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Robust fault reconstruction using multiple sliding mode observers in cascade: development and design

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Abstract— **This paper presents new results in observer-based robust fault reconstruction for uncertain systems using cascaded sliding mode observers. Signals from an observer are used as the output of a fictitious system whose input is the fault. Then an observer is implemented for the fictitious system. This process is repeated until the first Markov parameter of the fictitious system is full rank. The result is that robust fault reconstruction can be carried out for a wider class of systems. An example verifies the effectiveness of the proposed scheme.**

I. INTRODUCTION

Fault reconstruction [6], [5], [15], is an important area of research activity. However, most fault reconstruction schemes are designed about a model, which usually does not perfectly represent the system – as certain dynamics are either unknown or do not fit exactly into the framework of the model. These dynamics are usually represented as a class of disturbances within the model [14]. The disturbances corrupt the reconstruction, and could produce a nonzero reconstruction when there are no faults, or worse, mask the effect of a fault. Therefore, schemes need to be designed so that the reconstruction is robust to disturbances. Edwards *et al.*[6], [5] used a sliding mode observer [4] to reconstruct faults, in which there was no explicit consideration of the disturbances or uncertainty. Tan & Edwards [18] built on the work in [5], [6] and presented a design algorithm for the observer, using Linear Matrix Inequalities (LMIs) [2], such that the $\mathcal{L}_2$ gain from the disturbances to the fault reconstruction is minimized. Saif & Guan [15] aggregated the faults and disturbances to form a new 'fault' vector and used a linear unknown input observer to reconstruct the new 'fault' vector. A necessary condition in [6], [5], [18], [15] is that the first Markov parameter of the system connecting the fault to the output must be full rank. This limits the class of systems to which [6], [5], [18], [15] are applicable.

Recently, there have been developments in fault reconstruction for systems whose first *Markov parameter is not full rank*. Floquet & Barbot [7] transformed the system into an 'output information' form such that existing sliding mode observer techniques can be implemented to estimate the states in finite time and reconstruct the faults. However, in [7] there is no explicit consideration of disturbances or uncertainty. Higher order sliding mode schemes have been suggested by [1], [3], [9]. The work in [9] uses the concept of 'strong observability' together with higher order sliding mode observers. Strong observability has also been exploited in [1] using a hierarchy of observers. Chen & Saif used a bank of high-order sliding-mode differentiators to obtain derivatives of the outputs and then estimates the faults from these signals [3]. Floquet *et.al* suggest the use of exact differentiators to generate derivatives of the measurements to 'create' additional outputs [8] to circumvent relative degree assumptions. However all the work in [7], [3], [8], [1], [9] does not consider uncertainty – unless the faults and uncertainty are augmented and treated as 'unknown inputs'. In this case the number of disturbances plus faults must not exceed the number of outputs. This results in strong constraints which must be satisfied, and hence a smaller class of systems for which the results are applicable.

Ng *et al.*[13] extended the work of Tan & Edwards [18] exploiting two sliding mode observers in cascade; known signals from the first observer were considered as outputs of a 'fictitious' system which has a full rank (first) Markov parameter; then a second sliding mode observer is designed based on the fictitious system to reconstruct the fault. This enables robust fault reconstruction for systems where the number of disturbances and faults exceed the number of outputs (which cannot be achieved by [7], [3]).

This paper builds on the work of [13] by using multiple observers in cascade. The use of sliding mode observers in a cascade framework for unknown input estimation is not new [16], [11], [10], [12]. However the work in [10] assumes full state measurement, whilst [11] does not consider any external disturbances. Although [16] considers both faults and uncertainties, they are aggregated and are both treated as unknown inputs – this introduces considerable conservatism. In this paper the faults and disturbances are treated differently. Using similar techniques as in [13], measurable signals from an observer are used as outputs of a fictitious system; the next observer is designed for the fictitious system and the known signals from this observer are used as outputs of another fictitious system. The process is repeated until a fictitious system whose (first) Markov parameter is full rank is obtained. The technique proposed in [18] is then used to robustly reconstruct the fault. This results in robust fault reconstruction applicable to a wider class of systems than in [13]. The final fictitious system is found to be in the same framework as [18] which minimizes the $\mathcal{L}_2$ gain from the disturbances to the fault reconstruction; this enables the algorithm to be applicable for systems when the number of outputs are less than the sum of faults and disturbance channels. In addition, it is also found that the design of previous observers do not affect the sliding motion of the final observer, which implies that the $\mathcal{L}_2$ gain from the disturbances to the fault reconstruction is not affected.

II. THE ROBUST FAULT RECONSTRUCTION SCHEME

Consider the following system

$$\dot{x}^1 = A^1 x^1 + M^1 f^1 + Q^1 \xi^1 \qquad (1)$$

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$$y^1 = C^1 x^1 \tag{2}$$

where $x^1 \in \mathbb{R}^{n^1}$ are the states, $y^1 \in \mathbb{R}^p$ are the outputs and $f^1 \in \mathbb{R}^q$ are unknown faults. The signals $\xi^1 \in \mathbb{R}^h$ are uncertainties or dynamics that represent the mismatch between the linear model (1) and the real plant. Assume without loss of generality $rank(M^1) = q, rank(C^1) = p$ and $rank(C^1 M^1) = \bar{r}^1 < q$, implying that $\bar{r}^1 \leq \min\{p, q\}$. Since $rank(C^1) = p$, then C^1 can be written without loss of generality in the form $C^1 = \left[\begin{array}{cc} 0 & I_p \end{array}\right]$.

