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Monotonically Convergent Iterative Learning Control for Uncertain Time-Delay Systems: An LMI Approach

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Abstract— **This paper deals with the robust iterative learning control (ILC) problem for time-delay systems (TDS) subject to matched parameter uncertainties. Based on two-dimensional (2-D) approach, a stability analysis of the ILC process is developed in the sense that the control error converges monotonically as a function of iteration. It shows that a sufficient condition for the ILC stability can be given in terms of linear matrix inequalities (LMIs), which derives learning gains directly. Simulation results show that the ILC system under a law using gains solved by the LMI approach is robustly stable and monotonically convergent.**

I. Introduction

Iterative learning control (ILC) is a technique for systems that operate repetitively over a fixed time interval. It improves the control signal through iterated trials in order to achieve a perfect output tracking. This creates a two-dimensional (2-D) process in nature, with time and iteration as two independent directions [1]-[2]. Due to this feature, 2-D approaches to ILC have attracted considerable attention. Some of them are based on Roesser systems [3]-[8], and the others are based on linear repetitive processes [9]-[12]. Such approaches take the entire dynamics of ILC into account, which can also derive stability conditions for the ILC process straightforwardly from the 2-D linear systems/repetitive processes theory. However, ILC based on 2-D approaches has seen relatively little activity for time-delay systems (TDS), especially for the TDS subject to parameter uncertainties.

In ILC, the field of TDS has been studied for a long time. In [13], it is argued that ILC is a natural method to overcome the main difficulty in addressing TDS, that is, the state should be properly considered in an infinite dimensional space. This viewpoint coincides with the purpose of many promising ILC design approaches. In [14], a holding mechanism is adopted in ILC, which is to deal with input-delay uncertainty of linear time-invariant (LTI) systems. The higher-order ILC has been investigated for nonlinear systems with unknown but constant state delays [15]. For LTI systems with both model and delay uncertainties, a frequency-domain method to design ILC with Simth predictor has been employed in [16]. For more details, see also [17] for ILC with an initial rectifying action and [18] for feedback-based ILC. In addition to the plant dynamics, in ILC, the learning transient behavior has also been considered an important practical issue, and in order to obtain reasonable transients, the monotonic convergence is desirable [19]. But, to the best of our knowledge, there are only limited studies on monotonically convergent ILC for uncertain TDS. This is just the motivation of the present study.

In this paper, robust stability analysis of ILC for uncertain TDS is developed in the sense that the control error converges monotonically as the learning process proceeds from trial to trial. The updating laws under consideration are the PD-type ILC. In [18], it has been illustrated that the P-type error term can improve the transient learning behavior of TDS; however, strict proofs and explicit conditions are not provided to verify this point. This paper presents a systematic analysis to derive ILC with the monotonic convergence in order to refine a good learning transient behavior, which expresses the convergence conditions in terms of LMIs. Specifically, using 2-D analysis, an LMI approach is employed to develop the stability of ILC based on transforming the monotonic convergence of control error into a performance index. It shows that this approach to ILC can also be applicable to TDS with uncertain parameters that satisfy the matching conditions.

Notations: The superscript "T" represents for matrix transposition. Moreover, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $M > 0$ (respectively, $M < 0$) denotes a matrix M which is symmetric and positive (respectively, negative) definite; I and 0 denote the identity matrix and the null matrix with the required dimensions, respectively. Also, an asterisk $*$ is used to represent a term that is induced by symmetry in symmetric block matrices or long matrix expressions, and matrices (if their dimensions are not explicitly stated) are assumed to be compatible for algebraic operations.

II. Problem Formulation

Let $t \in [0,T]$ be the continuous-time index and $k \in \mathbb{Z}_+$ be the discrete-iteration index. Then let us consider a dynamic equation modeled in the following 2-D form:

$$\frac{\partial x(t,k)}{\partial t} = Ax(t,k) + A_\tau x(t-\tau,k) + Bu(t,k) \tag{1}$$

where $x(t,k) \in \mathbb{R}^n$ is the state, $u(t,k) \in \mathbb{R}^m$ is the control input, $\tau \geq 0$ is the delay parameter, and A, B, A_τ are matrices of appropriate dimensions. If τ is time-varying, it is assumed that $0 \leq \tau(t) \leq \tau$, $t \in [0,T]$. Also, if A, B, A_τ are uncertain,

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they are assumed to take the following forms:

$$A = A_0 + \Delta A(t),\ \ B = B_0 + \Delta B(t),\ \ A_\tau = A_{\tau 0} + \Delta A_\tau(t) \quad (2)$$

where A_0, B_0, $A_{\tau 0}$ are constant matrices, and $\Delta A(t)$, $\Delta B(t)$, $\Delta A_\tau(t)$ are time-varying uncertain matrices satisfying

$$\begin{bmatrix}\Delta A(t) & \Delta B(t) & \Delta A_\tau(t)\end{bmatrix} = E\Sigma(t)\begin{bmatrix}F_1 & F_2 & F_3\end{bmatrix}. \quad (3)$$

In (3), E and F_i $(i = 1,2,3)$ are constant matrices of appropriate dimensions, and $\Sigma(t)$ is an unknown time-varying matrix which satisfies $\Sigma^{\mathrm{T}}(t)\Sigma(t) \le I$.

