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Ellipsoidal Approximations to Attraction Domains of Linear Systems with Bounded Control

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Abstract— **We are concerned with characterization of attraction domains for linear state-space systems. Considered are bounded linear state feedback control and saturated control. Attraction domains for such systems are described in terms of invariant ellipsoids using LMI-based techniques and semidefinite programming (SDP). For systems with saturated control, the ideology of absolute stability is adopted. An application to nonlinear systems is provided.**

I. INTRODUCTION

The need in characterizing attraction domains of dynamic systems arises in various practical problems, for example, in the analysis of the behavior and control design for mechanical systems. As a rule, the control resources are limited in most of the realistic applications, the model of the system is not known precisely, and moreover, the system is affected by uncontrollable exogenous disturbances.

A general approach to solving certain types of such problems for linear systems was first formulated by A.M. Formalsky in [1]; for nonlinear systems, the notion of attraction domain goes back to classical works of LaSalle and Lefshetz [2], also see survey [3]. Among the recent publications, the monograph [4] is worth mentioning, which accumulates most of the theoretical achievements and computational experience available in this subject area; the notion of set invariance is the cornerstone of the exposition in [4].

In the last fifteen years, the methods of the theory of linear matrix inequalities (LMIs) have become a popular tool for characterization of reachability and attraction domains of dynamic systems; e.g., see [5]. The salient feature of the LMI-based approach is that it covers diverse formulations of analysis and design problems; moreover, it is applicable to high-dimensional systems and uses simple and user-friendly computational tools compatible with the standard MATLAB environment.

Since recently, much attention has been paid to characterization of attraction domains for systems driven by saturated linear control, in particular, [4]–[11], and especially, [6]–[10].

In the present paper, we follow this line of research. In order to synthesize a bounded control that maximizes the attraction domain for the closed-loop system, we are further attempted at combining the potential of the invariant ellipsoid approach with the ideas of the theory of absolute stability, [11]. The main attention is paid to the problem with saturated control, for which a new LMI-based technique is applied. The results in this paper are considered as a successive step towards solution of more realistically formulated problems, primarily those related to nonlinear systems, accounting for exogenous disturbances, imprecise knowledge of the model, etc.

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II. DEFINITIONS AND PROBLEM STATEMENT

We consider a linear continuous time invariant system

$$\dot{x} = Ax + Bu, \quad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \tag{1}$$

where the matrices A, B are known, the pair (A, B) is controllable, and

$$\|u\| \le u_{\max} \tag{2}$$

is a bounded state feedback control, where $u_{\max}$ is known. Specifically, below we consider the two cases, where the admissible control is taken either as a *linear* state feedback $u = Kx$ or a *saturated* linear feedback $u = \mathrm{sat}(Kx)$.

For any fixed admissible control, consider the closed-loop system and introduce the following definition.

Definition 1: The *attraction domain* $\mathcal{A}$ of the closed-loop system (1), (2) is the set of all initial conditions $x_0 \doteq x(0)$ having the property $x(x_0, t) \to 0$ as $t \to \infty$.

In words, the attraction domain is the set of points in the phase space, from which a system can be driven to the origin by a bounded control. Hence, the goal is to design an admissible control that maximizes this domain.

First, in the sequel we will be interested only in unstable open-loop systems; otherwise, use of the trivial bounded control $u = 0 \cdot x \equiv 0$ obviously yields the largest possible attraction domain, i.e., the whole space $\mathbb{R}^n$. Interestingly, $\mathcal{A} = \mathbb{R}^n$ can be obtained for stable systems by using saturation controls, which are not identically zero.

It is well known (e.g., see [4]) that the set $\mathcal{A}$ does not admit an exact closed-form description even for the simple case of bounded linear control; for instance, the efficient application of the classical approach via supporting functions is limited to the two-dimensional case, $n = 2$. We therefore adopt an approximate description keeping in mind the important property of the attraction domain; namely, its *invariance*, [4]. Invariance means that $x_0 = x(0) \in \mathcal{A}$ implies $x(x_0, t) \in \mathcal{A}$ for all $t > 0$, i.e., for any initial point in $\mathcal{A}$, the trajectory of the systems remains inside $\mathcal{A}$ at all times. In what follows we construct inner approximation also in the form of invariant sets having, however, a simpler structure; namely, invariant ellipsoids (also referred to as *holdable* ellipsoids, [5]).

Definition 2: The ellipsoid

$$\mathcal{E}(P) = \{x \in \mathbb{R}^n\colon\ x^{\mathsf{T}} P^{-1} x \le 1\}, \qquad P \succ 0, \tag{3}$$

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centered at the origin is said to be *invariant attractive* for the closed-loop system (1), (2), if for any $x_0 \in \mathcal{E}$ it holds $x(x_0, t) \to 0$ and $x(x_0, t) \in \mathcal{E}$ for all $t \geq 0$.

