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Nonlinear Optimal Control of Spacecraft Approaching a Tumbling Target

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Abstract—**This paper addresses the control of spacecraft to approach and align with a tumbling target. In order to complete the task, the spacecraft is required to perform large position and attitude maneuvers with sufficient accuracy. In addition, the flexible motion induced by large angular maneuvers needs to be minimized. The primary contribution of this work is to consider the control of position, attitude, and flexible motion in one unified optimal control framework. The 6-DOF rigid body dynamics and coupled flexible structure dynamics are highly nonlinear and lead to a challenging control problem. The $\theta - D$ nonlinear optimal control technique is employed to design an integrated controller for this problem by solving the associated Hamilton-Jacobi-Bellman (HJB) equation in an approximate analytical form via a perturbation process. The closed-form controller offered by this method is easy to implement onboard especially for this problem with a large state-space. Numerical results show that the proposed controller exhibits good tracking performance even under large moment of inertia uncertainties.**

I. Introduction

With the fast growing space activities, there is an increasing interest in autonomous on-orbit servicing satellites for economical and other reasons. The Orbital Express Program [1] and SUMO program [2] developed by DARPA are technology demonstration missions that aim to offer the potential for spacecraft salvage, repair, refueling, and debris removal. A key enabling technology in these missions is autonomous rendezvous and capturing. In this paper, we consider the problem of driving a spacecraft to a proximity position with respect to a target and synchronizing the spacecraft attitude with the tumbling target's attitude. The two vehicles will eventually have no relative motion and then subsequent service operations can be safely performed with a normal docking or capture mechanism. The spacecraft kinematics and dynamics are highly nonlinear and thus traditional linear control designs are unsuitable, especially when large angular maneuvers are required.

Many nonlinear control techniques have been investigated in the past to address the attitude control problem such as sliding mode [3], backstepping [4], and adaptive quaternion feedback [5]. The State Dependent Riccati Equation (SDRE) technique was used in [6] to design both position and attitude controllers. However, the SDRE method needs to solve the Riccati equation repetitively at every integration step. It may raise an implementation issue if the system order is higher. Ma et al. [7] designed a feed-forward optimal control strategy for spacecraft to approach a tumbling satellite by minimizing time/fuel consumption. A restricted planar rigid body motion was considered.

Since the spacecraft tracking a tumbling target may involve large or rapid angular motions, effective vibration control of flexible structures becomes essential to ensure the successful control task. There are many research works on vibration control of spacecraft structures [8-10]. However, very few works consider the coupling of the flexible motion with the six degree of freedom position and attitude dynamics.

In this paper, the control of spacecraft to approach a tumbling target is formulated as a unified optimal control problem incorporating control of position, attitude, and flexible structures. The $\theta - D$ technique [11,12] is employed to design this integrated control law. This method can provide a closed-form feedback controller based on an approximate solution to the Hamilton-Jacobi-Bellman (HJB) equation. The resultant closed-form solution offers a great advantage for online implementations.

Rest of the paper is organized as follows. Section II describes the equations of motion of the problem including six degree-of-freedom rigid body dynamics and the model of flexible structures. The $\theta - D$ technique is reviewed in Section III and is employed to design the spacecraft controllers in Section IV. Simulation results are presented in Section V. Some concluding remarks are given in Section VI.

II. Equations of Motion and Problem Formulation

In this study, we consider a spacecraft mission performing proximity flight around a tumbling target such as space debris or mal-functioned satellite that needs to be served. The control objective is to have the spacecraft positioned at a certain distance from the target while synchronizing the spacecraft body frame with the target body frame during the operation. The body fixed coordinate frames of the spacecraft and the target are represented by $\boldsymbol{B}_s\{\hat{\boldsymbol{b}}_{s_x}, \hat{\boldsymbol{b}}_{s_y}, \hat{\boldsymbol{b}}_{s_z}\}$ and $\boldsymbol{B}_t\{\hat{\boldsymbol{b}}_{t_x}, \hat{\boldsymbol{b}}_{t_y}, \hat{\boldsymbol{b}}_{t_z}\}$ respectively, which is shown in Figure 1. The inertial coordinate system is denoted by $\boldsymbol{N}_I\{\hat{\boldsymbol{n}}_x, \hat{\boldsymbol{n}}_y, \hat{\boldsymbol{n}}_z\}$.

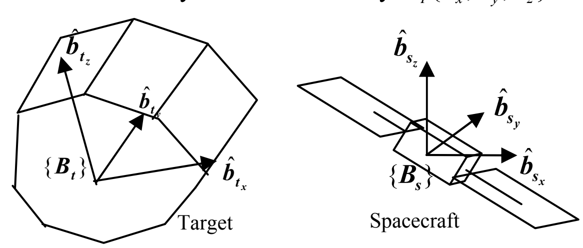

Fig. 1: Spacecraft and target coordinate frames

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A. Rigid Body Dynamics

The above task involves both translational and rotational maneuvers with highly nonlinear kinematics and dynamics. The six degree-of-freedom (6-DOF) motion of a spacecraft neglecting gravity can be described by [13]:

Kinematics:
$$\dot{\mathbf{r}}_s = -\tilde{\omega}_s \mathbf{r}_s + \mathbf{v}_s \tag{1}$$

$$\dot{\mathbf{q}}_s = \frac{1}{2}\Omega\left(\boldsymbol{\omega}_s\right)\mathbf{q}_s \tag{2}$$

where $\mathbf{r}_s \in R^3, \mathbf{v}_s \in R^3, \boldsymbol{\omega}_s = \begin{bmatrix} \omega_{s_x} & \omega_{s_y} & \omega_{s_z} \end{bmatrix}^T$ are spacecraft position, velocity, and body angular velocity respectively; Note that they are defined in the spacecraft body frame. $\boldsymbol{q}_s = \begin{bmatrix} q_{s_0} & q_{s_1} & q_{s_2} & q_{s_3} \end{bmatrix}^T$ is the quaternion vector to describe the spacecraft attitude with respect to the inertial frame. The elements of the quaternion are defined as:

$$q_{s_0} \triangleq \cos(\varphi/2), \quad q_{s_i} \triangleq c_{s_i} \sin(\varphi/2), i = 1,2,3 \tag{3}$$

where φ is the rotation angle about the Euler-axis, and $(c_{s_1}, c_{s_2}, c_{s_3})$ are the direction cosines of the Euler axis with respect to the reference frame.