For the dynamic equation (1), consider the output trajectory $y(t,k) \in \mathbb{R}^r$ given by

$$y(t,k) = Cx(t,k) \quad (4)$$

where $C \in \mathbb{R}^{r\times n}$ is a constant matrix. Then for system (1)-(4), we assume that $y_d(t)$, $t \in [0,T]$ is a realizable desired output trajectory, and $x_d(t)$, $t \in [-\tau,T]$ and $u_d(t)$, $t \in [0,T]$ are the corresponding desired state and control input trajectories, respectively. Therefore, we can define the sate, control input and output tracking error vectors as: $\delta x(t,k) = x_d(t) - x(t,k)$, $\delta u(t,k) = u_d(t) - u(t,k)$ and $e(t,k) = y_d(t) - y(t,k)$. Also, we assume the standard ILC reset condition: $x(t,k) = x_d(t)$ for all $t \in [-\tau,0]$ and $k \in \mathbb{Z}_+$. For the sake of discussions, an extension of $e(t,k)$ is given by $e(t,k) = C\delta x(t,k)$, $t \in [-\tau,0]$.

• *ILC Laws Description:* The updating law considered in this paper is a PD-type ILC implemented by

$$u(t,k+1) = u(t,k) + Ke(t,k) + K_\tau e(t-\tau,k) + \Gamma\frac{\partial e(t,k)}{\partial t} \quad (5)$$

or a typical PD-type ILC given by

$$u(t,k+1) = u(t,k) + Ke(t,k) + \Gamma\frac{\partial e(t,k)}{\partial t} \quad (6)$$

where $u(t,0) = u_0(t)$, $t \in [0,T]$ is an $\mathscr{L}_{2,[0,T]}$-norm bounded initial control input, and K, K_τ, Γ are $m\times r$ gain matrices.

Remark 1: If the ILC law (5) uses $K_\tau = 0$, it becomes just the one (6). Hence, the properties of (5) contain those of (6). However, (5) requires full knowledge of the time delay τ, and obviously, (6) does not. Noting the two facts, we will use (5) for analysis regardless of τ in the following, and in the case where τ is unavailable, (5) is used as just the ILC law (6).

• *Problem Statement:* Given system (1), let updating law (5) or (6) be applied. The problem addressed in this paper is to select gain matrices such that i) the output tracking error converges to zero as $k \to \infty$, and ii) the following monotonic convergence of the control input error is achieved

$$\|\delta u(t,k+1)\|_{2,[0,T]} < \gamma \|\delta u(t,k)\|_{2,[0,T]} \quad (7)$$

where $\gamma \in (0,1]$ represents the convergence rate, and for any $k \in \mathbb{Z}_+$, the norm $\|\delta u(t,k)\|_{2,[0,T]}$ is defined by

$$\|\delta u(t,k)\|_{2,[0,T]} = \left(\int_0^T \delta u^{\mathrm{T}}(t,k)\delta u(t,k)dt\right)^{1/2}.$$

III. LMI Approach to Stability Analysis of ILC

In this section, the 2-D analysis approach to ILC is first employed to derive an expression for the error system of the state error and the control input error. With this error system, the stability analysis of ILC is presented, for which sufficient conditions are developed in terms of LMIs.

A. 2-D Analysis

Similar to [15], the state error is given by

$$\begin{aligned}\frac{\partial \delta x(t,k)}{\partial t} &= \dot{x}_d(t) - \frac{\partial x(t,k)}{\partial t}\\ &= A\delta x(t,k) + A_\tau \delta x(t-\tau,k) + B\delta u(t,k).\end{aligned} \quad (8)$$

For the ILC system (1) and (5), the input error satisfies

$$\begin{aligned}\delta u(t,k+1) &= \delta u(t,k) - Ke(t,k) - K_\tau e(t-\tau,k) - \Gamma\frac{\partial e(t,k)}{\partial t}\\ &= -(KC + \Gamma CA)\,\delta x(t,k) - (K_\tau C + \Gamma CA_\tau)\\ &\quad \times \delta x(t-\tau,k) + (I - \Gamma CB)\,\delta u(t,k)\end{aligned} \quad (9)$$

where the fact $e(t,k) = C\delta x(t,k)$ is used, and (8) is inserted. Hence, if one defines

$$\hat{C} \triangleq -KC - \Gamma CA,\ \hat{C}_\tau \triangleq -K_\tau C - \Gamma CA_\tau,\ D \triangleq I - \Gamma CB \quad (10)$$

then (9) can be simply written as

$$\delta u(t,k+1) = \hat{C}\delta x(t,k) + \hat{C}_\tau \delta x(t-\tau,k) + D\delta u(t,k). \quad (11)$$

Thus, (8) and (11) describe the ILC dynamic equations along two independent axes, i.e., 2-D ILC processes [6], [7].

Remark 2: If τ is a known time delay, then following [7], one can formulate the error equations (8) and (11) into a 2-D continuous-discrete Roesser's type linear system. If there do not exist the matrix uncertainties, then using the 2-D system theory, one can further derive the stability result of the ILC system (1) and (5) as: $\lim_{k\to\infty}\begin{bmatrix}\delta x^{\mathrm{T}}(t,k) & \delta u^{\mathrm{T}}(t,k)\end{bmatrix}^{\mathrm{T}} = 0$, $t \in [0,T]$ holds if and only if (iff) the matrix D is stable, i.e., D locates all its eigenvalues inside the unit circle. Furthermore, it can be shown that $\lim_{k\to\infty} e(t,k) = 0$, $t \in [0,T]$. That is, the perfect output tracking can be guaranteed. However, only the asymptotic stability of ILC can be derived, which not only restricts the time delay τ to time-invariant cases but also can not address such time-varying system uncertainties as in (3).