The objective is to reconstruct f^1 whilst minimizing the effects of ξ^1 on the fault reconstruction. If $h + q > p$ and $\bar{r}^1 < q$, then the approaches suggested in [6], [5], [15], [18], [16], [1], [3], [9], [7], [8] are *not applicable*. In this situation, this paper proposes the cascade observer scheme shown in Figure 1.

Firstly partition the matrices from (1) as

$$A^1 = \left[\begin{array}{cc} A_1^1 & A_2^1 \\ A_3^1 & A_4^1 \end{array}\right], \; M^1 = \left[\begin{array}{c} M_1^1 \\ M_2^1 \end{array}\right], \; Q^1 = \left[\begin{array}{c} Q_1^1 \\ Q_2^1 \end{array}\right] \begin{array}{l} \updownarrow n^1 - p \\ \updownarrow p \end{array}$$

where A_1^1 is square. Since by assumption $C^1 = \left[\begin{array}{cc} 0 & I_p \end{array}\right]$ and $rank(C^1 M^1) = \bar{r}^1$, then it follows that $rank(M_2^1) = \bar{r}^1$. In the representation above, Q^1 has no particular structure.

A. Summary of fault reconstruction algorithm

The fault reconstruction method proposed in this paper can be summarized in the following steps. Set $i = 1$ and enter the following algorithm:

1) Consider the generic uncertain faulty system

$$\dot{x}^i = A^i x^i + M^i f^i + Q^i \xi^i \tag{3}$$
$$y^i = C^i x^i \tag{4}$$

and define $\bar{r}^i := rank(C^i M^i)$.

   a) If $rank(C^i M^i) = rank(M^i)$, set $i = k$ and jump to step 7.
   b) If $rank(C^i M^i) < rank(M^i)$ and $i = n^1$, then the method in this paper cannot be used to reconstruct the faults[1] and terminate the algorithm.

If neither (a) nor (b) are satisfied, proceed to the next step.

2) For the case when $i = 1$, define the following

$$\bar{M}_{11}^0 := M_1^1, \bar{M}_{12}^0 := M_2^1, m^1 := p, \bar{r}^0 := 0 \tag{5}$$

$$\tilde{A}_{13}^0 := A_3^1, \tilde{A}_{11}^0 := A_1^1, \; \bar{A}_\Omega^0 = \alpha^0 = \bar{M}_{22}^0 = \phi \tag{6}$$

where ϕ is the empty matrix. Then A^i and M^i can be expanded as

$$\left[\begin{array}{cc|ccc} \bar{A}_\Omega^{i-1} & 0 & \star & 0 & 0 \\ \star & \tilde{A}_{11}^{i-1} & \star & 0 & 0 \\ \hline \star & \tilde{A}_{13}^{i-1} & \star & 0 & 0 \\ \star & 0 & \star & -\alpha^{i-1} I & 0 \\ \star & \star & \star & 0 & -\alpha^{i-1} I \end{array}\right] \begin{array}{l} \updownarrow (i-1)h \\ \updownarrow n^i - p - (i-1)h \\ \updownarrow m^i \\ \updownarrow p - m^i - \bar{r}^{i-1} \\ \updownarrow \bar{r}^{i-1} \end{array} \tag{7}$$

and

$$M^i = \left[\begin{array}{cc} 0 & 0 \\ \bar{M}_{11}^{i-1} & 0 \\ \hline \bar{M}_{12}^{i-1} & 0 \\ 0 & 0 \\ 0 & \alpha^{i-1} \bar{M}_{22}^{i-1} \end{array}\right] \begin{array}{l} \updownarrow (i-1)h \\ \updownarrow n^i - p - (i-1)h \\ \updownarrow m^i \\ \updownarrow p - \bar{r}^{i-1} - m^i \\ \updownarrow \bar{r}^{i-1} \end{array} \tag{8}$$

3) Define two orthogonal matrices $D^i \in \mathbb{R}^{m^i \times m^i}$ and $T_2^i \in \mathbb{R}^{(q - \bar{r}^{i-1}) \times (q - \bar{r}^{i-1})}$ such that

$$\left[\begin{array}{cc} I & 0 \\ 0 & (D^i)^{-1} \end{array}\right] \left[\begin{array}{c} \bar{M}_{11}^{i-1} \\ \bar{M}_{12}^{i-1} \end{array}\right] (T_2^i)^{-1} = \left[\begin{array}{cc} M_{11}^i & M_{12}^i \\ \hline 0 & 0 \\ 0 & M_{22}^i \end{array}\right] \tag{9}$$

where $M_{22}^i \in \mathbb{R}^{r^i \times r^i}$ is square and invertible. Define $T_1^i := T_{11}^i \times diag\left\{I_{n^i - p}, (D^i)^{-1}, I_{p - m^i}\right\}$ where T_{11}^i is defined as

$$\left[\begin{array}{cc|c|ccc} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & -M_{12}^i (M_{22}^i)^{-1} & 0 & 0 \\ \hline 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \begin{array}{l} \updownarrow h \\ \updownarrow n^i - p - h \\ \updownarrow m^i - r^i \\ \updownarrow p - m^i - \bar{r}^{i-1} \\ \updownarrow r^i \\ \updownarrow \bar{r}^{i-1} \end{array} \tag{10}$$

Define

$$\tilde{A}_3^i := (D^i)^{-1} \tilde{A}_{13}^{i-1} = \left[\begin{array}{c} \tilde{A}_{31}^i \\ \tilde{A}_{32}^i \end{array}\right] \begin{array}{l} \updownarrow m^i - r^i \\ \updownarrow r^i \end{array} \tag{11}$$

and

$$\tilde{A}_1^i := \tilde{A}_{11}^{i-1} - M_{12}^i (M_{22}^i)^{-1} \tilde{A}_{32}^i \tag{12}$$