The definition of $\tilde{\omega}_s$ and $\Omega(\boldsymbol{\omega}_s)$ can be referred to [13].

Dynamics:
$$\dot{\mathbf{v}}_s + \tilde{\boldsymbol{\omega}}_s \mathbf{v}_s = \frac{\mathbf{F}_s}{m_s} \tag{4}$$

$$\dot{\boldsymbol{\omega}}_s + J_s^{-1}\tilde{\boldsymbol{\omega}}_s J_s \boldsymbol{\omega}_s = J_s^{-1}\boldsymbol{\Gamma}_s \tag{5}$$

where m_s and J_s are spacecraft mass and moment of inertia matrix respectively. Control force $\mathbf{F}_s = \begin{bmatrix} F_{s_x} & F_{s_y} & F_{s_z} \end{bmatrix}^T$ and control torque $\boldsymbol{\Gamma}_s = \begin{bmatrix} \Gamma_{s_x} & \Gamma_{s_y} & \Gamma_{s_z} \end{bmatrix}^T$ are defined about the spacecraft body axes $\boldsymbol{B}_s\{\hat{\boldsymbol{b}}_{s_x}, \hat{\boldsymbol{b}}_{s_y}, \hat{\boldsymbol{b}}_{s_z}\}$.

The kinematic and dynamic equations of the 6-DOF tumbling target can be written in the target body frame in the same fashion as Eqs. (1),(2) and Eqs. (4), (5) in the absence of control forces and control torques.

The errors in position, velocity, angular velocity, and quaternions between the spacecraft and the target can be described by:

$$\mathbf{r}_e = \mathbf{r}_s - T_I^{B_s}\mathbf{r}_t^I \tag{6}$$

$$\mathbf{v}_e = \mathbf{v}_s - T_I^{B_s}\mathbf{v}_t^I \tag{7}$$

$$\boldsymbol{\omega}_e = \boldsymbol{\omega}_s - T_I^{B_s}\boldsymbol{\omega}_t^I \tag{8}$$

$$\boldsymbol{q}_e = Q_t^{-1}\boldsymbol{q}_s \tag{9}$$

where $\mathbf{r}_t \in R^3, \mathbf{v}_t \in R^3$, $\boldsymbol{\omega}_t \in R^3$ are the target position, velocity, and body angular velocity respectively defined in the target body frame; $\boldsymbol{q}_t$ is the target quaternion vector. The target states can be transformed to the inertial frame by

$$\begin{bmatrix} \mathbf{r}_t^I & \mathbf{v}_t^I & \boldsymbol{\omega}_t^I \end{bmatrix}^T = T_{B_t}^I \begin{bmatrix} \mathbf{r}_t & \mathbf{v}_t & \boldsymbol{\omega}_t \end{bmatrix} \tag{10}$$

where superscript *'I'* denotes inertial frame and $T_{B_t}^I$ is the target body-to-inertial coordinate transformation matrix that can be obtained from the target quaternions [13]. $T_I^{B_s}$ is the coordinate transformation matrix from the inertial frame to the spacecraft body frame and can be obtained in terms of the spacecraft quaternions [13]. Q_t is defined as:

$$Q_t \triangleq \begin{bmatrix} q_{t_0} & -q_{t_3} & q_{t_2} & q_{t_1} \\ q_{t_3} & q_{t_0} & -q_{t_1} & q_{t_2} \\ -q_{t_2} & q_{t_1} & q_{t_0} & q_{t_3} \\ -q_{t_1} & -q_{t_2} & -q_{t_3} & q_{t_0} \end{bmatrix} \tag{11}$$

B. Coupling of Flexible Structure Dynamics

During the slewing maneuver of the spacecraft to track the tumbling target, it is important to suppress the induced vibrations in order to ensure accurate attitude control. In this paper, control of flexible motion is also considered together with the rigid body dynamics and is formulated as a unified optimal control problem.

The specific model considered in this study consists of a rigid hub with two appendages attached symmetrically to the center hub representing the lightweight flexible structures. Without loss of generality, deflection of flexible appendages is assumed to occur only about the spacecraft body $\hat{\boldsymbol{b}}_{s_y}$ axis (pitch motion) as shown in Fig. 2. Furthermore, it's assumed that the deflection is anti-symmetric such that the center of mass of the spacecraft does not shift. This assumption has been validated to be a good one in Ref. [10].

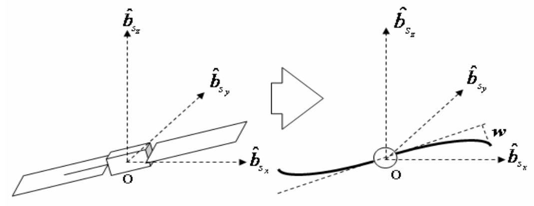

Fig. 2. Spacecraft and appendage allocation

The flexible structure considered is a continuous simple elastic structure such as Euler-Bernoulli beam. The control force and control torque are placed only on the rigid hub.