From Remark 2, it is clear that the 2-D analysis approach adopted in [7] is not sufficient to achieve our ILC objective. To handle this problem, we take the error equations (8) and (11) as a system from $\delta u(t,k)$ to $\delta u(t,k+1)$, which is simply denoted by

$$\begin{aligned}\delta U(s,k+1) &\triangleq \left[\begin{array}{c|c} \overbrace{\begin{pmatrix}A & A_\tau\end{pmatrix}}^{\text{state}} & \overbrace{B}^{\text{input}} \\ \hline \begin{pmatrix}\hat{C} & \hat{C}_\tau\end{pmatrix} & D\end{array}\right]\delta U(s,k)\\ &\triangleq T^u_{k+1,k}(s)\delta U(s,k)\end{aligned} \quad (12)$$

with $\delta U(s,k) \triangleq \mathscr{L}[\delta u(t,k)]$. To reach (7), we thus only need to guarantee the operator $T^u_{k+1,k}(s)$ satisfying $\left\|T^u_{k+1,k}\right\|_{[0,T]} \le \gamma$, since $\|\delta U(s,k)\|_2 = \|\delta u(t,k)\|_2$. The following analysis will realize this idea in detail.

B. Convergence Analysis

First of all, let us consider system (1) and (4) by neglecting the matrix uncertainties in (2), i.e., matrices A, B and A_τ are known constant. According to the previous development, one can state the following result:

Theorem 1: Consider system (1) and (4), and let updating law (5) be applied. If, for a prescribed scalar $\gamma \in (0,1]$, there exist positive-definite matrices $P>0$ and $Q>0$, and matrices X_i, $i=1,2,3$ that satisfy

$$\Xi_1 = \begin{bmatrix} -I & X_1C+X_3CA & -I+X_3CB & X_2C+X_3CA_\tau \\ * & A^{\mathrm{T}}P+PA+Q & PB & PA_\tau \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -Q \end{bmatrix} < 0 \tag{13}$$

then the state error $\delta x(t,k)$, the control input error $\delta u(t,k)$ and the tracking error $e(t,k)$ converge to zero as $k\to\infty$ for all $t\in[0,T]$, and the monotonic convergence result shown in (7) is achieved for all $k\in\mathbb{Z}_+$. If the LMI (13) holds, then learning gain matrices K, K_τ and Γ are given by

$$K=X_1,\ \ K_\tau = X_2,\ \ \Gamma = X_3. \tag{14}$$

Proof: First, let us show (7). For the prescribed scalar γ and any iteration $k\in\mathbb{Z}_+$, define the performance index by

$$J(k) = \int_0^T \left[\delta u^{\mathrm{T}}(t,k+1)\delta u(t,k+1) - \gamma^2 \delta u^{\mathrm{T}}(t,k)\delta u(t,k)\right] dt. \tag{15}$$

Clearly, (7) holds iff $J(k)<0$ is satisfied for all $k\in\mathbb{Z}_+$. Also, for system (12), define the Lyapunov functional candidate by

$$V(\delta x_\tau(t,k)) = \delta x^{\mathrm{T}}(t,k)P\delta x(t,k) + \int_{t-\tau}^t \delta x^{\mathrm{T}}(s,k)Q\delta x(s,k)ds \tag{16}$$

where $\delta x_\tau(t,k)$ represents the function $\delta x(t,k)$ defined on the interval $[t-\tau,t]$. Obviously, $V(\delta x_\tau(t,k))$ is positive-definite, and particularly, one has $V(\delta x_\tau(0,k))=0$. Using (8), we thus know that $\dot V(\delta x_\tau(t,k)) \triangleq \partial V(\delta x_\tau(t,k))/\partial t$ is expressed by

$$\begin{aligned} &\dot V(\delta x_\tau(t,k)) \\ &= \delta x^{\mathrm{T}}(t,k)A^{\mathrm{T}}P\delta x(t,k) + \delta u^{\mathrm{T}}(t,k)B^{\mathrm{T}}P\delta x(t,k) \\ &\quad + \delta x^{\mathrm{T}}(t-\tau,k)A_\tau^{\mathrm{T}}P\delta x(t,k) + \delta x^{\mathrm{T}}(t,k)PA\delta x(t,k) \\ &\quad + \delta x^{\mathrm{T}}(t,k)PB\delta u(t,k) + \delta x^{\mathrm{T}}(t,k)PA_\tau\delta x(t-\tau,k) \\ &\quad + \delta x^{\mathrm{T}}(t,k)Q\delta x(t,k) - \delta x^{\mathrm{T}}(t-\tau,k)Q\delta x(t-\tau,k) \\ &= \xi^{\mathrm{T}}(t,k)\Pi_1\xi(t,k) \end{aligned} \tag{17}$$

where $\xi(t,k)$ and Π_1 are denoted as

$$\begin{aligned} \xi(t,k) &= \left[\delta x^{\mathrm{T}}(t,k)\ \ \delta u^{\mathrm{T}}(t,k)\ \ \delta x^{\mathrm{T}}(t-\tau,k)\right]^{\mathrm{T}} \\ \Pi_1 &= \begin{bmatrix} A^{\mathrm{T}}P+PA+Q & PB & PA_\tau \\ * & 0 & 0 \\ * & * & -Q \end{bmatrix}. \end{aligned} \tag{18}$$