Perform the coordinate transformation

$$x^i \mapsto T_1^i x^i, \; f^i \mapsto f^{i+1} := \underbrace{\left[\begin{array}{cc} T_2^i & 0 \\ 0 & I_{\bar{r}^{i-1}} \end{array}\right]}_{T_f^i} f^i \tag{13}$$

then the matrix triple (A^i, M^i, C^i) will have the form

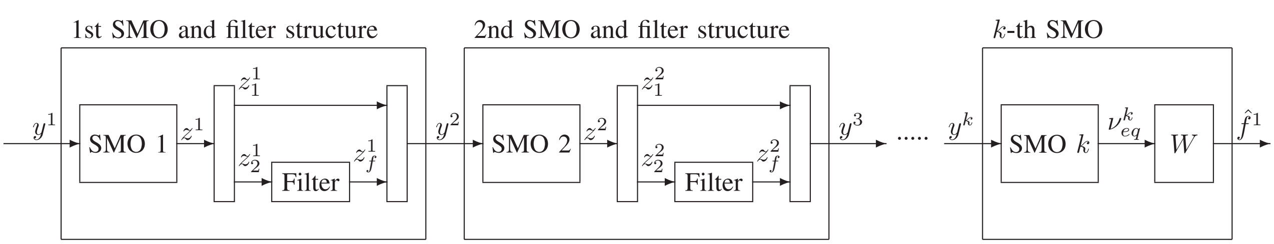

Fig. 1. The proposed FDI scheme formed from a cascaded sliding mode observer/filter structure

[1]The justification of this will be given in Proposition 2 in the appendix.

$$\begin{bmatrix} A_1^i & A_2^i \\ A_3^i & A_4^i \end{bmatrix} = \left[\begin{array}{cc|c} \bar{A}_\Omega^{i-1} & 0 & \star \\ \star & \tilde{A}_1^i & \star \\ \hline \star & \tilde{A}_{31}^i & \star \\ \star & 0 & \star \\ \star & \star & \star \end{array}\right] \begin{array}{l} \updownarrow (i-1)h \\ \updownarrow n^i-p-(i-1)h \\ \updownarrow m^i-r^i \\ \updownarrow p-m^i-\bar{r}^{i-1} \\ \updownarrow \bar{r}^i \end{array} \tag{14}$$

$$\begin{bmatrix} M_1^i \\ M_2^i \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ M_{11}^i & 0 \\ \hline 0 & 0 \\ 0 & \bar{M}_{22}^i \end{array}\right] \begin{array}{l} \updownarrow (i-1)h \\ \updownarrow n^i-p-(i-1)h \\ \updownarrow p-\bar{r}^i \\ \updownarrow \bar{r}^i \end{array} \tag{15}$$

$$C^i = \begin{bmatrix} 0 & C_2^i \end{bmatrix} \tag{16}$$

where

$$\bar{M}_{22}^i = \begin{bmatrix} M_{22}^i & 0 \\ 0 & \alpha^{i-1}\bar{M}_{22}^{i-1} \end{bmatrix} \begin{array}{l} \updownarrow r^i \\ \updownarrow \bar{r}^{i-1} \end{array}$$

and

$$C_2^i = diag\left\{D^i, I_{p-m^i}\right\} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{array}{l} \updownarrow m^i-r^i \\ \updownarrow p-\bar{r}^{i-1}-m^i \\ \updownarrow r^i \\ \updownarrow \bar{r}^{i-1} \end{array}$$

4) Assume ξ^i satisfies

$$\dot{\xi}^i = A_\Omega^i \xi^i + B_\Omega^i \xi^{i+1} \tag{17}$$

Augment (17) with (3) to obtain

$$\dot{\bar{x}}^i = \bar{A}^i \bar{x}^i + \bar{M}^i f^{i+1} + \bar{Q}^i \xi^{i+1} \tag{18}$$

$$y^i = \bar{C}^i \bar{x}^i \tag{19}$$

where

$$\bar{A}^i = \left[\begin{array}{ccc|c} A_\Omega^i & 0 & 0 & 0 \\ \star & \bar{A}_\Omega^{i-1} & 0 & \star \\ \star & \star & \tilde{A}_1^i & \star \\ \hline \star & \star & \tilde{A}_{31}^i & \star \\ \star & \star & 0 & \star \\ \star & \star & \star & \star \end{array}\right] \begin{array}{l} \updownarrow h \\ \updownarrow (i-1)h \\ \updownarrow n^i-p-(i-1)h \\ \updownarrow m^i-r^i \\ \updownarrow p-m^i-\bar{r}^{i-1} \\ \updownarrow \bar{r}^i \end{array}$$

$$\bar{M}^i = \left[\begin{array}{cc} 0 & 0 \\ M_{11}^i & 0 \\ \hline 0 & 0 \\ 0 & \bar{M}_{22}^i \end{array}\right] \begin{array}{l} \updownarrow ih \\ \updownarrow n^1-p-(i-1)h \\ \updownarrow p-\bar{r}^i \\ \updownarrow \bar{r}^i \end{array}$$