Appendage deflection can be expressed in term of a series of assumed admissible mode shape function [10].

$$w(t,x) = \sum_{k=1}^{n} \eta_k(t)\phi_k(\bar{x} - r) \tag{12}$$

where $\eta_k(t)$ is the kth time-varying mode coefficient and $\phi_k(\bar{x} - r)$ is the pre-determined admissible shape function. n is the mode number; r is the hub radius; $\bar{x}$ is the distance from the center of the hub.

The rotation angle about the spacecraft $\hat{\boldsymbol{b}}_{s_y}$ axis is denoted by ψ and the torque placed on the hub is Γ_{s_y}. The pitch equation of motion of the flexible structures can be derived via Lagrangian procedure [10]

$$\left(J_{s_y} - \boldsymbol{\eta}^T \mathrm{M}^* \boldsymbol{\eta}\right)\ddot{\psi} + \mathrm{M}_{\psi\eta}^T \ddot{\boldsymbol{\eta}} - 2\dot{\psi}\dot{\boldsymbol{\eta}}^T \mathrm{M}^* \boldsymbol{\eta} = \boldsymbol{\Gamma}_{s_y} \tag{13}$$

$$\mathrm{M}_{\eta\eta}\ddot{\boldsymbol{\eta}} + \mathbf{M}_{\psi\eta}\ddot{\psi} + \mathrm{C}_{\eta\eta}\dot{\boldsymbol{\eta}} + \left[\mathrm{K}_{\eta\eta} + \dot{\psi}^2 \mathrm{M}^*\right]\boldsymbol{\eta} = 0 \tag{14}$$

where $\left[\mathrm{M}_{\eta\eta}\right]_{kp} = 2\int_r^{r+L} \phi_k(\bar{x} - r)\phi_p(\bar{x} - r)\rho d\bar{x}$

$$\left[\mathbf{M}_{\psi\eta}\right]_k = 2\int_r^{r+L} \bar{x}\phi_k(\bar{x} - r)\rho d\bar{x} \; ; \left[\mathrm{C}_{\eta\eta}\right]_{kp} = 2\int_r^{r+L} CJ_{s_y}\ddot{\phi}_k(\bar{x} - r)\ddot{\phi}_p(\bar{x} - r)d\bar{x}$$

$$\left[\mathrm{K}_{\eta\eta}\right]_{kp} = 2\int_r^{r+L} EJ_{s_y}\ddot{\phi}_k\left(\bar{x}-r\right)\phi_p\left(\bar{x}-r\right)d\bar{x};\ \mathrm{M}^* = \tilde{\mathrm{M}} - \mathrm{M}_{\eta\eta}$$

$$\left[\tilde{\mathrm{M}}\right]_{kp} = 2\int_r^{r+L}\frac{1}{2}\rho\left[\left(r+L\right)^2 - r^2 - \bar{x}^2\right]\dot{\phi}_k\left(\bar{x}-r\right)\dot{\phi}_p\left(\bar{x}-r\right)\rho d\bar{x}$$

J_{s_y} is the moment of inertia about the spacecraft body $\hat{\boldsymbol{b}}_{s_y}$ axis; L is the undeformed length of the appendage; C and E are the damping coefficient and modulus of elasticity respectively; p is the mode number; ρ is mass per length.

When spacecraft maneuvers are relatively fast, we can simplify the equation (13) by neglecting the quadratic terms [10]. Also note that $\dot{\psi} = \omega_{s_y}$. The equation (13) becomes:

$$J_{s_y}\dot{\omega}_{s_y} + \mathbf{M}_{\psi\eta}^T\ddot{\boldsymbol{\eta}} = \Gamma_{s_y} \quad (15)$$

The rigid body rotational equation of motion coupling the flexible mode can be written as

$$J_s\dot{\boldsymbol{\omega}}_s + \tilde{\omega}_s\left\{J_s\boldsymbol{\omega}_s + \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T M_{\psi\eta}^T\dot{\boldsymbol{\eta}}\right\} + \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T\mathbf{M}_{\psi\eta}^T\ddot{\boldsymbol{\eta}} = \boldsymbol{\Gamma}_s \quad (16)$$

$$\mathrm{M}_{\eta\eta}\ddot{\boldsymbol{\eta}} + \mathbf{M}_{\psi\eta}\dot{\omega}_{s_y} + \mathrm{C}_{\eta\eta}\dot{\boldsymbol{\eta}} + \left[\mathrm{K}_{\eta\eta} + \omega_{s_y}^2\mathrm{M}^*\right]\boldsymbol{\eta} = 0 \quad (17)$$

As can be seen, the flexible modes are coupled with the attitude dynamics and in turn coupled with the position and attitude kinematics. These equations are highly nonlinear and pose a challenging control problem.

In the next two sections, we propose the $\theta - D$ suboptimal control technique to solve this nonlinear control problem.