With the denotation $\xi(t,k)$ and based on (11), we also get

$$\delta u^{\mathrm{T}}(t,k+1)\delta u(t,k+1) = \xi^{\mathrm{T}}(t,k)\begin{bmatrix}\hat C^{\mathrm{T}} \\ D^{\mathrm{T}} \\ \hat C_\tau^{\mathrm{T}}\end{bmatrix}\left[\hat C\ \ D\ \ \hat C_\tau\right]\xi(t,k). \tag{19}$$

Using (17) and (19), we now have that the index $J(k)$ satisfies

$$\begin{aligned} J(k) &= \int_0^T \left[\delta u^{\mathrm{T}}(t,k+1)\delta u(t,k+1) - \gamma^2\delta u^{\mathrm{T}}(t,k)\delta u(t,k)\right. \\ &\qquad \left. + \dot V(\delta x_\tau(t,k))\right] dt - \left[V(\delta x_\tau(T,k)) - V(\delta x_\tau(0,k))\right] \\ &= \int_0^T \xi^{\mathrm{T}}(t,k)\Pi_2\xi(t,k)dt - V(\delta x_\tau(T,k)) \\ &< \int_0^T \xi^{\mathrm{T}}(t,k)\Pi_2\xi(t,k)dt \end{aligned} \tag{20}$$

where the matrix Π_2 is computed as follows

$$\begin{aligned} \Pi_2 &= \begin{bmatrix}\hat C^{\mathrm{T}} \\ D^{\mathrm{T}} \\ \hat C_\tau^{\mathrm{T}}\end{bmatrix}\left[\hat C\ \ D\ \ \hat C_\tau\right] + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0\end{bmatrix} + \Pi_1 \\ &= \begin{bmatrix} A^{\mathrm{T}}P+PA+Q+\hat C^{\mathrm{T}}\hat C & PB+\hat C^{\mathrm{T}}D & PA_\tau + \hat C^{\mathrm{T}}\hat C_\tau \\ * & -\gamma^2 I + D^{\mathrm{T}}D & D^{\mathrm{T}}\hat C_\tau \\ * & * & -Q+\hat C_\tau^{\mathrm{T}}\hat C_\tau \end{bmatrix}. \end{aligned} \tag{21}$$

If the LMI (13) holds, then selecting learning gain matrices in (14), and hence considering the definition (10), we obtain

$$\Pi_3 = \begin{bmatrix} -I & -\hat C & -D & -\hat C_\tau \\ * & A^{\mathrm{T}}P+PA+Q & PB & PA_\tau \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -Q \end{bmatrix} < 0. \tag{22}$$

Applying the Schur complement formula to (22), one obtains $\Pi_2<0$. With this fact, it is immediate from (20) that $J(k)<0$, $\forall k\in\mathbb{Z}_+$. Namely, (7) is proved.

Next, it will be shown that zero tracking errors of $\delta x(t,k)$, $\delta u(t,k)$ and $e(t,k)$ are achieved as $k\to\infty$. From (7), one has

$$\begin{aligned} \|\delta u(t,k)\|_{2,[0,T]} &< \gamma\|\delta u(t,k-1)\|_{2,[0,T]} \\ &< \gamma^k\|\delta u(t,0)\|_{2,[0,T]}. \end{aligned} \tag{23}$$

Since $\gamma\in(0,1]$ and $\delta u(t,0) = u_d(t)-u_0(t)$ is $\mathscr{L}_{2,[0,T]}$-norm bounded, one can derive that $\|\delta u(t,k)\|_{2,[0,T]}$ is bounded for all $k\in\mathbb{Z}_+$, and is monotonically decreasing as $k\to\infty$. Hence, $\lim_{k\to\infty}\delta u(t,k)\triangleq \delta u(t,\infty)$, $\forall t\in[0,T]$ exists, which together with (11) ensures that $\lim_{k\to\infty}\delta x(t,k)\triangleq\delta x(t,\infty)$, $\forall t\in[0,T]$ exists. Consequently, it follows that $\lim_{k\to\infty}\xi(t,k)\triangleq\xi(t,\infty)$, $\forall t\in[0,T]$ exists. If $\gamma\in(0,1)$, then it is clear from (23) that $\delta u(t,\infty)=0$, $\forall t\in[0,T]$. Otherwise, if $\gamma=1$, it is immediate from using (15) that $\lim_{k\to\infty}J(k)=0$. Then taking $k\to\infty$ on both sides of (20) and noting $\Pi_2<0$, we have

$$0 = \lim_{k\to\infty}J(k) \le \int_0^T \xi^{\mathrm{T}}(t,\infty)\Pi_2\xi(t,\infty)dt \le 0$$

which leads to $\xi(t,\infty)=0$, $\forall t\in[0,T]$, and as a consequence, $\delta u(t,\infty)=0$, $\forall t\in[0,T]$. That is, the zero tracking of control input error is achieved if $\gamma\in(0,1]$. In addition, if we denote