Define

$$\bar{A}_\Omega^i := \begin{bmatrix} A_\Omega^i & 0 \\ \star & \bar{A}_\Omega^{i-1} \end{bmatrix}$$

then $\bar{A}^i$ can be re-expressed as

$$\bar{A}^i = \left[\begin{array}{cc|c} \bar{A}_\Omega^i & 0 & 0 \\ \star & \tilde{A}_1^i & \star \\ \hline \bar{Q}_{21}^i & \tilde{A}_{31}^i & \star \\ \star & 0 & \star \\ \star & \star & \star \end{array}\right] \begin{array}{l} \updownarrow ih \\ \updownarrow n^i-p-(i-1)h \\ \updownarrow m^i-r^i \\ \updownarrow p-m^i-\bar{r}^{i-1} \\ \updownarrow \bar{r}^i \end{array} \tag{20}$$

5) Define $m^{i+1} := rank(\tilde{A}_{31}^i)$. If $m^{i+1} < q-\bar{r}^i$, then the fault can never be fully reconstructed[2] and terminate the algorithm. Otherwise, proceed with the following:

[2]The justification for this will be given in Proposition 3 in the appendix.

Let U_1^i and U_2^i be invertible matrices of dimension $m^i - r^i$ and $n^i - p - (i-1)h$ respectively such that

$$U_1^i \tilde{A}_{31}^i (U_2^i)^{-1} = \begin{bmatrix} 0 & I_{m^{i+1}} \\ 0 & 0 \end{bmatrix}, \; U_1^i \bar{Q}_{21}^i = \begin{bmatrix} \bar{Q}_{211}^i \\ \bar{Q}_{212}^i \end{bmatrix} \tag{21}$$

where $\bar{Q}_{211}^i, \bar{Q}_{212}^i$ are matrices with no particular structure. Also partition

$$U_2^i \tilde{A}_1^i (U_2^i)^{-1} = \begin{bmatrix} \tilde{A}_{11}^i & \tilde{A}_{12}^i \\ \tilde{A}_{13}^i & \tilde{A}_{14}^i \end{bmatrix} \begin{array}{l} \updownarrow n^i-p-m^{i+1}-(i-1)h \\ \updownarrow m^{i+1} \end{array} \tag{22}$$

Introduce the following transformation $\bar{x}^i \mapsto \bar{T}^i \bar{x}^i$ where $\bar{T}^i := diag\left\{I_{ih}, U_2^i, U_1^i, I_{p+r^i-m^i}\right\} \bar{T}_1^i$ with

$$\bar{T}_1^i := \left[\begin{array}{ccc|c} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \bar{Q}_{211}^i & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right] \begin{array}{l} \updownarrow ih \\ \updownarrow n^i-p-(i-1)h-m^{i+1} \\ \updownarrow m^{i+1} \\ \updownarrow p \end{array} \tag{23}$$

Then $\bar{A}^i, \bar{M}^i, \bar{C}^i$ are transformed to be

$$\begin{bmatrix} \bar{A}_1^i & \bar{A}_2^i \\ \bar{A}_3^i & \bar{A}_4^i \end{bmatrix} = \left[\begin{array}{ccc|c} \bar{A}_\Omega^i & 0 & 0 & \star \\ \star & \tilde{A}_{11}^i & \tilde{A}_{12}^i & \star \\ \star & \tilde{A}_{13}^i & \tilde{A}_{14}^i & \star \\ \hline 0 & 0 & I & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{array}\right] \begin{array}{l} \updownarrow ih \\ \updownarrow n^i-p-m^{i+1}-(i-1)h \\ \updownarrow m^{i+1} \\ \updownarrow m^{i+1} \\ \updownarrow p-m^{i+1}-\bar{r}^i \\ \updownarrow \bar{r}^i \end{array} \tag{24}$$

$$\begin{bmatrix} \bar{M}_1^i \\ \bar{M}_2^i \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ \bar{M}_{11}^i & 0 \\ \bar{M}_{12}^i & 0 \\ \hline 0 & 0 \\ 0 & \bar{M}_{22}^i \end{array}\right] \begin{array}{l} \updownarrow ih \\ \updownarrow n^i-p-m^{i+1}-(i-1)h \\ \updownarrow m^{i+1} \\ \updownarrow p-\bar{r}^i \\ \updownarrow \bar{r}^i \end{array} \tag{25}$$

$$\bar{C}^i = \begin{bmatrix} 0 & \bar{C}_2^i \end{bmatrix}, \; |\bar{C}_2^i| \neq 0 \tag{26}$$

Finally partition

$$\bar{A}_3^i = \begin{bmatrix} \bar{A}_{31}^i \\ \bar{A}_{32}^i \end{bmatrix} \begin{array}{l} \updownarrow m^{i+1} \\ \updownarrow p-m^{i+1} \end{array} \tag{27}$$

which from (24) results in $\bar{A}_{31}^i = \begin{bmatrix} 0 & I_{m^{i+1}} \end{bmatrix}$.