III. REVIEW OF θ - D CONTROL TECHNIQUE

The θ - D nonlinear control technique addresses the class of nonlinear time-invariant systems described by

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + B\boldsymbol{u} \quad (18)$$

with the cost functional:

$$J = \frac{1}{2}\int_0^\infty[\boldsymbol{x}^T Q\boldsymbol{x} + \boldsymbol{u}^T R\boldsymbol{u}]dt \quad (19)$$

where $\boldsymbol{x}\in\Omega\subset R^n, f:\Omega\to R^n, B\in R^{n\times m}, \boldsymbol{u}:\Omega\to R^m, Q\in R^{n\times n}, R\in R^{m\times m}$; Q is a positive semi-definite matrix and R is a positive definite matrix; B is a constant matrix and $f(\mathbf{0})=\mathbf{0}$; Ω is a compact subset in R^n;

The optimal solution to this infinite-horizon nonlinear regulator problem can be obtained by solving the Hamilton-Jacobi-Bellman (HJB) partial differential equation [14]:

$$\frac{\partial V^T}{\partial \boldsymbol{x}}f(\boldsymbol{x}) - \frac{1}{2}\frac{\partial V^T}{\partial \boldsymbol{x}}BR^{-1}B^T\frac{\partial V}{\partial \boldsymbol{x}} + \frac{1}{2}\boldsymbol{x}^T Q\boldsymbol{x} = 0 \quad (20)$$

where $V(\boldsymbol{x})$ is the optimal cost , i.e.

$$V(\boldsymbol{x}) = \min_{\boldsymbol{u}}\frac{1}{2}\int_0^\infty(\boldsymbol{x}^T Q\boldsymbol{x} + \boldsymbol{u}^T R\boldsymbol{u})dt \quad (21)$$

Optimal control is obtained from the necessary condition as

$$\boldsymbol{u} = -R^{-1}B^T\frac{\partial V}{\partial \boldsymbol{x}} \quad (22)$$

The $\theta - D$ control technique provides an approximate solution to the above HJB equation (20) such that a suboptimal closed-form feedback controller can be obtained.

The $\theta - D$ control method can be summarized by the following procedure [11].

Write the nonlinear state equation as a linear-like structure:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + B\boldsymbol{u} = F(\boldsymbol{x})\boldsymbol{x} + B\boldsymbol{u} = \left[A_0 + \theta\frac{A(\boldsymbol{x})}{\theta}\right]\boldsymbol{x} + B\boldsymbol{u} \quad (23)$$

where A_0 is a constant matrix such that (A_0, B) is a controllable pair and $[F(\boldsymbol{x}), B]$ is pointwise controllable.

Add a perturbation power series $\sum_{i=1}^{\infty}D_i\theta^i$ to the original cost function (19)

$$J = \frac{1}{2}\int_0^\infty\left[\boldsymbol{x}^T\left(Q + \sum_{i=1}^{\infty}D_i\theta^i\right)\boldsymbol{x} + \boldsymbol{u}^T R\boldsymbol{u}\right]dt \quad (24)$$

Assuming a power series expansion $\frac{\partial V}{\partial \boldsymbol{x}} = \sum_{i=0}^{\infty}T_i(\boldsymbol{x},\theta)\theta^i\boldsymbol{x}$ and solving the new optimal control problem (23) and (24) through the HJB equation yield a suboptimal control

$$\boldsymbol{u} = -R^{-1}B^T\sum_{i=0}^{\infty}T_i(\boldsymbol{x},\theta)\theta^i\boldsymbol{x} \quad (25)$$

where $T_i(\boldsymbol{x},\theta)$ ($i = 0,\cdots,n,\cdots$) is a symmetric matrix and is solved recursively by the following algorithm.

$$T_0A_0 + A_0^TT_0 - T_0BR^{-1}B^TT_0 + Q = 0 \quad (26a)$$

$$T_1(A_0 - BR^{-1}B^TT_0) + (A_0^T - T_0BR^{-1}B^T)T_1 = -\frac{T_0A(\boldsymbol{x})}{\theta} - \frac{A^T(\boldsymbol{x})T_0}{\theta} - D_1 \quad (26b)$$

$$\vdots$$

$$T_n(A_0 - BR^{-1}B^TT_0) + (A_0^T - T_0BR^{-1}B^T)T_n = -\frac{T_{n-1}A(\boldsymbol{x})}{\theta} - \frac{A^T(\boldsymbol{x})T_{n-1}}{\theta} + \sum_{j=1}^{n-1}T_jBR^{-1}B^TT_{n-j} - D_n \quad (26c)$$

Equation (26a) is an algebraic Riccati equation and the rest of equations are Lyapunov equations that are *linear* in terms of T_i ($i = 1,\cdots,n$). Since all the coefficients of T_i are the same constant matrices, i.e. $A_0 - BR^{-1}B^TT_0$ and $A_0^T - T_0BR^{-1}B^T$, closed-form solution for $T_i(\boldsymbol{x},\theta)$ can be easily obtained by solving Eqs. (26a)-(26c) successively [11].

The D_i matrix is constructed in the form of:

$$D_1 = k_1e^{-l_1t}\left[-\frac{T_0A(\boldsymbol{x})}{\theta} - \frac{A^T(\boldsymbol{x})T_0}{\theta}\right] \quad (27a)$$

$$\vdots$$

$$D_n = k_ne^{-l_nt}\left[-\frac{T_{n-1}A(\boldsymbol{x})}{\theta} - \frac{A^T(\boldsymbol{x})T_{n-1}}{\theta} + \sum_{j=1}^{n-1}T_jBR^{-1}B^TT_{n-j}\right] \quad (27b)$$

where k_i and $l_i > 0$ are design parameters; D_i satisfies

$$-\frac{T_{i-1}A(\boldsymbol{x})}{\theta} - \frac{A^T(\boldsymbol{x})T_{i-1}}{\theta} + \sum_{j=1}^{i-1}T_jBR^{-1}B^TT_{i-j} - D_i = \varepsilon_i\left[-\frac{T_{i-1}A(\boldsymbol{x})}{\theta} - \frac{A^T(\boldsymbol{x})T_{i-1}}{\theta} + \sum_{j=1}^{i-1}T_jBR^{-1}B^TT_{i-j}\right] \quad (28)$$

where $\varepsilon_i = 1 - k_ie^{-l_it}$ ($i = 1,\cdots n$).