$$\Pi_4 = \begin{bmatrix} A^{\mathrm{T}}P+PA+Q & PB & PA_\tau \\ * & -\gamma^2 I & 0 \\ * & * & -Q \end{bmatrix} \tag{24}$$

then applying the Schur complement formula to (13), we can obtain $\Pi_4<0$. Using (17), we can further obtain

$$\dot V(\delta x_\tau(t,k)) - \gamma^2\delta u^{\mathrm{T}}(t,k)\delta u(t,k) = \xi^{\mathrm{T}}(t,k)\Pi_4\xi(t,k) \le 0$$

which, together with (16) and the condition $V(\delta x_\tau(0,k)) = 0$, $k \in \mathbb{Z}_+$, yields

$$\begin{aligned}\delta x^{\mathrm{T}}(t,k)P\delta x(t,k) &\le V(\delta x_\tau(t,k))\\ &\le \int_0^t \gamma^2 \delta u^{\mathrm{T}}(s,k)\delta u(s,k)ds \\ &\le \gamma^2 \|\delta u(t,k)\|^2_{2,[0,T]}.\end{aligned} \tag{25}$$

Since $P > 0$, it follows immediately on taking $k \to \infty$ on both sides of (25) that $\delta x(t,\infty) = 0$, $\forall t \in [0,T]$. Noticing the fact that $e(t,k) = C\delta x(t,k)$, one can complete the proof. ■

Remark 3: From (21), it is obvious that if $\Pi_2 < 0$, $-\gamma^2 I + D^{\mathrm{T}}D < 0$, $\gamma \in (0,1]$. This implies that the matrix D is stable. As mentioned in Remark 2, the zero tracking errors can thus be shown with the 2-D system theory. That is, after showing (7), Theorem 1 can also be derived by using the 2-D analysis approach to ILC as used in [7].

Noticing the facts in Remark 1, one can state the following result using the ILC law (6) for unknown but constant delays:

Corollary 1: Consider system (1) and (4), and let updating law (6) be applied. If, for a prescribed scalar $\gamma \in (0,1]$, there exist positive-definite matrices $P > 0$ and $Q > 0$, and matrices X_1 and X_2 that satisfy

$$\begin{bmatrix} -I & X_1C + X_2CA & -I + X_2CB & X_2CA_\tau \\ * & A^{\mathrm{T}}P + PA + Q & PB & PA_\tau \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -Q \end{bmatrix} < 0 \tag{26}$$

then the state error $\delta x(t,k)$, the control input error $\delta u(t,k)$ and the tracking error $e(t,k)$ converge to zero as $k \to \infty$ for all $t \in [0,T]$, and the monotonic convergence result shown in (7) is achieved for all $k \in \mathbb{Z}_+$. If the LMI (26) holds, then learning gain matrices K and Γ are given by

$$K = X_1, \quad \Gamma = X_2. \tag{27}$$

Proof: This proof is omitted since it follows the lines of the proof of Theorem 1. ■

Actually, the result in Corollary 1 is applicable to systems with time-varying delays. If the delay τ satisfies

$$0 \le \tau(t) < \infty, \quad \dot{\tau}(t) \le \rho < 1 \tag{28}$$

then one can state the following result:

Corollary 2: Consider system (1) and (4) subject to time-varying delay of (28), and let updating law (6) be applied. If, for a prescribed scalar $\gamma \in (0,1]$, there exist positive-definite matrices $P > 0$ and $Q > 0$, and matrices X_1 and X_2 that satisfy

$$\begin{bmatrix} -I & X_1C + X_2CA & -I + X_2CB & X_2CA_\tau \\ * & A^{\mathrm{T}}P + PA + Q & PB & PA_\tau \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -(1-\rho)Q \end{bmatrix} < 0 \tag{29}$$

then the state error $\delta x(t,k)$, the control input error $\delta u(t,k)$ and the tracking error $e(t,k)$ converge to zero as $k \to \infty$ for all $t \in [0,T]$, and the monotonic convergence result shown in (7) is achieved for all $k \in \mathbb{Z}_+$. If the LMI (29) holds, then learning gain matrices K and Γ are given by (27).

Proof: Let us consider again the Lyapunov functional candidate $V(\delta x_\tau(t,k))$ defined in (16) by noticing that τ is time-varying. Following the steps of (17) and inserting (28), we can derive that $V(\delta x_\tau(t,k))$ now satisfies

$$\begin{aligned}&\dot{V}(\delta x_\tau(t,k))\\ &= \delta x^{\mathrm{T}}(t,k)A^{\mathrm{T}}P\delta x(t,k) + \delta u^{\mathrm{T}}(t,k)B^{\mathrm{T}}P\delta x(t,k)\\ &\quad + \delta x^{\mathrm{T}}(t-\tau,k)A_\tau^{\mathrm{T}}P\delta x(t,k) + \delta x^{\mathrm{T}}(t,k)PA\delta x(t,k)\\ &\quad + \delta x^{\mathrm{T}}(t,k)PB\delta u(t,k) + \delta x^{\mathrm{T}}(t,k)PA_\tau\delta x(t-\tau,k)\\ &\quad + \delta x^{\mathrm{T}}(t,k)Q\delta x(t,k) - \delta x^{\mathrm{T}}(t-\tau,k)Q\delta x(t-\tau,k)\left[1-\dot{\tau}(t)\right]\\ &\le \delta x^{\mathrm{T}}(t,k)\left(A^{\mathrm{T}}P + PA + Q\right)\delta x(t,k)\\ &\quad + \delta x^{\mathrm{T}}(t,k)PB\delta u(t,k) + \delta x^{\mathrm{T}}(t,k)PA_\tau\delta x(t-\tau,k)\\ &\quad + \delta u^{\mathrm{T}}(t,k)B^{\mathrm{T}}P\delta x(t,k) + \delta x^{\mathrm{T}}(t-\tau,k)A_\tau^{\mathrm{T}}P\delta x(t,k)\\ &\quad - \delta x^{\mathrm{T}}(t-\tau,k)(1-\rho)Q\delta x(t-\tau,k)\\ &= \xi^{\mathrm{T}}(t,k)\widetilde{\Pi}_1\xi(t,k)\end{aligned}$$