6) A sliding mode observer [4] for the system (18) - (19) is

$$\dot{\hat{\bar{x}}}^i = \bar{A}^i \hat{\bar{x}}^i - \bar{G}_l^i \bar{e}_y^i + \bar{G}_n^i \bar{\nu}^i \tag{28}$$

$$\hat{y}^i = \bar{C}^i \bar{x}^i \tag{29}$$

where $\hat{\bar{x}}^i \in \mathbb{R}^{\bar{n}^i}$ is the estimate of $\bar{x}^i$ and $\bar{e}_y^i = \hat{y}^i - y^i$ is the output estimation error. The matrices $\bar{G}_l^i, \bar{G}_n^i \in \mathbb{R}^{\bar{n}^i \times p}$ are observer gains (to be designed). In the coordinate system of (24) - (26), $\bar{G}_n^i$ will be assumed to have the structure

$$\bar{G}_n^i = \begin{bmatrix} -\bar{L}^i \\ I_p \end{bmatrix} (\bar{P}_o^i \bar{C}_2^i)^{-1}, \; \bar{L}^i = \begin{bmatrix} \bar{L}_o^i & 0 \end{bmatrix} \tag{30}$$

where $\bar{P}_o^i \in \mathbb{R}^{p \times p}$ is a symmetric positive definite (s.p.d.) matrix, $\bar{L}^i \in \mathbb{R}^{(\bar{n}^i-p) \times p}$ and $\bar{L}_o^i \in \mathbb{R}^{(\bar{n}^i-p) \times m^{i+1}}$. The term $\bar{\nu}^i$ is a nonlinear discontinuous term defined by

$$\bar{\nu}^i = -\bar{\rho}^i \frac{\bar{e}_y^i}{\|\bar{e}_y^i\|}, \; \bar{\rho}^i \in \mathbb{R}_+ \tag{31}$$

Define $\bar{e}^i := \hat{\bar{x}}^i - \bar{x}^i$ as the state estimation error, and combine (18), (19) and (28) - (29) to obtain the error system

$$\dot{\bar{e}}^i = (\bar{A}^i - \bar{G}_l^i\bar{C}^i)\bar{e}^i + \bar{G}_n^i\bar{\nu}^i - \bar{M}^i f^{i+1} - \bar{Q}^i\xi^{i+1} \quad (32)$$

Proposition 1: Consider a s.p.d. matrix

$$\bar{P}^i = \begin{bmatrix} \bar{P}_1^i & \bar{P}_1^i\bar{L}^i \\ (\bar{P}_1^i\bar{L}^i)^T & (\bar{C}_2^i)^T\bar{P}_o^i\bar{C}_2^i + (\bar{L}^i)^T\bar{P}_1^i\bar{L}^i \end{bmatrix} \quad (33)$$

where $\bar{P}_1^i \in \mathbb{R}^{(\bar{n}^i-p)\times(\bar{n}^i-p)}$. Assume that

$$\bar{P}^i(\bar{A}^i - \bar{G}_l^i\bar{C}^i) + (\bar{A}^i - \bar{G}_l^i\bar{C}^i)^T\bar{P}^i < 0 \quad (34)$$

Then, for a large enough $\bar{\rho}^i$ in (31), an ideal sliding motion takes place on $\bar{\mathbb{S}}^i = \left\{\bar{e}^i : \bar{C}^i\bar{e}^i = 0\right\}$ in finite time.

Proof: See Lemma 1 and Proposition 2 from Tan & Edwards [18]. ■

Apply a change of coordinates T_L^i to the triple in (24) - (26) and $\bar{G}_n^i$ in (30) where

$$T_L^i := \begin{bmatrix} I_{\bar{n}^i-p} & \bar{L}^i \\ 0 & \bar{C}_2^i \end{bmatrix}$$

then the matrices $\bar{A}^i, \bar{M}^i, \bar{C}^i, \bar{Q}^i$ from (24) - (26) and $\bar{G}_n^i$ are transformed to have the structures

$$\bar{A}^i \rightarrow \begin{bmatrix} \bar{A}_1^i + \bar{L}_o^i\bar{A}_{31}^i & \star \\ \bar{C}_2^i\bar{A}_3^i & \star \end{bmatrix}, \ \bar{M}^i \rightarrow \begin{bmatrix} \bar{M}_1^i \\ \bar{C}_2^i\bar{M}_2^i \end{bmatrix} \quad (35)$$

$$\bar{C}^i \rightarrow \begin{bmatrix} 0 & I_p \end{bmatrix}, \ \bar{Q}^i \rightarrow \begin{bmatrix} \bar{Q}_1^i \\ 0 \end{bmatrix}, \ \bar{G}_n^i \rightarrow \begin{bmatrix} 0 \\ (\bar{P}_o^i)^{-1} \end{bmatrix} \quad (36)$$

Assume that a sliding motion is taking place on $\bar{\mathbb{S}}^i$ so that $\bar{e}_y^i = \dot{\bar{e}}_y^i = 0$, then (32) can be partitioned in the new coordinates associated with (35) - (36) as

$$\begin{aligned} \dot{\bar{e}}_1^i &= (\bar{A}_1^i + \bar{L}_o^i\bar{A}_{31}^i)\bar{e}_1^i - \bar{M}_1^i f^{i+1} - \bar{Q}_1^i\xi^{i+1} && (37) \\ 0 &= \bar{C}_2^i\bar{A}_3^i\bar{e}_1^i - \bar{C}_2^i\bar{M}_2^i f^{i+1} + (\bar{P}_o^i)^{-1}\bar{\nu}_{eq}^i && (38) \end{aligned}$$

where $\bar{\nu}_{eq}^i$ is the equivalent output error injection required to maintain a sliding motion [6], [5] and can be approximated to any degree of accuracy [6] by replacing $\bar{\nu}^i$ with

$$\bar{\nu}^i = -\bar{\rho}^i\frac{\bar{e}_y^i}{\|\bar{e}_y^i\| + \bar{\delta}^i} \quad (39)$$

where $\bar{\delta}^i$ is a positive scalar. As the term $\bar{e}_y^i$ is a measurable signal, the signal $\bar{\nu}_{eq}^i$ is computable online and is available for use in an online FDI scheme [6], [5].