ε_i is a small number used to overcome the large control problem because the state dependent term $A(\boldsymbol{x})$ on the right-hand side of the equations (26b,c) may cause large magnitude of $T_i(\boldsymbol{x},\theta)$ if the initial states are large. ε_i is also required in the proof of convergence and stability of the above algorithm [11].

Remark 3.1 θ is just an intermediate variable. The introduction of θ is for the convenience of power series expansion, and it is cancelled when $T_i(\boldsymbol{x},\theta)$ multiply θ^i in the final control calculations, i.e., equation (25).

Theoretic results concerning the convergence of the series $\sum_{i=0}^{\infty} T_i(\boldsymbol{x},\theta)\theta^i$, closed-loop stability, and optimality of truncating the series can be found in Ref. [11].

IV. INTEGRATED SPACECRAFT CONTROL DESIGN

In order to employ the $\theta - D$ optimal control, the system dynamics need to be written in the state-space representation. To this end, we can eliminate $\dot{\omega}_{s_y}$ from Eqs. (16) and (17) to get the differential equation for the flexible mode $\boldsymbol{\eta}$:

$$\ddot{\boldsymbol{\eta}} = M_1 \cdot \boldsymbol{\omega}_s - C_1 \cdot \dot{\boldsymbol{\eta}} - \mathrm{K}_1 \cdot \boldsymbol{\eta} - M_2 \cdot \boldsymbol{\Gamma}_s \qquad (29)$$

where: $M_1 = \mathrm{M}_0^{-1}\mathbf{M}_{\psi\boldsymbol{\eta}}\begin{bmatrix}0 & 1 & 0\end{bmatrix} J_s^{-1}\tilde{\omega}_s J_s$

$$\mathrm{C}_1 = \mathrm{M}_0^{-1}\left\{C_{\eta\eta} - M_{\psi\eta}\begin{bmatrix}0 & 1 & 0\end{bmatrix} J_s^{-1}\tilde{\omega}_s \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T M_{\psi\eta}^T\right\}$$

$$\mathrm{K}_1 = \mathrm{M}_0^{-1}\left[\mathrm{K}_{\boldsymbol{\eta\eta}} + \omega_{s_y}^2 \mathrm{M}^*\right], M_2 = \mathrm{M}_0^{-1}\mathbf{M}_{\psi\boldsymbol{\eta}}\begin{bmatrix}0 & 1 & 0\end{bmatrix} J_s^{-1}$$

$$\mathrm{M}_0 = \mathrm{M}_{\boldsymbol{\eta\eta}} - \mathbf{M}_{\psi\boldsymbol{\eta}}\begin{bmatrix}0 & 1 & 0\end{bmatrix} J_s^{-1}\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \mathbf{M}_{\psi\boldsymbol{\eta}}^T$$

Define $\mathbf{s}_1 = \boldsymbol{\eta}, \mathbf{s}_2 = \dot{\boldsymbol{\eta}}$. Equation (29) can be written in the state-space form:

$$\dot{\mathbf{s}}_1 = \mathbf{s}_2 \qquad (30)$$

$$\dot{\mathbf{s}}_2 = M_1 \cdot \boldsymbol{\omega}_s - \mathrm{C}_1 \cdot \mathbf{s}_2 - \mathrm{K}_1 \cdot \mathbf{s}_1 - M_2 \cdot \boldsymbol{\Gamma}_s \qquad (31)$$

To improve the steady-state tracking performance, an integral state of $\mathbf{r}_s$ is augmented into the state space, i.e.

$$\dot{\mathbf{r}}_{sI} = \mathbf{r}_s \qquad (32)$$

In the flexible structure model, the first three modes are considered. This will be shown to be a good approximation. The state-space variables for this spacecraft control problem are chosen to be $\boldsymbol{x} = \begin{bmatrix}\mathbf{r}_s & \mathbf{v}_s & \mathbf{q}_e & \boldsymbol{\omega}_s & \mathbf{r}_{sI} & \mathbf{s}_1 & \mathbf{s}_2\end{bmatrix}_{22\times 1}^T$ (33)

where $\mathbf{r}_s$ and $\mathbf{v}_s$ are governed by the position kinematic and dynamic equations (1) and (4). The attitude error kinematics is governed by

$$\dot{\mathbf{q}}_e = \frac{1}{2}\begin{bmatrix} q_{0e}\varepsilon_{\omega_t} - q_{1e}\omega_{e_x} - q_{2e}\omega_{e_y} - q_{3e}\omega_{e_z} \\ q_{0e}\omega_{e_x} + q_{1e}\varepsilon_{\omega_t} + q_{2e}\omega_{e_z} - q_{3e}\omega_{e_y} \\ q_{0e}\omega_{e_y} - q_{1e}\omega_{e_z} + q_{2e}\varepsilon_{\omega_t} + q_{3e}\omega_{e_x} \\ q_{0e}\omega_{e_z} + q_{1e}\omega_{e_y} - q_{2e}\omega_{e_x} + q_{3e}\varepsilon_{\omega_t} \end{bmatrix} \qquad (34)$$

where $\varepsilon_{\omega_t} = -0.0001$ is added to the quaternion error dynamics since the target body rates are unstable.