where $\widetilde{\Pi}_1$ is defined by

$$\widetilde{\Pi}_1 = \begin{bmatrix} A^{\mathrm{T}}P + PA + Q & PB & PA_\tau \\ * & 0 & 0 \\ * & * & -(1-\rho)Q \end{bmatrix}.$$

The remaining of the proof can be established by considering the performance index $J(k)$ in (15) and following the same lines of the proof of Theorem 1. ■

Next, let us consider the uncertain system (1)-(4) and state the following result:

Theorem 2: Consider the uncertain system (1)-(4), and let updating law (5) be applied. If, for two scalars $\gamma \in (0,1]$ and $\lambda > 0$, there exist positive-definite matrices $P > 0$ and $Q > 0$, and matrices X_i, $i = 1,2,3$ that satisfy

$$\Xi_2 = \begin{bmatrix} -I & 0 & X_1C + X_3CA_0 & -I + X_3CB_0 & \gamma\lambda X_3CE & X_2C + X_3CA_{\tau 0} \\ * & -I & \frac{1}{\lambda}F_1 & \frac{1}{\lambda}F_2 & 0 & \frac{1}{\lambda}F_3 \\ * & * & A_0^{\mathrm{T}}P + PA_0 + Q & PB_0 & \gamma\lambda PE & PA_{\tau 0} \\ * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -Q \end{bmatrix} < 0 \tag{30}$$

then for all $t \in [0,T]$, the state error, the input error and the tracking error converge to zero as $k \to \infty$, and the monotonic convergence result shown in (7) holds for all $k \in \mathbb{Z}_+$. If the LMI (30) is satisfied, then learning gain matrices K, K_τ and Γ can be computed by (14).

To show Theorem 2, the following lemma is adopted from the literature.

Lemma 1: [20]: Given symmetric matrices $X, Y, Z \in \mathbb{R}^{n\times n}$ satisfying $X \ge 0$, $Y < 0$, $Z \ge 0$, if for any nonzero vector $0 \ne \zeta \in \mathbb{R}^n$, the following inequality holds

$$\left(\zeta^{\mathrm{T}}Y\zeta\right)^2 - 4\zeta^{\mathrm{T}}X\zeta\zeta^{\mathrm{T}}Z\zeta > 0 \tag{31}$$

then there exists a scalar $\lambda > 0$ such that

$$\lambda^2 X + \lambda Y + Z < 0. \tag{32}$$

Proof: (*Proof of Theorem 2:*) First, let us show $J(k) < 0$. From the proof of Theorem 1, it can be seen that (20) holds also for equation (1) with uncertainties of (2) and (3), but Π_2 and Π_3 are no longer LMIs. In this case, if for any nonzero vector μ, $\mu^{\mathrm{T}}\Pi_2\mu < 0$ is satisfied, then (20) still ensures that $J(k) < 0$. Obviously, this result can be achieved if $\xi^{\mathrm{T}}\Pi_3\xi < 0$ holds for any nonzero vector ξ. This can be seen from the fact that if one chooses $\xi = \begin{bmatrix}\mu_1^{\mathrm{T}} & \mu^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$ and $\mu_1 = \begin{bmatrix}\hat{C} & D & \hat{C}_\tau\end{bmatrix}\mu$, then $\xi^{\mathrm{T}}\Pi_3\xi = \mu^{\mathrm{T}}\Pi_2\mu$ holds for this particular choice. Note that $\xi^{\mathrm{T}}\Pi_3\xi < 0$ iff $\xi^{\mathrm{T}}\Xi_1\xi < 0$. Therefore, to derive the proof of $J(k) < 0$, we only need to prove that $\xi^{\mathrm{T}}\Xi_1\xi < 0$ holds for any nonzero vector ξ iff $\Xi_2 < 0$.

