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$\mathcal{H}_2$ Analysis and Synthesis of Networked Dynamic Systems

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***Abstract*— This work provides a general framework for the analysis and synthesis of a class of linear networked dynamic systems (NDS). We focus our attention on NDS where the underlying connection topology couples the agents at their outputs. A distinction is made between NDS with homogeneous agent dynamics and NDS with heterogeneous agent dynamics. In the homogeneous setting, the $\mathcal{H}_2$ norm expression reduces to the Frobenius norm of the underlying connection topology incidence matrix, $E(\mathcal{G})$, scaled by the $\mathcal{H}_2$ norm of the agents comprising the NDS. In the heterogeneous case, the $\mathcal{H}_2$ norm becomes the weighted Frobenius norm of the incidence matrix, where the weights appear on the nodes of the graph. The $\mathcal{H}_2$ norm characterization is then used to synthesize NDS with certain $\mathcal{H}_2$ performance. Specifically, a semi-definite programming solution is presented to design a local controller for each agent when the underlying topology is fixed. A solution using Kruskal's algorithm for finding a minimum weight spanning tree is used to design the optimal NDS topology given fixed agent dynamics.**

I. Introduction

Networked dynamic systems (NDS) are a collection of multiple dynamic systems that are coupled together through a network. These types of systems are found in a range of applications that involve, for example, the coordination of multiple space, air, and land vehicles [1], [2], [3], [4]. Studying system theoretic notions from the perspective of the underlying topology can lead to interpretations that explicitly characterize the effects of the network on the behavior of the system.

For linear and time-invariant systems, all the essential systems theoretic properties can be derived from the quadruple system matrices (A, B, C, D). When considering mutli-agent systems, the underlying connection topology, $\mathcal{G}$, can typically be embedded into the system matrices. It is then enlightening to consider how certain properties of the system explicitly depend on that topology. Therefore, when studying linear NDS, one should consider the quintuple $(A, B, C, D, \mathcal{G})$ and explicitly describe the dependance of the underlying topology on the system properties. Recent examples of such network-centric analysis include relating closed-loop stability properties of NDS to the spectral properties of the graph Laplacian [5], relating controllability in consensus seeking systems to graph symmetry [6], and graph-centric observability properties of relative sensing NDS [7].

In this work we focus on a class of linear NDS where the underlying connection topology couples the agents at their outputs. Such systems are prevalent in formation flying applications where relative sensing is used to measure inter-agent distances [8].

The main contribution of this paper is a graph-centric characterization of the system $\mathcal{H}_2$ norm for both analysis and synthesis purposes. A distinction is made between NDS with homogeneous agent dynamics and NDS with heterogeneous agent dynamics. Although the homogenous case is actually a subset of the heterogeneous case, it is more illuminating to consider these cases separately due to the algebraic simplicity of the former case.

For the synthesis portion of this paper we consider two general design scenarios that can be akin to an inner-loop control design for an NDS. In the first case, we focus on the design of a local $\mathcal{H}_2$ controller for each agent when the underlying connection topology is given and fixed. In addition to satisfying local performance objectives (such as those typically found in $\mathcal{H}_2$ synthesis), the proposed synthesis procedure also satisfies a global NDS objective related to the underlying connection topology. A semi-definite program is derived as a solution method for this problem.

The second synthesis objective focuses on the design of the connection topology that optimizes the $\mathcal{H}_2$ performance of the NDS. Topology design can be considered a problem in combinatorial optimization, which can be a prohibitively hard to solve when the number of agents is large. The results of this paper shows that the problem can be solved using Kruskal's minimum spanning tree algorithm. It should also be noted that the design of the underlying topology in the context of systems theoretic properties, such as the $\mathcal{H}_2$ norm, has received little attention in the literature.

II. Preliminaries and Notations

A. Graphs and their Algebraic Representation

We make use of results from algebraic graph theory. The reader is referred to [9] for a detailed treatment of the subject and we present here only a minimal summary of relevant constructs and results. An undirected (simple) graph $\mathcal{G}$ is specified by a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$ whose elements characterize the incidence relation between distinct pairs of $\mathcal{V}$.