Let $\boldsymbol{\xi} = T_I^{B_s} T_{B_t}^I \boldsymbol{\omega}_t$. The angular velocity tracking error $\boldsymbol{\omega}_e$ expressed in the spacecraft body frame can be written as

$$\omega_{e_x} = \omega_{s_x} - \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T \boldsymbol{\xi}\, \omega_{e_y} = \omega_{s_y} - \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \boldsymbol{\xi}\, \omega_{e_z} = \omega_{s_z} - \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \boldsymbol{\xi} \qquad (35)$$

Using Eq. (29) in Eq. (16) yields the state equation for $\boldsymbol{\omega}_s$

$$\begin{aligned}\dot{\boldsymbol{\omega}}_s &= M_3\boldsymbol{\omega}_s + coeff \cdot \mathrm{K}_1 \cdot \mathbf{s}_1 + \left(J_s^{-1} + coeff \cdot M_2\right)\boldsymbol{\Gamma}_s \\ &\quad + \left(coeff \cdot \mathrm{C}_1 - J_s^{-1}\tilde{\omega}_s\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T M_{\psi\eta}^T\right)\cdot \mathbf{s}_2\end{aligned} \qquad (36)$$

where $M_3 = -J_s^{-1}\tilde{\omega}_s J_s - coeff \cdot M_1$; $coeff = J_s^{-1}\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \mathbf{M}_{\psi\boldsymbol{\eta}}^T$

The state-space equations (1), (4), (34), (36), (32), (30), and (31) describe the entire spacecraft dynamics. The $\theta - D$ technique can be employed to integrate the control of position, attitude, and flexible structures.

In the $\theta - D$ formulation, the nonlinear state-space equations need to be written in the form of (23). $F(\boldsymbol{x})$ is chosen to be:

$$\begin{bmatrix} -\tilde{\omega}_s & \mathbf{I}_{33} & \mathbf{0}_{34} & \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{0}_{33} \\ \mathbf{0}_{33} & -\tilde{\omega}_s & \mathbf{0}_{34} & \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{0}_{33} \\ \mathbf{0}_{43} & \mathbf{0}_{43} & \frac{1}{2}Q_1 & \frac{1}{2}Q_2 & \mathbf{0}_{43} & \mathbf{0}_{43} & \mathbf{0}_{43} \\ \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{0}_{34} & M_3 & \mathbf{0}_{33} & coeff \cdot \mathrm{K}_1 & \begin{array}{c} coeff \cdot \mathrm{C}_1 \\ -J_s^{-1}\tilde{\omega}_s\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T M_{\psi\eta}^T \end{array} \\ \mathbf{I}_{33} & \mathbf{0}_{33} & \mathbf{0}_{34} & \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{0}_{33} \\ \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{0}_{34} & \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{I}_{33} \\ \mathbf{0}_{33} & \mathbf{0}_{33} & \mathbf{0}_{34} & M_1 & \mathbf{0}_{33} & -\mathrm{K}_1 & -\mathrm{C}_1 \end{bmatrix}$$

where

$$Q_1 = \begin{bmatrix} \varepsilon_{\omega t} & a\xi & b\xi & c\xi \\ -a\xi & \varepsilon_{\omega t} & -c\xi & b\xi \\ -b\xi & c\xi & \varepsilon_{\omega t} & a\xi \\ -c\xi & -b\xi & a\xi & \varepsilon_{\omega t} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -q_{1e} & -q_{2e} & -q_{3e} \\ q_{0e} & -q_{3e} & q_{2e} \\ q_{3e} & q_{0e} & -q_{1e} \\ -q_{2e} & q_{1e} & q_{0e} \end{bmatrix}$$

and $a = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T, b = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T, c = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$.

The A_0 and $A(\boldsymbol{x})$ are chosen such that $A_0 = F(\boldsymbol{x}(t_0))$, and $A(\boldsymbol{x}) = F(\boldsymbol{x}) - F(\boldsymbol{x}(t_0))$. The control coefficient matrix B is:

$$B(\boldsymbol{x}) = \begin{bmatrix} \mathbf{0}_{33} & \mathbf{0}_{33} \\ \frac{1}{m}\mathbf{I}_{33} & \mathbf{0}_{33} \\ \mathbf{0}_{43} & \mathbf{0}_{43} \\ \mathbf{0}_{33} & J_s^{-1} + coeff \cdot M_2 \\ \mathbf{0}_{33} & \mathbf{0}_{33} \\ \mathbf{0}_{33} & \mathbf{0}_{33} \\ \mathbf{0}_{33} & -M_2 \end{bmatrix} \qquad (37)$$

The cost function is chosen to be in the form of (19).

A satisfactory state weighting matrix is chosen to be:

$$Q = diag\left(\begin{bmatrix} w_{r_s} & w_{v_s} & w_{q_e} & w_{\omega_s} & w_{r_{sI}} & w_{s_1} & w_{s_2}\end{bmatrix}\right) \qquad (38)$$

$w_{r_s} = diag\left(\begin{bmatrix}1 & 100 & 1\end{bmatrix}\right)$; $w_{v_s} = diag\left(\begin{bmatrix}1 & 1 & 1\end{bmatrix}\right)$

$w_{q_e} = diag\left(\begin{bmatrix}1, & 8\times10^{11}, & 3\times10^{11}, & 1\times10^{12}\end{bmatrix}\right)$; $w_{\omega_s} = diag\left(\begin{bmatrix}1\times10^{2} & 1\times10^{3} & 1\times10^{2}\end{bmatrix}\right)$

$w_{r_{sI}} = diag\left(\begin{bmatrix}2\times10^{11} & 4\times10^{11} & 2\times10^{12}\end{bmatrix}\right)$, $w_{s_1} = diag\left(\begin{bmatrix}1\times10^{9} & 3\times10^{16} & 3\times10^{14}\end{bmatrix}\right)$,