Here, $P \succ 0$ denotes the positive-definiteness of the matrix P.

The goal is to find a bounded stabilizing control which *maximizes* the size of the attraction ellipsoid $\mathcal{E} \subseteq \mathcal{A}$.

As mentioned in the Introduction, the convenience of invariant ellipsoids is explained by their direct connection to quadratic Lyapunov functions and the availability of a well-developed apparatus of linear matrix inequalities, which is the main tool for the construction of ellipsoidal approximations of attraction domains.

Namely, in the sections to follow, we formulate the problem above as a convex optimization problem subject to LMI constraints, which is easy to solve numerically. The performance index (the size of $\mathcal{E}(P)$) in such problems can be taken in different forms, e.g. such as $-\ln\det P^{-1}$ (associated with the maximization of the volume of $\mathcal{E}(P)$), $\operatorname{tr} P$ (associated with the sum of squared semiaxes), or its minimal eigenvalue (associated with the radius of the inscribed ball), etc. In the sequel, the trace criterion $\operatorname{tr} P$ is adopted as a one retaining the *SDP structure* of the problem (SDP, semidefinite program, optimization of a linear function subject to LMI constraints), which is then solvable by means of widely available MATLAB-based software.

III. Linear Control

We consider system (1) with *linear* state feedback control

$$u = Kx, \quad \|u\| \leq u_{\max}, \tag{4}$$

and find a stabilizing matrix $K \in \mathbb{R}^{m\times n}$ that maximizes the attraction ellipsoid for the closed-loop system (1), (4).

Solution to this problem is provided by the theorem below.

Theorem 1: Let P, Y be solutions of the following SDP:

$$\begin{gathered}\max \operatorname{tr}(P) \quad \text{s.t.} \quad AP + PA^{\mathsf T} + BY + Y^{\mathsf T}B^{\mathsf T} \prec 0; \\ \begin{pmatrix} P & Y^{\mathsf T} \\ Y & u_{\max}^2 I \end{pmatrix} \succeq 0, \end{gathered} \tag{5}$$

in the matrix variables $P = P^{\mathsf T}$ and Y. Then the control $u = Kx$, $K = YP^{-1}$,

(i) stabilizes system (1),

(ii) is bounded $\|u\| \leq u_{\max}$ on the ellipsoid $\mathcal{E}(P)$ (3), and

(iii) this ellipsoid is maximal with respect to the performance $\operatorname{tr}(P)$ among all invariant ellipsoids of the system.

The function $V(x) = x^{\mathsf T}P^{-1}x$ is a quadratic Lyapunov function for the closed-loop system $\dot{x} = (A + BK)x$.

The first LMI ensures that the control $u = Kx$ is stabilizing, the second one guarantees its boundedness, and the performance optimization yields the best one among such controls. These LMI constraints above are well-known, e.g., see [5]; Theorem 1 explicitly reduces finding the maximal invariant ellipsoid to an SDP problem.

It is readily seen that the solutions P, Y of (5) may lead to the controller $K = YP^{-1}$ such that the closed-loop system with matrix $A_c = A+BK$ is at the stability margin. In other words, the ellipsoid thus obtained is *not* attractive (although being invariant) in the sense of Definition 2; indeed, the system trajectories do not tend to the origin.

In order to ensure $x(t) \to 0$, we require that the Lyapunov function $V(x)$ decrease at a certain rate, $\dot{V}(x) \leq -\alpha\|x\|^2$ for some $\alpha > 0$; namely:

$$(A + BK)^{\mathsf T}P^{-1} + P^{-1}(A + BK) \preceq -\alpha I.$$

This requirement is closely related to the decay rate constraint formulated as $\dot{V}(x) \leq -\alpha V$ in [5] or to the so-called β-contractivity condition exploited in [4].

Pre- and post-multiplying the last matrix inequality by P and changing to the new variables P and $Y = PK$, we re-write it as the following LMI:

$$\begin{pmatrix} AP + PA^{\mathsf T} + BY + Y^{\mathsf T}B^{\mathsf T} & P \\ P & -I/\alpha \end{pmatrix} \preceq 0,$$

which is now to be used instead of the first LMI constraint in (5). We therefore arrive at the following formulation.