We make extensive use of the $|\mathcal{V}| \times |\mathcal{E}|$ incidence matrix, $E(\mathcal{G})$, for a graph with arbitrary orientation. The columns of $E(\mathcal{G})$ are indexed by the edges, and the i-th row entry takes the value one if it is the initial node of the corresponding edge, negative one if it is the terminal node, and zero otherwise. The degree of vertex i, d_i, is the cardinality of the set of vertices adjacent to it. The diagonal matrix $\Delta(\mathcal{G})$ contains the degree of each vertex on its diagonal.

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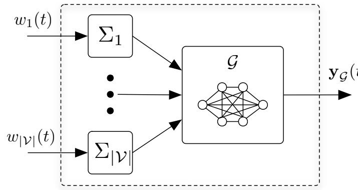

Fig. 1. Global NDS layer block diagram

The (graph) Laplacian of $\mathcal{G}$,

$$L(\mathcal{G}) := E(\mathcal{G})E(\mathcal{G})^T = \Delta(\mathcal{G}) - A(\mathcal{G}), \tag{1}$$

is a rank deficient positive semi-definite matrix. The adjacency matrix, $A(\mathcal{G})$, is the symmetric $|\mathcal{V}| \times |\mathcal{V}|$ matrix with zero on the diagonal and one in the ij-th position if node i is adjacent to node j.

B. Matrix Kronecker Products

Some important results on the Kronecker product are presented here. The Kronecker product of two matrices A and B is written as $A \otimes B$.

Theorem 2.1 ([10]): Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ each have a singular value decomposition of $A = U_A\Sigma_A V_A^T$ and $B = U_B\Sigma_B V_B^T$. The singular value decomposition of the Kronecker product of A and B is then $A \otimes B = (U_A \otimes U_B)(\Sigma_A \otimes \Sigma_B)(V_A^T \otimes V_B^T)$.

An immediate consequence of Theorem 2.1 is the following result on the matrix 2-norm, $\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$. We also make extensive use of the following Kronecker product matrix multiplication property, $(A \otimes B)(C \otimes D) = (AC \otimes BD)$, where the matrices are all of commensurate dimensions.

III. Homogeneous and Heterogeneous NDS

A NDS consists of two system layers. The first can be considered as the local agent layer corresponding to the dynamics of the individual agents in the ensemble. The second layer is a global NDS layer that represents the complete interconnected system. This section develops a general linear model for NDS that includes both the local and global layers.

We identify two classes of NDS in this paper: 1) homogeneous NDS, and 2) heterogeneous NDS. For both cases, we will work with linear and time-invariant systems,

$$\Sigma_i : \left\{ \begin{array}{rcl} \dot{x}_i(t) &=& A_i x_i(t) + B_i u_i(t) + \Gamma_i w_i(t) \\ z_i(t) &=& C_i^z x_i(t) + D_i^z u_i(t) \\ y_i(t) &=& C_i^y x_i(t), \end{array} \right. \tag{2}$$

where each agent is indexed by the sub-script i. Here, $x_i(t)$ represents the state, $u_i(t)$ the control, $w_i(t)$ an exogenous input (e.g. disturbances), $z_i(t)$ the controlled variable, and $y_i(t)$ is the measured output. In the homogeneous case, it is assumed that each dynamic agent in the NDS is described by the same set of linear state-space dynamics (e.g. $(A_i, B_i, \Gamma_i, C_i^z, D_i^z, C_i^y) = (A_j, B_j, \Gamma_j, C_j^z, D_j^z, C_j^y)$, for all i, j). When working with homogeneous NDS, we drop the sub-script for all state-space and operator representations of the system.

As we are focusing on the $\mathcal{H}_2$ properties of this system, we assume no feedforward term of the control $u_i(t)$ and no noises in the measurements (e.g. strictly proper system). Additionally, we assume a minimal realization for each agent with the outputs of each agent being compatible (e.g., system outputs correspond to the same physical quantity). It should be noted that in a heterogeneous system, the dimension of each agent need not be the same. However, without loss of generality we assume each agent to have the same dimension.

We denote the open-loop map from $w_i(t)$ to $y_i(t)$ as $T_i^{w\mapsto y}$, and the closed-loop map from $w_i(t)$ to $z_i(t)$ as $T_i^{w\mapsto z}$. The $\mathcal{H}_2$ synthesis problem for a local agent is to design a feedback controller of the form $u_i(t) = K_i y_i(t)$ that minimizes the closed-loop system norm, $\|T_i^{w\mapsto z}\|_2$.