Define $w^i := -\bar{e}_1^i$ and re-arrange (37) - (38) to obtain

$$\dot{w}^i = (\bar{A}_1^i + \bar{L}_o^i\bar{A}_{31}^i)w^i + \bar{M}_1^i f^{i+1} + \bar{Q}_1^i\xi^{i+1} \quad (40)$$

$$(\bar{P}_o^i\bar{C}_2^i)^{-1}\bar{\nu}_{eq}^i = \bar{A}_3^i w^i + \bar{M}_2^i f^{i+1} \quad (41)$$

Define

$$z^i := (\bar{P}_o^i\bar{C}_2^i)^{-1}\bar{\nu}_{eq}^i = \begin{bmatrix} z_1^i \\ z_2^i \end{bmatrix} \begin{matrix} \updownarrow m^{i+1} \\ \updownarrow p-m^{i+1} \end{matrix}$$

Note, as argued above, z_1^i and z_2^i are available in real time. Substituting for $\bar{A}_3^i$ from (27) results in

$$\begin{aligned} z_1^i &= \begin{bmatrix} 0 & I_{m^{i+1}} \end{bmatrix} w^i && (42) \\ z_2^i &= \bar{A}_{32}^i w^i + \begin{bmatrix} 0 & 0 \\ 0 & \bar{M}_{22}^i \end{bmatrix} f^{i+1} && (43) \end{aligned}$$

Define a signal z_f^i, that is also available in real time, as an output from a stable filter

$$\dot{z}_f^i := -\alpha^i z_f^i + \alpha^i z_2^i \quad (44)$$

where $\alpha^i \in \mathbb{R}_+$. From (43) and (44):

$$\dot{z}_f^i = -\alpha^i z_f^i + \alpha^i\bar{A}_{32}^i w^i + \begin{bmatrix} 0 & 0 \\ 0 & \alpha^i\bar{M}_{22}^i \end{bmatrix} f^{i+1} \quad (45)$$

Combining (40), (42) and (45) the following state-space system

$$\dot{x}^{i+1} = A^{i+1}x^{i+1} + M^{i+1}f^{i+1} + Q^{i+1}\xi^{i+1} \quad (46)$$

$$y^{i+1} = C^{i+1}x^{i+1} \quad (47)$$

can be obtained where

$$x^{i+1} := \begin{bmatrix} w^i \\ z_f^i \end{bmatrix}, \ y^{i+1} := \begin{bmatrix} z_1^i \\ z_f^i \end{bmatrix}$$

$$C^{i+1} := \begin{bmatrix} 0 & I_p \end{bmatrix} \quad (48)$$

and

$$\begin{aligned} A^{i+1} &:= \begin{bmatrix} \bar{A}_1^i + \bar{L}_o^i\bar{A}_{31}^i & 0 \\ \alpha^i\bar{A}_{32}^i & -\alpha^i I_{p-m^{i+1}} \end{bmatrix} && (49) \\ M^{i+1} &:= \begin{bmatrix} \bar{M}_1^i \\ \begin{bmatrix} 0 & 0 \\ 0 & \alpha^i\bar{M}_{22}^i \end{bmatrix} \end{bmatrix} && (50) \\ Q^{i+1} &:= \begin{bmatrix} \bar{Q}_1^i \\ 0 \end{bmatrix} && (51) \end{aligned}$$

Notice that (46) is in the form of (1) and C^{i+1} and C^i have the same structure. It is clear that $f^{i+1} \in \mathbb{R}^q, \xi^{i+1} \in \mathbb{R}^h$. Let $x^{i+1} \in \mathbb{R}^{n^{i+1}}, y^{i+1} \in \mathbb{R}^p$ and define $\bar{r}^{i+1} := rank(C^{i+1}M^{i+1})$. Note that $\bar{r}^{i+1} \leq q$. It can be seen that

$$n^{i+1} = n^i + h - m^{i+1} \quad (52)$$

Increment the counter i by 1 and return to step 1.

7) Since $rank(C^kM^k) = rank(M^k)$, then the robust fault reconstruction approach from [18] may be adopted to estimate f^k, which minimizes the effect of the disturbance ξ^k. Define $\hat{f}^k$ to be the estimate of f^k, then the reconstruction of f^1 can be obtained from

$$\hat{f}^1 := (T_f^k)^{-1}...(T_f^2)^{-1}(T_f^1)^{-1}\hat{f}^k \quad (53)$$

where the T_f^i are defined in (13).

Key observation: Notice from the structure of A^i in (7), the matrix $\bar{L}_o^{i-1}$ appears only in the last p columns of A^i. From the structure of C^i in (48), it is clear that $\bar{L}_o^{i-1}$ affects only the p output states of x^i, and hence $\bar{L}_o^{i-1}$ will not affect the reduced order sliding motion of observer i and also all subsequent observers. From [18], the quality of the fault reconstruction depends on the sliding motion of observer k, which is independent of $\bar{L}_o^i$ from previous observers. Hence, only the design of the k-th observer will affect the quality of the fault reconstruction, and only the k-th observer needs to be designed using the method in [18]; all previous observers can be designed using simpler methods, such as in [17].

III. Design example

The method proposed in this paper will now be demonstrated using a model of a 2-cart system. The first cart is connected to a rigid wall via a damper, and is connected to a second cart by a spring. An external force is then applied to the second cart via an actuator. Assume both carts have a nominal mass of $a = 1\ kg$, the damper constant $b = 2\ Ns/m$ and the spring constant $c = 1\ N/m$. Denote x_1, x_2 respectively as the displacement of carts 1 and 2, and u as the applied force. The equations of motion are therefore

$$\ddot{x}_1 = -\frac{c}{a}x_1 - \frac{b}{a}\dot{x}_1 + \frac{c}{a}x_2 \quad (54)$$

$$\ddot{x}_2 = \frac{c}{a}x_1 - \frac{c}{a}x_2 + \frac{1}{a}u \quad (55)$$

Assume that the positions of both carts are measurable and that the force on the second cart is achieved from the force command via an actuator modelled as a first order lag with a time constant $\tau = 0.2$. The states are the force, velocity of the first cart, velocity of the second cart, position of the first cart and position of the second cart. Assume the actuator is potentially faulty.