Theorem 2: Let P, Y be solutions of the SDP

$$\max \operatorname{tr}(P) \text{ s.t. } \begin{pmatrix} AP{+}PA^{\mathsf T}{+}BY{+}Y^{\mathsf T}B^{\mathsf T} & P \\ P & -I/\alpha \end{pmatrix} \preceq 0;$$

$$\begin{pmatrix} P & Y^{\mathsf T} \\ Y & u_{\max}^2 I \end{pmatrix} \succeq 0$$

in the matrix variables $P = P^{\mathsf T}$ and Y, where $\alpha > 0$. Then the control $u = Kx$, $K = YP^{-1}$, is stabilizing, the matrix P defines the attraction ellipsoid $\mathcal{E}(P)$ for the closed-loop system such that the control input is bounded over it, $\|u\| \leq u_{\max}$, and this ellipsoid is maximal over all linear static state controllers such that the Lyapunov function $V(x) = x^{\mathsf T}P^{-1}x$ decreases not slower than $-\alpha\|x\|^2$.

It is noted that in the problem considered, the bound magnitude $u_{\max}$ can be specified arbitrarily, since the constraints are always compatible (for smaller $u_{\max}$ we simply obtain a smaller attraction ellipsoid). Hence, in the exposition to follow we take $u_{\max} = 1$.

IV. Saturated Control

The ellipsoid obtained above corresponds to linear controls $u = Kx$. Let us expand the class. Namely, for system (1), we consider scalar controls (i.e., $K^{\mathsf T} \in \mathbb{R}^n$) having the form

$$u \;=\; \operatorname{sat}(Kx) \;=\; \begin{cases} -1 & \text{for } \; Kx < -1; \\ Kx & \text{for } \; |Kx| \leq 1; \\ +1 & \text{for } \; Kx > 1; \end{cases}$$

i.e., those defined over the whole phase space.

Likewise the previous section, we are aimed at designing a control of such form yielding the maximal invariant attractive ellipsoid. Since the closed-loop system is nonlinear, we make use of the approach adopted in the theory of absolute stability.

Let us briefly describe the main idea underlying these constructions. Assume that the desired controller K is found. For $\gamma \geq 1$, consider the sector bounded by the straight lines

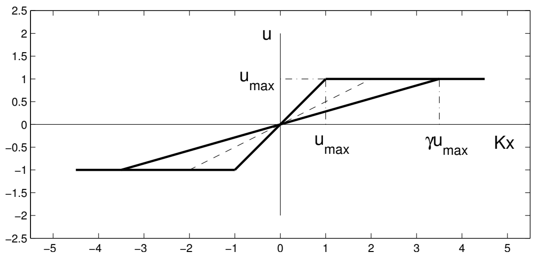

Fig. 1. Control with saturation.

$u = Kx$ and $u = Kx/\gamma$, and all bounded nonlinearities $-1 < u \leq 1$ belonging to this sector, see Fig. 1. Let us find a quadratic Lyapunov function

$$V(x) = x^{\mathsf{T}} P_{sat}^{-1} x, \qquad P_{sat} \succ 0,$$

having the property $\dot{V}(x) < 0$ along the trajectories of the system subjected to any nonlinearity in the sector. It then suffices to find such a function for the two "limiting" linear systems: the one associated with $u = Kx$ (i.e., $\gamma = 1$), and the other corresponding to the maximal possible value of γ (maximal opening of the sector). On top of that, we require that the desired ellipsoid $\{x^{\mathsf{T}} P_{sat}^{-1} x \leq 1\}$ be contained in the stripe $\mathcal{S} = \{|Kx| \leq \gamma\}$. Writing down these conditions in the form of quadratic constraints, using the lossless version of S-theorem with two constraints (see [12]) and changing to the new variables P, Y (similarly to what was done in Theorems 1 and 2), we arrive at the following result (below, the notation $\mu = 1/\gamma$ is used).

Theorem 3: Let Y_{sat}, P_{sat} be solutions of the SDP

$$\max \operatorname{tr}(P) \text{ s.t. } \begin{pmatrix} AP+PA^{\mathsf{T}}+BY+Y^{\mathsf{T}}B^{\mathsf{T}} & P \\ P & -I/\alpha \end{pmatrix} \preceq 0;$$

$$\begin{pmatrix} AP+PA^{\mathsf{T}}+\mu(BY+Y^{\mathsf{T}}B^{\mathsf{T}}) & P \\ P & -I/\alpha \end{pmatrix} \preceq 0;$$

$$\begin{pmatrix} P & Y^{\mathsf{T}} \\ Y & \frac{1}{\mu^2} I \end{pmatrix} \succeq 0;$$

in the matrix variables $P = P^{\mathsf{T}}$, Y, for some value of the scalar parameter $\mu \in (0,\ 1]$. Then the control $u = \operatorname{sat}(K_{sat}x)$, $K_{sat} = Y_{sat} P_{sat}^{-1}$, is stabilizing, and among all saturated controls it provides the maximal attraction ellipsoid $\mathcal{E}_{sat} = \{x : x^{\mathsf{T}} P_{sat}^{-1} x \leq 1\} \subset \mathcal{S} \doteq \{|K_{sat}x| \leq 1/\mu\}$ for the closed-loop system with a given rate of decrease of the Lyapunov function $V(x) = x^{\mathsf{T}} P_{sat}^{-1} x$ along the trajectories.