The parallel interconnection of all agents is described with the following state-space description:

$$\begin{array}{rcl} \dot{\mathbf{x}}(t) &=& \mathbf{Ax(t)} + \mathbf{Bu(t)} + \mathbf{\Gamma w(t)} \\ \mathbf{z}(t) &=& \mathbf{C^z x(t)} + \mathbf{D^z u(t)} \\ \mathbf{y}(t) &=& \mathbf{C^y x(t)}, \end{array} \tag{3}$$

with $\mathbf{x}(t)$, $\mathbf{u}(t)$, $\mathbf{w}(t)$, $\mathbf{z}(t)$, and $\mathbf{y}(t)$ denoting, respectively, the concatenated state vector, control vector, exogenous input vector, controlled vector, and output vector of all the agents in the NDS. The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{\Gamma}$, $\mathbf{C}^z$, $\mathbf{D}^z$, and $\mathbf{C}^y$ are the block diagonal aggregation of each agent's state-space matrices.

The global NDS layer we examine for the duration of this paper is motivated by the relative sensing problem. The sensed output of the NDS is the vector $\mathbf{y}_{\mathcal{G}}(t)$ containing the relative state information of each agent and its neighbors. For example, the output sensed across an edge $e = (i, i')$ would be of the form $y_i(t) - y_{i'}(t)$. This can be compactly written as

$$\mathbf{y}_{\mathcal{G}}(t) = (E(\mathcal{G})^T \otimes I)\mathbf{y}(t). \tag{4}$$

The global layer is visualized in the block diagram in Figure 1.

When considering the analysis of the global layer, we are interested in studying the map from the agent's exogoneous inputs to the NDS sensed output, which we denote by the operator $T_{hom}^{w\mapsto\mathcal{G}}$ for homogeneous NDS, and $T_{het}^{w\mapsto\mathcal{G}}$ for heterogeneous NDS. Using the above notations and the Kronecker properties of §II-B, we can express the homogeneous and heterogeneous NDS in a compact form, shown in equations (5) and (6).

IV. $\mathcal{H}_2$ System Norm of NDS

The $\mathcal{H}_2$ norm of a system is an important performance metric in the analysis and design of feedback systems. This section aims to explicitly characterize the affect of the network on the $\mathcal{H}_2$ norm of the system.

The $\mathcal{H}_2$ norm of a system can be calculated in a variety of ways. One description involves the observability grammian

$$\Sigma_{hom}(\mathcal{G}) \quad \begin{cases} \dot{\mathbf{x}}(t) &= (I_{|\mathcal{V}|}\otimes A)\mathbf{x}(t) + (I_{|\mathcal{V}|}\otimes B)\mathbf{u}(t) + (I_{|\mathcal{V}|}\otimes \Gamma)\mathbf{w}(t) \\ \mathbf{z}(t) &= (I_{|\mathcal{V}|}\otimes C^z)\mathbf{x}(t) + (I_{|\mathcal{V}|}\otimes D^z)\mathbf{u}(t) \\ \mathbf{y}(t) &= (I_{|\mathcal{V}|}\otimes C^y)\mathbf{x}(t) \\ \mathbf{y}_{\mathcal{G}}(t) &= (E(\mathcal{G})^T\otimes C^y)\mathbf{x}(t) \end{cases} \quad (5)$$

$$\Sigma_{het}(\mathcal{G}) \quad \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{\Gamma}\mathbf{w}(t) \\ \mathbf{z}(t) &= \mathbf{C}^z\mathbf{x}(t) + \mathbf{D}^z\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}^y\mathbf{x}(t) \\ \mathbf{y}_{\mathcal{G}}(t) &= (E(\mathcal{G})^T\otimes I)\mathbf{C}^y\mathbf{x}(t) \end{cases} \quad (6)$$

of the system. The observability grammian for an individual agent based on the dynamics in (2) is defined as

$$Y_o^{(i)} = \int_0^\infty e^{A_i^T t}(C_i^y)^T C_i^y e^{A_i t}dt\,. \quad (7)$$

The observability grammian can be calculated by solving a system of linear equations, described by the Lyapunov equation $A_i^T Y_o^{(i)} + Y_o^{(i)} A_i + (C_i^y)^T C_i^y = 0$.

