diag{0.1,0.1}

In this problem, the first three terms in the $\theta - D$ control and observer algorithm (8) and (16) are sufficient for good performance. The k_i and l_i parameters for both control and observer are chosen to be $k_1 = k_2 = 1$, and $l_1 = l_2 = 0.01$.

Figures 1 and 2 show the responses of the angle of attack, pitch rate, and flight path angle respectively. As can be seen, the estimated states converge to the actual states very quickly. The optimal control regulates the states to zero with good transient responses. Figure 3 gives the canard deflection and control command (optimal control), which does not demand large control.

The second scenario is to track a command flight path angle γ_c. The commanded γ_c starts from 0 and gradually increases to 45^0, holds until 15 seconds, and then returns to 0 at 20 seconds as shown in Fig. 4. The initial condition is at zero and the initial estimated state starts slightly off the actual, i.e. $x_e(0) = \begin{bmatrix} 0 & 2^0 & 0 & 2^0 & 0 \end{bmatrix}^T$.

In order to ensure a good tracking performance, an integral state of the flight path angle is augmented into the original state space, i.e.

$$\dot{\gamma}_I = \gamma \quad (59)$$

The state space for this tracking problem becomes

$$\boldsymbol{x} = \begin{bmatrix} \Delta V & \Delta\alpha & q & \gamma & \Delta\delta & \gamma_1 \end{bmatrix}^T \quad (60)$$

and the associated A_L and A_{NL} matrices are changed accordingly. The optimal control is applied as a servomechanism formulation for this tracking problem [13],

$$u(\hat{x}) = -R^{-1}B^T \sum_{i=0}^{\infty} T_i(\hat{x},\theta)\theta^i \left[\hat{x} - \hat{x}_r\right]^T \quad (61)$$

where $\hat{x}_r = \begin{bmatrix} 0 & 0 & 0 & \gamma_c & 0 & \int \gamma_c \end{bmatrix}^T$ is the command state vector. The k_i and l_i parameters are the same as those in the first scenario. Weights used for control and estimation are chosen to be

Q=diag(1,1,1,100,1,200) , R=10, W=I_5, V=diag(0.01,0.01)

Figure 4 and 5 show the flight path angle tracking and angle of attack response respectively. As can be seen, the $\theta - D$ control and observer exhibit excellent tracking performance. For comparison, the extended Kalman filter (EKF) is used as a nonlinear observer for the $\theta - D$ optimal control. The results in the first scenario are not much different from using the $\theta - D$ observer. However, in the second flight path angle tracking scenario, the EKF exhibits much worse performance as seen in Figs 6 and 7.

V. Conclusions

In this paper, a new nonlinear observer was developed from the dual of the $\theta - D$ control technique. This observer does not involve a linearization process required by EKF and address the nonlinearity directly. The major advantage of this observer is that the observer gain can be obtained as a closed-form expression and consequently does not need complex on-line computations compared to the SDREF technique. The $\theta - D$ observer combined with the $\theta - D$ optimal control is applied to the flight control of a highly maneuverable aircraft with high nonlinearity and unstable dynamics. The effectiveness of this technique has been demonstrated by regulating the high angle of attack and tracking flight path angle command.

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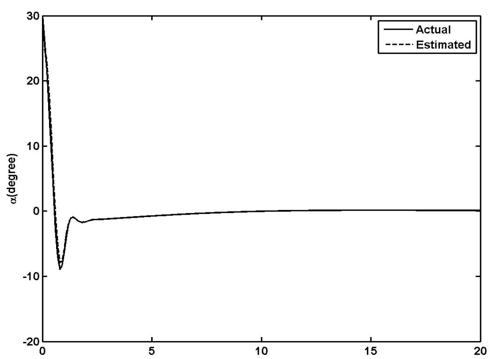

Fig. 1: Angle of attack response

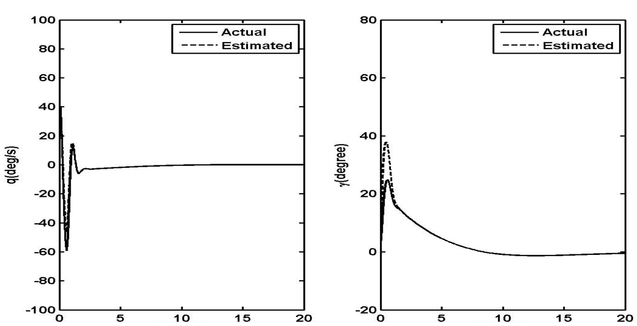

Fig. 2: Pitch rate and flight path angle response

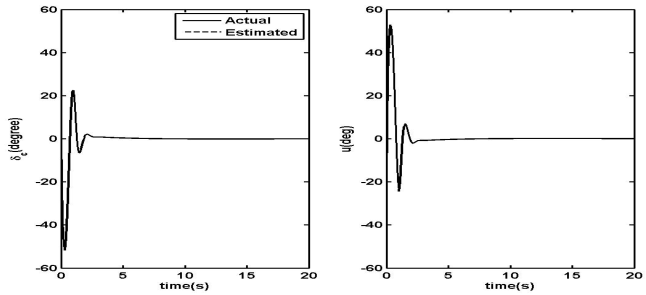

Fig. 3: Canard deflection and control command

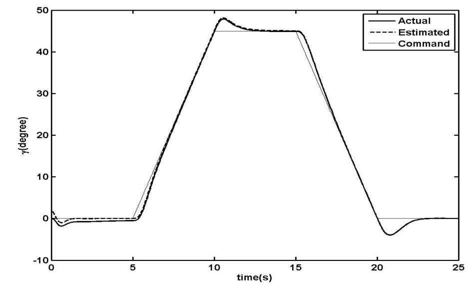

Fig. 4: Flight path angle tracking

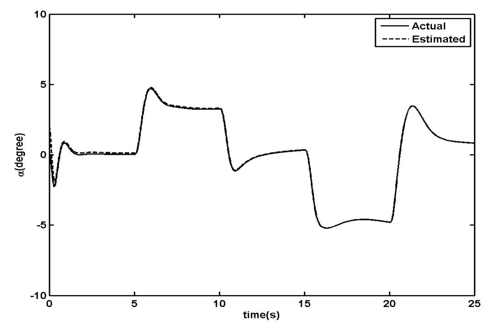

Fig. 5: Angle of attack tracking

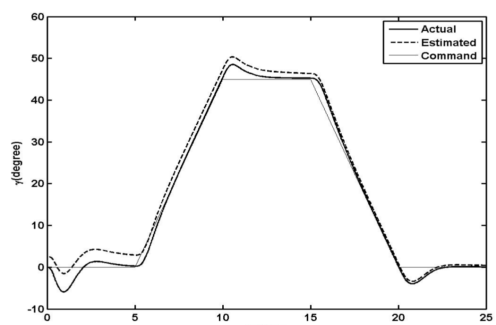

Fig. 6: Flight path angle tracking using EKF

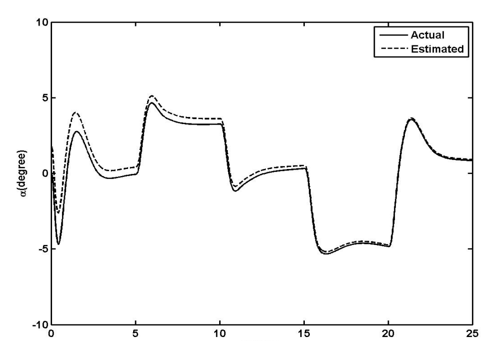

Fig.7: Angle of attack response using EKF