In the above approach we assumed μ being fixed. It is of interest to maximize the size of the ellipsoid with respect to μ. The conjecture is that the maximum is attained at $\mu = 1$, i.e. saturated control does not enlarge the attraction ellipsoid.

The result of Theorem 3 is similar to the ones reported in the literature, e.g., see [6]. However, there is an essential difference. First, in proving our result, we make use of the lossless version of S-theorem with 2 constraints, thus reducing the conservatism of the ellipsoidal approximations obtained. Second, maximization of the attraction ellipsoid $\mathcal{E}(P)$ in the above-mentioned literature is understood as maximal expansion $\beta\mathcal{X}_R$ of a certain reference set $\mathcal{X}_R$ contained in the ellipsoid. For the natural case where $\mathcal{X}_R$ is a given ellipsoid $\mathcal{E}(P_0)$, we cover this formulation by simply imposing an extra LMI constraint $P \geq \beta P_0$ and maximizing the variable β. The same goes for the case where $\mathcal{X}_R$ is a polygon with given vertices x_i by a standard representation of the requirement $x_i \in \mathcal{E}(P)$ in LMI format. Third, our approach admits an extension to the presence of uncertainty in the matrix coefficients.

V. Attraction Domain for a Nonlinear System

The technique described above can be extended to more general nonlinear systems. Consider the closed-loop system in the form

$$\dot{x} = Ax + B\varphi(c^T x)$$

where $B, c \in \mathbb{R}^n$, and $\varphi : \mathbb{R}^1 \to \mathbb{R}^1$ is a nonlinear function. Such system description is standard for the absolute stability framework [15], [16]. An LMI technique for checking absolute stability has been proposed in [5]. However in contrast with the absolute stability theory we do not assume φ to be a sector nonlinearity; for instance this function can tend to ∞. In that case, we can not guarantee stability for all initial points $x(0)$. We suppose that the matrix A is Hurwitz while $\varphi(0) = 0, \varphi'(0) = 0$. The issue of interest is the largest attraction ellipsoid satisfying the the conditions above. The technique to obtain such ellipsoid is similar to the one developed in Section IV with function u replaced with φ. For simplicity of exposition, we restrict ourselves to an illustration of the approach as applied to the problem borrowed from [13].

Consider a nonlinear two-dimensional system $\dot{x} = g(x)$, $x \in \mathbb{R}^2$, with $g_1(x) = x_2$ and $g_2(x) = 0.2347136969 - 0.0633\sin(x_1+0.0405) - 0.582\sin(x_1+0.4103) - 0.7143x_2$. Using the general mathematical programming formulation, the authors of [13] obtained the attraction ellipse $\mathcal{E}_{mp} = \mathcal{E}(P_{mp})$ with matrix

$$P_{mp} = \begin{pmatrix} 11.4161 & -7.8977 \\ -7.8977 & 10.5771 \end{pmatrix}.$$

This ellipse was obtained by maximizing the area (i.e., the performance index was taken as $f(P) = \det P$).

Within our approach, we first linearize the system and represent it in the equivalent form

$$\dot{x} = Ax + Bu,$$

where $A = (A_{ij}) = \partial g_i(x)/\partial x_j|_{x=0}$, $i, j = 1, 2$, and $Bu = g(x) - Ax$, whence

$$A = \begin{pmatrix} 0 & 1 \\ -0.5969 & -0.7143 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(i.e., the A matrix is stable, $\lambda(A) = -0.3572 \pm j0.6851$), and $u = u(x_1) = 0.2347136969 - 0.0633\sin(x_1+0.0405) - 0.582\sin(x_1+0.4103) + 0.5969x_1$ represents a scalar nonlinearity.

Next, we assume that the nonlinear *disturbance* u is (i) bounded and (ii) confined to the sector with "opening" μ.

More accurately, we specify a value of $x_1^{\max}$, set $\mu = u(x_1^{\max})/x_1^{\max}$ and consider nonlinearities in the sector $0 \leq u(x_1) \leq \mu x_1$, keeping in mind that $0 \leq x_1 \leq x_1^{\max}$, see Fig. 2.