Using (2), (3) and (13), we know that $\Xi_1 = \Pi_5 + \Pi_6$, where

$$\Pi_5 = \begin{bmatrix} -I & X_1C + X_3CA_0 & -I + X_3CB_0 & X_2C + X_3CA_{\tau 0} \\ * & A_0^{\mathrm{T}}P + PA_0 + Q & PB_0 & PA_{\tau 0} \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -Q \end{bmatrix}$$

$$\Pi_6 = \begin{bmatrix} 0 & X_3CE\Sigma(t)F_1 & X_3CE\Sigma(t)F_2 & X_3CE\Sigma(t)F_3 \\ * & \left\{\begin{array}{c}[E\Sigma(t)F_1]^{\mathrm{T}}P \\ +PE\Sigma(t)F_1\end{array}\right\} & PE\Sigma(t)F_2 & PE\Sigma(t)F_3 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}.$$

If we denote $\xi \triangleq \begin{bmatrix}\xi_1^{\mathrm{T}} & \xi_2^{\mathrm{T}} & \xi_3^{\mathrm{T}} & \xi_4^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$, the inequality $\xi^{\mathrm{T}}\Xi_1\xi < 0$ holds iff it follows for any $\Sigma^{\mathrm{T}}(t)\Sigma(t) \le I$ that

$$\begin{aligned}\xi^{\mathrm{T}}\Xi_1\xi = \xi^{\mathrm{T}}\Pi_5\xi + 2\left(\xi_1^{\mathrm{T}}X_3CE + \xi_2^{\mathrm{T}}PE\right)\Sigma(t) \\ \times (F_1\xi_2 + F_2\xi_3 + F_3\xi_4) < 0.\end{aligned} \tag{33}$$

Now, if one takes

$$\Sigma(t) = \frac{\left(E^{\mathrm{T}}C^{\mathrm{T}}X_3^{\mathrm{T}}\xi_1 + E^{\mathrm{T}}P\xi_2\right)(F_1\xi_2 + F_2\xi_3 + F_3\xi_4)^{\mathrm{T}}}{\left\|E^{\mathrm{T}}C^{\mathrm{T}}X_3^{\mathrm{T}}\xi_1 + E^{\mathrm{T}}P\xi_2\right\|_2 \left\|F_1\xi_2 + F_2\xi_3 + F_3\xi_4\right\|_2} \tag{34}$$

then $\xi^{\mathrm{T}}\Xi_1\xi$ in (33) reaches the maximum value. Hence, (33) holds for any $\Sigma(t)$ satisfying $\Sigma^{\mathrm{T}}(t)\Sigma(t) \le I$ iff it holds for $\Sigma(t)$ in (34). Thus, inserting (34) into (33), we derive that (33) is equivalent to

$$\xi^{\mathrm{T}}\Pi_5\xi + 2\sqrt{\xi^{\mathrm{T}}X\xi}\sqrt{\xi^{\mathrm{T}}Z\xi} < 0 \tag{35}$$

where X and Z are given by

$$\begin{aligned} X &= \begin{bmatrix} X_3CEE^{\mathrm{T}}C^{\mathrm{T}}X_3^{\mathrm{T}} & X_3CEE^{\mathrm{T}}P & 0 & 0 \\ PEE^{\mathrm{T}}C^{\mathrm{T}}X_3^{\mathrm{T}} & PEE^{\mathrm{T}}P & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ Z &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & F_1^{\mathrm{T}}F_1 & F_1^{\mathrm{T}}F_2 & F_1^{\mathrm{T}}F_3 \\ 0 & F_2^{\mathrm{T}}F_1 & F_2^{\mathrm{T}}F_2 & F_2^{\mathrm{T}}F_3 \\ 0 & F_3^{\mathrm{T}}F_1 & F_3^{\mathrm{T}}F_2 & F_3^{\mathrm{T}}F_3 \end{bmatrix}. \end{aligned} \tag{36}$$

Clearly, (35) holds iff $\Pi_5 < 0$ and

$$\left(\xi^{\mathrm{T}}\Pi_5\xi\right)^2 - 4\xi^{\mathrm{T}}X\xi\xi^{\mathrm{T}}Z\xi > 0. \tag{37}$$

From (36), it is obvious that $X \ge 0$ and $Z \ge 0$. Thus, using Lemma 1, it follows that (37) holds iff there exists a scalar $\lambda > 0$ that satisfies

$$\Pi_5 + \lambda^2 X + \lambda^{-2} Z < 0 \tag{38}$$

which is equivalent to

$$\Pi_5 + W^{\mathrm{T}}W < 0 \tag{39}$$

with W given by

$$W = \begin{bmatrix} \lambda E^{\mathrm{T}}C^{\mathrm{T}}X_3^{\mathrm{T}} & \lambda E^{\mathrm{T}}P & 0 & 0 \\ 0 & \frac{1}{\lambda}F_1 & \frac{1}{\lambda}F_2 & \frac{1}{\lambda}F_3 \end{bmatrix}.$$

According to the Schur complement formula, (39) becomes

$$V = \begin{bmatrix} -I & W \\ W^{\mathrm{T}} & \Pi_5 \end{bmatrix} < 0. \tag{40}$$

If one takes

$$\Upsilon = \begin{bmatrix} 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ \gamma I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

then $\Xi_2 = \Upsilon V \Upsilon^{\mathrm{T}}$ holds. Hence, (40) is equivalent to $\Xi_2 < 0$. Summarizing the above analysis, we can conclude that the LMI (30) is equivalent to $\xi^{\mathrm{T}}\Xi_1\xi < 0$ for any nonzero vector ξ. This ensures that $J(k) < 0$, $\forall k \in \mathbb{Z}_+$. The remaining of this proof is omitted since it is immediate from Theorem 1. ■