Further suppose that the spring and damper constants are imprecisely known; the actual values being $1.02\ N/s$ and $1.8\ Nm/s$ respectively. Hence the state equation becomes

$$\dot{x}^1 = (A^1 + \triangle A^1)x + M^1 f \quad (56)$$

where $\triangle A^1$ is the discrepancy between the known matrix A^1 and its actual value. The first, fourth and fifth rows of the system matrix do not contain any uncertainty due to the nature of the state equations. Hence, any parametric uncertainty will appear in the second and third and fourth rows of A^1. Equation (56) can be placed in the framework of (1) by writing $\triangle A^1 x^1 = Q^1\xi^1$ where $Q^1 \in \mathbb{R}^{5\times 2}$.

The disturbance ξ^1 will be generated by the states x^1, which are in turn generated by the fault f^1. Notice that the method in [7] cannot be used on this system as there is no consideration of the disturbance ξ^1. If the signals f^1 and ξ^1 are augmented to form a new 'fault' vector, as in [15], this results in the new 'fault' vector having 3 components. The number of outputs in this system is only 2, and hence the method in [7] is still not applicable. It can be established that $n^1 = 5, p = 2, q = 1, h = 2, \bar{r}^1 = 0$.

A. Design of observer 1

Here the filter matrices that describe the characteristics of ξ^1 are chosen as $\bar{A}_\Omega^1 = -10I_2, \bar{B}_\Omega^1 = 10I_2$, and an augmented system of dimension $\bar{n}^1 = n^1 + h = 7$ is produced (as in (18)). For this example it can be shown that $m^2 = 2$.

The gains of the first sliding mode observer $\bar{G}_l^1, \bar{G}_n^1$ were designed using the method in [17], based on a sub-optimal Linear Quadratic Gaussian (LQG) approach. For full details of the algorithm, see [17]. In this design the weighting matrices have been chosen as $\bar{W}^1 = 0.01I_7, \bar{V}^1 = I_2$ and the corresponding observer parameters $\bar{G}_l^1 \in \mathbb{R}^{7\times 2}$, $\bar{P}_o^1 \in \mathbb{R}^{2\times 2}$ and $\bar{L}^1 \in \mathbb{R}^{5\times 2}$ have been obtained from the LMI solver. Then $\bar{G}_n^1$ can be calculated using $\bar{P}_o^1$ and $\bar{L}^1$.

Since $p - m^2 = 0$ (because $C^1M^1 = 0$), the filter scalar α^1 does not exist. It follows that the system associated with the second observer will be of order $n^2 = \bar{n}^1 - m^2 = 5$ and the number of outputs are $p = 2$. The system matrices for the second observer A^2, M^2, C^2, Q^2 can be calculated using the parameters of the first observer.

B. Design of observer 2

For this example, $C^2M^2 = 0$, and hence $\bar{r}^2 = 0$ which results in $r^2 = 0$. To obtain the structures of (14) - (16), suitable coordinate transformations $T_1^2 \in \mathbb{R}^{5\times 5}, T_2^2 \in \mathbb{R}$ are found. Here the matrices A_Ω^2, B_Ω^2 that describe ξ^2 are chosen as $A_\Omega^2 = -10I_2, B_\Omega^2 = 10I_2$ and the augmented system (18) can then be formed for the case $i = 2$. It can be shown that $m^3 = 1$. To obtain the structure in (24) - (26) a suitable transformation matrix $\bar{T}^2 \in \mathbb{R}^{7\times 7}$ is found. The gains $\bar{G}_l^2$ and $\bar{G}_n^2$ have been designed using the method in [17]. The weighting matrices were chosen as $\bar{W}^2 = 0.01I_7, \bar{V}^2 = I_2$, and the corresponding gains $\bar{G}_l^2, \bar{G}_n^2 \in \mathbb{R}^{7\times 2}, \bar{P}_o^2 \in \mathbb{R}^{2\times 2}$ have been synthesized. The filter scalar α^2 was chosen as 10. It follows that the system for observer 3 will be of order $n^3 = \bar{n}^2 - m^3 = 6$ and the number of outputs is $p = 2$.

C. Design of observer 3

Now $rank(C^3M^3) = rank(M^3)$. Finally, a robust sliding mode observer can be designed based on A^3, M^3, C^3, Q^3 using the method in [18] which bounds the $\mathcal{L}_2$ gain from ξ^3 to the fault reconstruction. In [18], two parameters need to be chosen to tune the observer gains. The motivation for, and the effects of these parameters, is described in [18]. Here they have been chosen as $D_1 = I_2, \gamma_o = 10$. Implementing the algorithm in [18] yields an $\mathcal{L}_2$ gain of $\gamma = 1.156$.