$w_{s_2} = diag\left(\begin{bmatrix}1\cdot10^{9} & 5\cdot10^{12} & 5\cdot10^{12}\end{bmatrix}\right)$

The control weighting matrix is chosen to be

$$R = diag\left(\begin{bmatrix} 4\times10^5 & 4\times10^5 & 2\times10^5 & 2\times10^5 & 2\times10^4 & 1\times10^5 \end{bmatrix}\right) \quad (39)$$

In order to perform the command tracking, the $\theta - D$ controller is implemented as an integral servomechanism [12]

$$\boldsymbol{u} = -R^{-1}B^T \sum_{i=0}^{\infty} T_i(\boldsymbol{x},\theta)\theta^i \begin{bmatrix} \mathbf{r}_s - \mathbf{r}_c & \mathbf{v}_s - \mathbf{v}_c & \boldsymbol{q}_e & \boldsymbol{\omega}_e & \mathbf{r}_{sI} - \int \mathbf{r}_c \end{bmatrix}^T \quad (40)$$

where $\mathbf{r}_c$ and $\mathbf{v}_c$ are commanded position and velocity vectors described in the spacecraft body frame $\{\boldsymbol{B}_s\}$. In this study, the first three terms of $\sum_{i=0}^{\infty} T_i(\boldsymbol{x},\theta)\theta^i$ are used in (40) to produce satisfactory results. The parameters in the perturbation matrices D_1 and D_2 are chosen to be $k_1 = k_2 = 1$ and $l_1 = l_2 = 0.01$.

The spacecraft is required to position itself at a constant distance $\boldsymbol{d}_t$ with respect to the target and keep the same velocity $\mathbf{v}_t$ as that of the target. The variables $\boldsymbol{d}_t$ and $\mathbf{v}_t$ are defined in the target body frame $\{\boldsymbol{B}_t\}$. Hence, the commanded position $\mathbf{r}_c$ and velocity $\mathbf{v}_c$ can be obtained by

$$\mathbf{r}_c = T_I^{B_s} T_{B_t}^I \boldsymbol{d}_t, \quad \mathbf{v}_c = T_I^{B_s} T_{B_t}^I \mathbf{v}_t \quad (41)$$

Note that we are designing an optimal controller for this highly nonlinear system with 22 state variables. Since the integrated controller (40) can be obtained in ***closed-form*** via the $\theta - D$ algorithm, this controller possesses a great online computational advantage for real-time implementations.

V. SIMULATIONS AND RESULTS

A 6-DOF simulation of a spacecraft and a tumbling target was established to demonstrate the performance of the integrated optimal control design. The simulation scenario assumes that the target center of mass is located at the origin of the inertial frame with zero velocity.

The target is assumed to be axisymmetric about $\hat{\boldsymbol{b}}_{t_y}$ -axis of the target body frame, i.e. $J_{t_x} = J_{t_z}$.

The 2-1-2 Euler angles, Θ_t, of the tumbling target are given by

$$\Theta_t(\phi_t, \theta_t, \psi_t) = \begin{bmatrix} \dfrac{(1-\Lambda)\Omega_2 t}{\cos\theta_t} & \theta_t & \Lambda\Omega_2 t \end{bmatrix}^T rad \quad (42)$$

where $\Lambda = 1 - J_{t_y} / J_{t_x} = 0.5$; $\Omega_2 = 0.1 rad/s$ is the spin rate of the target; $\theta_t = 0.5 rad$ is the nutation angle.

The spacecraft is assumed to have a constant mass of 3000 kg. The moment of inertia without appendages is assumed to be

$$J_{s_h} = \begin{bmatrix} 3000 & -300 & -500 \\ -300 & 3000 & -400 \\ -500 & -400 & 3000 \end{bmatrix} kg\, m^2 \quad (43)$$

The flexible structure has the parameters of $r = 1\,m; L = 20\,m$; $\rho = 0.04096 kg\cdot m^2$; $E = 2.9661\, m/s^2$; $C = 0.0365 m/s$.

The flexible mode shape function is chosen to be [10]:

$$\phi_p(\bar{x} - r) = 1 - \cos\left[\frac{p\pi(\bar{x}-r)}{L}\right] + \frac{1}{2}(-1)^{p+1}\left[\frac{p\pi(\bar{x}-r)}{L}\right]^2 \quad (44)$$

The spacecraft is initially located at an inertial position of $\begin{bmatrix} 8.0 & 9.0 & 10.0 \end{bmatrix}^T m$ with zero velocity and body rate, and an orientation aligned with the inertial frame. The control objective is to drive the spacecraft to a fixed position of $\boldsymbol{d}_t = \begin{bmatrix} 3.5 & 0.0 & 0.0 \end{bmatrix}^T m$ in the target body frame and its attitude to be coincident with the target body frame. Thus a large angular maneuver as well as a complicated position change has to be performed by the spacecraft.

Figure 3 shows the spacecraft position and attitude trajectories with respect to the commanded trajectories, which are defined in the inertial frame. Figure 4 presents the angular velocity responses. As can be seen, the spacecraft is able to track the command precisely and quickly. Fig. 5 shows the histories of the control forces and torques. A relatively large control forces and torques are required initially to drive the spacecraft to the desired position and attitude quickly. Then, the control efforts decrease rapidly after some oscillations to keep the spacecraft at the desired position and attitude. Figure 6 presents the result of flexible motion. Maximum tip deformation is only about 11 centimeters. These results demonstrate that the satellite with the $\theta - D$ optimal control precisely tracks the tumbling target and effectively suppresses the vibration of the flexible structure as well.