Remark 4: From Theorems 1 and 2, it can be seen that the H_∞ control design approach and the robust H_∞ control design approach can be employed to develop sufficient conditions in terms of LMIs for the stability of the (uncertain) ILC process with time delays. With the satisfaction of certain LMIs, the learning gains can be determined directly. In addition, when τ is unknown but constant, Theorem 2 implies results similar to Corollary 1. As in Corollary 2, when τ is time-varying and satisfies (28), one can also derive the convergence results in Theorem 2 for the ILC law (6) if the following LMI holds:

$$\begin{bmatrix} -I & 0 & X_1C + X_2CA_0 & -I + X_2CB_0 & \gamma\lambda X_2CE & X_2CA_{\tau 0} \\ * & -I & \frac{1}{\lambda}F_1 & \frac{1}{\lambda}F_2 & 0 & \frac{1}{\lambda}F_3 \\ * & * & A_0^{\mathrm{T}}P + PA_0 + Q & PB_0 & \gamma\lambda PE & PA_{\tau 0} \\ * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -(1-\rho)Q \end{bmatrix} < 0. \tag{41}$$

Consequently, the learning gains of (6) are computed by (27).

IV. Simulation Results

This section examines an example for the uncertain system description (1)-(4), with $T = 1$ and $y(t,k) = [y_1(t,k) \;\; y_2(t,k)]^{\mathrm{T}}$ representing the output at iteration k. Here, the time-varying delay τ satisfying (28) is considered, i.e., $\tau = 0.2 + 0.2\sin(t)$, the constant matrices are given by

$$A_0 = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}, A_{\tau 0} = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and parameters in (3) are described by the uncertainty $\Sigma(t) = \mathrm{diag}\{\sin(10t), 1 - e^{-t}\}$ and the constant matrices as follows:

$$E = \begin{bmatrix} 0 & 0 \\ 0.8 & 0.8 \end{bmatrix},\; F_1 = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.2 \end{bmatrix},\; F_2 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix},\; F_3 = F_1.$$

Also, assume that the desired trajectory is described by

$$y_d(t) = \begin{bmatrix} y_{d_1}(t) \\ y_{d_2}(t) \end{bmatrix} = \begin{bmatrix} t^3(4 - 3t) \\ 12t^2(1 - t) \end{bmatrix},\;\; t \in [0, 1]$$

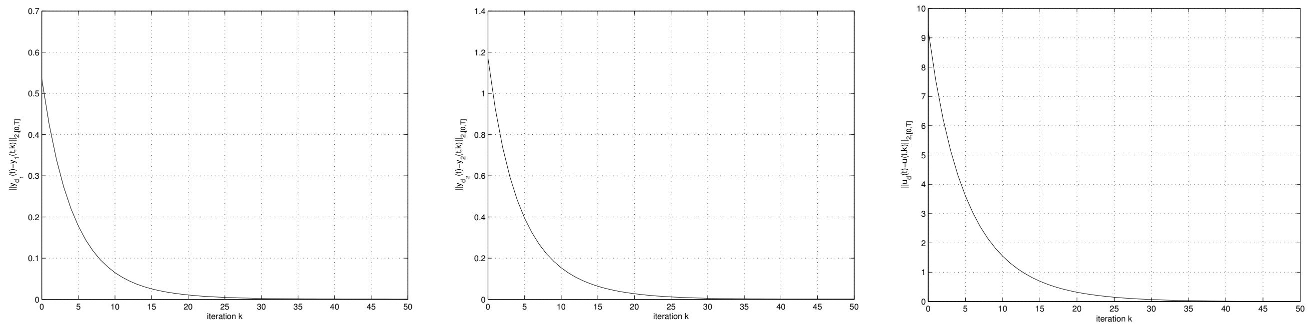

Fig. 1: ILC process of the law (6). Left: $\|y_{d_1}(t)-y_1(t,k)\|_{2,[0,1]}$. Middle: $\|y_{d_2}(t)-y_2(t,k)\|_{2,[0,1]}$. Right: $\|u_d(t)-u(t,k)\|_{2,[0,1]}$.

and the initial function is given as: $x_d(t)=0$ for all $t \leq 0$.

To apply the ILC law (6), the zero initial control input is adopted, i.e., $u(t,0)=0$ for $t \in [0,1]$. From Remark 4, the learning gain matrices can be determined using the LMI (41). For $\gamma=1$ and $\lambda=2$, and with the LMI toolbox of Matlab, we know that the LMI (41) is feasible (tmin $= -9.7743 \times 10^{-4}$), and the learning gain matrices are given by

$$K=[0.9840 \quad -5.2219], \quad \Gamma=[6.3466 \quad 0.1661].$$

Fig. 1 shows the test results, where we describe the tracking performance of the ILC process for the first 50 iterations. In this figure, we show the trend of the tracking errors, respectively, $\|y_{d_1}(t)-y_1(t,k)\|_{2,[0,1]}$ (left), $\|y_{d_2}(t)-y_2(t,k)\|_{2,[0,1]}$ (middle), and $\|u_d(t)-u(t,k)\|_{2,[0,1]}$ (right). Clearly, the ILC system is robustly stable, and the input error converges, in the sense of the $\mathscr{L}_2$-norm, monotonically to zero as a function of iteration.

V. Conclusions

In this paper, the stability analysis of the ILC problem for TDS has been presented when system parameters are subject to uncertainties. After providing the 2-D analysis of the ILC process, we have employed an approach to develop the robust stability conditions in terms of LMIs, which is based on the monotonic convergence of the control input error. It has been shown that the proposed approach can be applicable to TDS which may suffer unknown delays or time-varying delays. In particular, the effectiveness of our approach has been verified through simulation tests.

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