D. Simulation results

For the observers, the gains were chosen as $\bar{\rho}^1 = 30, \bar{\rho}^2 = 30, \bar{\rho}^3 = 60$ respectively and the smoothing constants were chosen as $\delta^1 = 10^{-4}, \delta^2 = 0.01, \delta^3 = 0.02$. The left subfigure of Figure 2 shows the fault injected into the second actuator, and Figure 3 shows the disturbances ξ^1 that arise from it. The fault reconstruction $\hat{f}^1$ from the cascade-observer method is shown in the right subfigure of Figure 2. It can be seen that the fault reconstruction rejects the effects of ξ^1 (which is not insignificant in magnitude) because the fault reconstruction scheme has been designed to minimize the upper bound of the $\mathcal{L}_2$ gain from ξ^3 to $\hat{f}^1$.

IV. Conclusion

This paper has presented a new scheme for robust fault reconstruction, using multiple observers in cascade. Signals from one observer are used as outputs of a fictitious system, and the next observer in the cascade is designed based on the fictitious system. The novelty of this scheme is that it can reconstruct faults in a wider class of systems compared to previous methods. In addition, the scheme is formulated in a framework which facilitates the minimization of disturbances on the fault reconstruction. This is particularly useful in cases when the number of outputs is less than the number

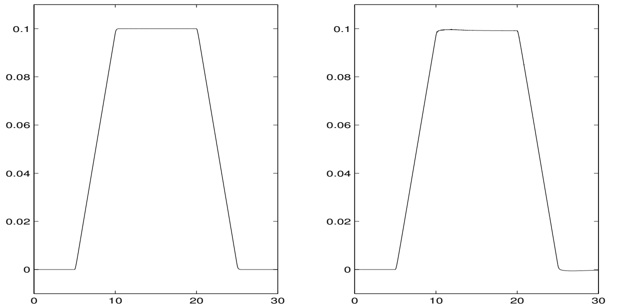

Fig. 2. *The left subfigure is the fault applied to the 2nd actuator, the right subfigure is its reconstruction.*

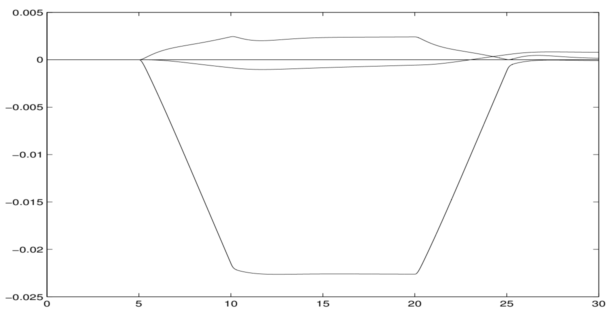

Fig. 3. *The components of ξ^1, namely $\triangle A^1 x^1$*

of disturbances and faults, a scenario that will render many other multiple observer methods inapplicable. An example verifies the effectiveness of the scheme. A method to obtain k from A^1, M^1 and C^1 is currently under investigation by the authors.

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APPENDIX

Proposition 2: If $rank(C^{n^1} M^{n^1}) < rank(M^{n^1})$ then the fault can never be fully reconstructed.

Proof: From (14), $\tilde{A}_1^i$ has $n^i - (i-1)h - p$ rows and therefore $n^i - (i-1)h - p \geq 0$. Substituting for n^i from (52) results in

$$n^1 - \Sigma_{j=2}^{i} m^j - p \geq 0 \tag{57}$$

Since $m^{i+1} = rank(\tilde{A}_{31}^i)$ and knowing that $\tilde{A}_{31}^i$ from (20) has $m^i - r^i$ rows, it is obvious that $m^{i+1} \leq m^i$ and hence $0 \leq m^i \leq m^{i-1} \leq ... \leq m^2 \leq m^1 = p$. It follows from (57) that $m^i = 0$ when $i > n^1$. From (9), it is clear that $r^i \leq m^i$ and therefore $r^i = 0$ when $i > n^1$. Then, since by definition $\bar{r}^{i-1} + r^i = \bar{r}^i$, $\bar{r}^i = \bar{r}^{n^1}$ when $i > n^1$ which results in $rank(C^i M^i) = rank(C^{n^1} M^{n^1})$ when $i > n^1$. This means that if observer n^1 is unable to reconstruct the fault, then subsequent observers will not be able to either, and the scheme in this paper is not feasible. ■

Proposition 3: If there exists an integer i such that the inequality $m^i < q - \bar{r}^{i-1}$ holds, then the fault can never be fully reconstructed.

Proof: Since $m^i := rank(\tilde{A}_{31}^{i-1})$ and the matrix $\tilde{A}_{31}^{i-1}$ has $m^{i-1} - r^{i-1}$ rows, it can be deduced that

$$m^i \leq m^{i-1} - r^{i-1} \tag{58}$$

Since $M_{22}^i \in \mathbb{R}^{r^i \times r^i}$ is obtained from an orthogonal transformation of $\bar{M}_{12}^{i-1}$ which has m^i rows, then

$$r^i \leq min\left\{m^i, q - \bar{r}^{i-1}\right\} \tag{59}$$

must hold. Suppose

$$m^i < q - \bar{r}^{i-1} \tag{60}$$

then from (59), $r^i < q - \bar{r}^{i-1}$. By definition $\bar{r}^{i-1} + r^i = \bar{r}^i$, and therefore (60) results in $\bar{r}^i < q$ which in turn implies that observer i is unable to reconstruct f. Rearranging (58) and inductively incrementing i gives $m^{i+1} + r^i < m^i$, which combined with (60) results in $m^{i+1} + r^i < q - \bar{r}^i$ which implies $m^{i+1} < q - \bar{r}^i$, which also implies that the next observer $i+1$ is also unable to reconstruct f. Hence, if (60) is true, then all subsequent observers will be unable to reconstruct f. ■