The robustness of the $\theta - D$ controller was also tested by assuming that the diagonal elements of the moment of inertia matrix J_s are perturbed by -50% uncertainties and the off-diagonal elements of J_s are perturbed by -30% uncertainties. Figures 7-8 provide the results under the same initial conditions of the states as the Figures 3-6. Note that the controller is designed based on the nominal moment of inertia. As can be seen, the position and attitude tracking are still so good that it is not distinguishable from Fig. 3. The only discernible difference is the control torque, which are smaller than the nominal case because the actual moment of inertia is smaller than the nominal one.

VI. CONCLUSIONS

In this paper, nonlinear control of spacecraft position and attitude to approach a tumbling target is addressed using the $\theta - D$ technique. The 6-DOF rigid body dynamics coupled with the flexible motion is considered in one unified optimal control framework. The $\theta - D$ technique can provide a closed-form solution to the resultant nonlinear optimal control problem and offers a great implementation advantage. The simulation results demonstrate that the controller is able to drive the spacecraft to the desired position and attitude accurately in close proximity of a tumbling target so that subsequent service operations can be conducted. The controller is also shown to be robust to moment of inertia uncertainties.

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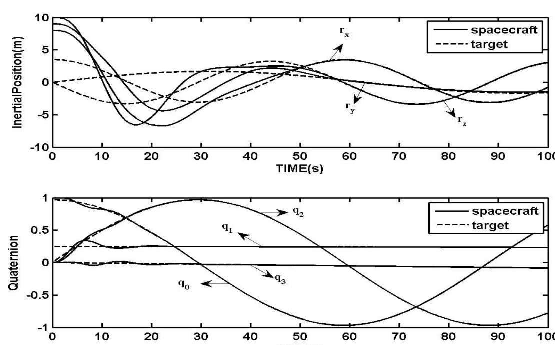

Fig. 3: Position and attitude tracking

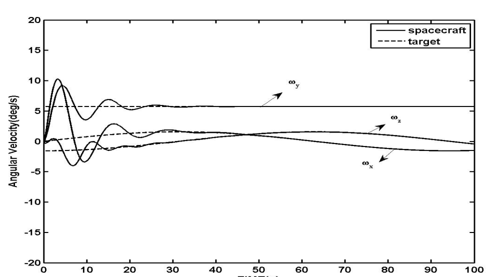

Fig. 4: Angular velocity tracking

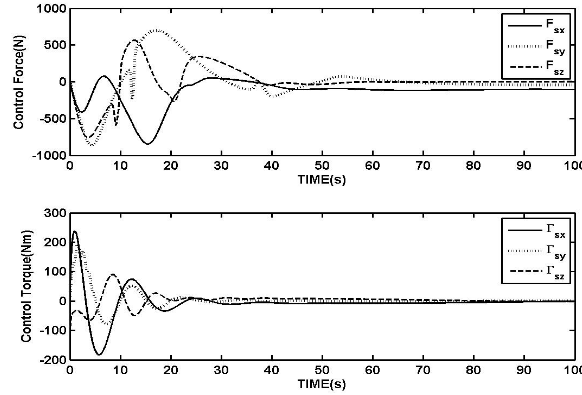

Fig. 5: Control forces and control torques

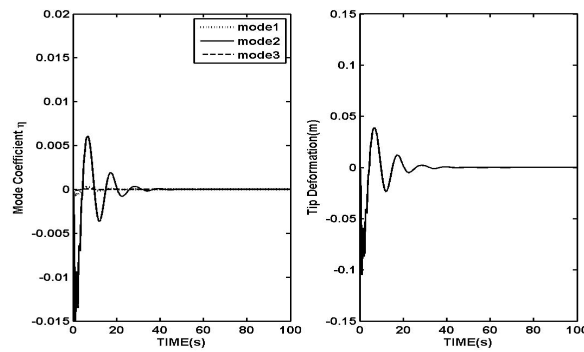

Fig. 6: Mode coefficient η and tip deformation $w(t)$

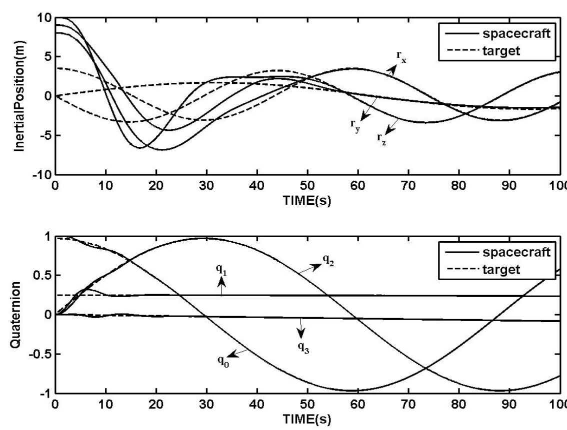

Fig. 7: Position and attitude tracking with moment of inertia uncertainties

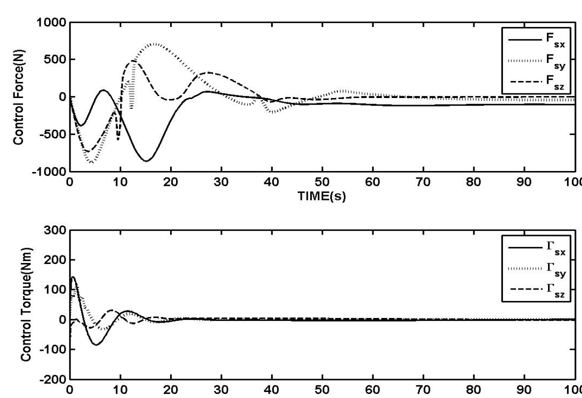

Fig. 8: Control forces and torques with moment of inertia uncertainties