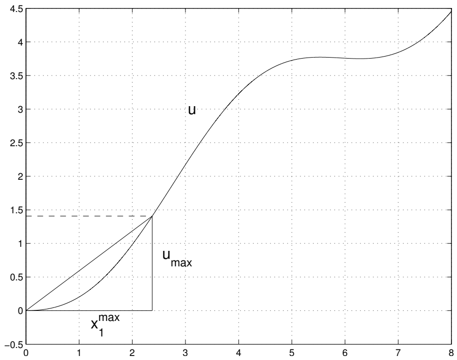

Fig. 2. Sector-confined bounded nonlinearity.

For the two systems associated with the two limiting values of the constraints (the two rays defining the sector), we build a common quadratic Lyapunov function. These two systems are given by

$$\dot{x} = Ax; \qquad \dot{x} = Ax + \mu BKx,$$

where the row vector $K = [1 \quad 0]$ distinguishes the first component of x, i.e., $Kx = x_1$.

Hence, for various values of $x_1^{\max}$ (defining the values of $u_{\max}$ and μ) we solve the SDP

$$\begin{aligned} \max \operatorname{tr}(P) \quad \text{s.t.} \quad & AP + PA^{\mathsf{T}} \preceq 0; \\ & (A+\mu BK)P + P(A+\mu BK)^{\mathsf{T}} \preceq 0; \\ & \begin{pmatrix} P & PK^{\mathsf{T}} \\ KP & \frac{u_{\max}^2}{\mu^2} I \end{pmatrix} \succeq 0, \end{aligned} \tag{6}$$

with respect to one matrix variable $P = P^{\mathsf{T}}$. The ellipse of interest $\mathcal{E} = \{x^{\mathsf{T}} P^{-1} x \leq 1\}$ corresponds to the value $x_1^{\max}$ such that the solution matrix P is maximal (e.g., with respect to trace).

The first two LMI constraints in the SDP above follow from those in Theorem 3 by noting that here, the controller K is already chosen (therefore, (6) is an *analysis* result).

After the calculations we find that the maximum is attained with $x_1 \approx 2.375$, and the resulting ellipse (the smaller one on Fig. 3) defined by the matrix

$$P = \begin{pmatrix} 5.6406 & -1.7389 \\ -1.7389 & 1.5297 \end{pmatrix},$$

turns out to be worse than the one obtained in [13]. The reason is seemingly that Theorem 3 provides only sufficient conditions, while the technique elaborated in [13] may lead to the exact optimal solution.

However, the example above was primarily aimed at showing the very capability of using the approach proposed in this paper. Moreover, the technique exploited above is very simple; it uses widely accepted MATLAB-based implementations of the methods for solving semidefinite programs (Yalmip and SeDuMI toolboxes), while the method in [13] is heavily based on solving a general problem of mathematical programming and requires a computationally intensive numerical "fitting" of the result. Finally, and most importantly, the technique proposed here can be extended to the many-dimensional case,—the situation where the method proposed in [13] can hardly be applied; in particular, the sign-definiteness condition of a matrix is extremely cumbersome to formulate in terms of the entries of the matrix.

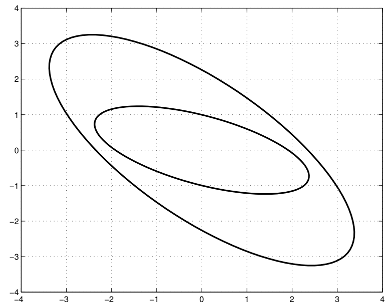

Fig. 3. Attraction ellipsoids for the nonlinear system.

VI. Generalizations

We mention several most important possible generalizations of the approach given here.

First, our technique immediately applies to the discrete time case by considering respective discrete-time Lyapunov matrix inequalities. Second, importantly, unknown-but-bounded exogenous disturbances affecting the system can be taken into account; this will be the subject of the subsequent papers. Third, the presence of norm-bounded uncertainty in the matrix coefficients of the system leading to robust versions of the results are possible (e.g., in the spirit of [14]). Finally, state constraints of the form $x \in \mathcal{E}(P_x)$ can be easily covered with our approach by imposing an extra LMI constraint of the form $P \geq P_x$.

VII. Conclusion

We proposed a design method for bounded saturated state feedback control which maximizes (an inner approximation to) the attraction domain of the closed-loop system. The approach consists of computing the estimates in the form of invariant ellipsoids by using LMI formulations and applying the techniques of the theory of absolute stability.

The results obtained testify to the simplicity and usefulness of the proposed approach; they should be considered as the first step towards more general results.

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