Another description involves the controllability grammian of the system. The controllability grammian for an individual agent (from the exogenous input channel) based on the dynamics in (2) is defined as

$$X_c^{(i)} = \int_0^\infty e^{A_i t}\Gamma_i\Gamma_i^T e^{A_i^T t}dt\,, \quad (8)$$

with the corresponding Lyapunov equation $A_i X_c^{(i)} + X_c^{(i)} A_i^T + \Gamma_i\Gamma_i^T = 0$.

The $\mathcal{H}_2$ norm of each agent from the exogenous input channel to the measured output can be expressed in terms of the grammians as

$$\|T_i^{w\mapsto y}\|_2 = \sqrt{\mathbf{tr}(\Gamma_i^T Y_o^{(i)}\Gamma_i)} = \sqrt{\mathbf{tr}(C_i X_c^{(i)} C_i^T)}. \quad (9)$$

Using the above description we can begin to understand how the underlying network topology influences the system norm. We separate our analysis into the homogeneous and heterogeneous cases.

A. Homogeneous NDS $\mathcal{H}_2$ Norm

The $\mathcal{H}_2$ norm of the homogeneous NDS described in (5) can be written in terms of the observability grammian. As mentioned in §III, when examining the global NDS layer, we consider the map $T_{hom}^{w\mapsto\mathcal{G}}$. Therefore, the expression for the observability grammian of the global NDS layer in (5) is

$$\begin{aligned}\mathbf{Y_o} &= \int_0^\infty e^{(I_N\otimes A)^T t}(E(\mathcal{G})^T\otimes C^y)^T(E(\mathcal{G})^T\otimes C^y)e^{(I_N\otimes A)t}dt \\ &= L(\mathcal{G})\otimes Y_o, \end{aligned} \quad (10)$$

where Y_o represents the observability grammian of a single agent in the network (described in (7)).

Using (10), we have the following characterization of the $\mathcal{H}_2$ norm,

$$\begin{aligned}\|T_{hom}^{w\mapsto\mathcal{G}}\|_2 &= \sqrt{\mathbf{tr}((I_N\otimes\Gamma)^T(L(\mathcal{G})\otimes Y_o)(I_N\otimes\Gamma))} \\ &= \|E(\mathcal{G})\|_F\|T^{w\mapsto y}\|_2, \end{aligned} \quad (11)$$

where $\|M\|_F$ denotes the Frobenius norm of the matrix M.

The expression in (11) gives an explicit characterization of how the network affects the overall gain of the NDS. In the homogeneous case, we can focus our attention on how the Frobenius norm of the incidence matrix changes with the addition or removal of an edge. Recall that the Frobenius norm of a matrix can be expressed as the sum of the vector 2-norm of each column.

In the case of the incidence matrix, each column, representing a single edge of the graph, always has the same structure. Therefore, the Frobenius norm of the incidence matrix can be expressed in terms of the number of edges in the graph, $|\mathcal{E}|$, as $\|E(\mathcal{G})\|_F = (2\,|\mathcal{E}|)^{1/2}$.

One immediate consequence of this description is that the NDS $\mathcal{H}_2$ norm is only dependent on the number of edges in the graph rather than the actual structure of the topology. This makes intuitive sense, as more edges would correspond to additional amplification of the disturbances entering the system.

If we consider only connected graphs, then we have immediate lower and upper bounds on the $\mathcal{H}_2$ norm of the system,

$$\|T_{hom}^{w\mapsto\mathcal{G}}\|_2^2 \geq 2\,\|T^{w\mapsto y}\|_2^2\,(|\mathcal{V}|-1). \quad (12)$$

The lower bound is attained with equality whenever the underlying graph is a spanning tree. It is clear from the definition of the Frobenius norm that the choice of tree is inconsequential (e.g. a star or a path).

If we assume that all graphs are simple, that is they do not have multiple edges between a single pair of nodes, then the upper bound for the system norm is achieved by the complete graph,

$$\|T_{hom}^{w\mapsto\mathcal{G}}\|_2^2 \leq 2\,\|T^{w\mapsto y}\|_2^2\,|\mathcal{V}|\,(|\mathcal{V}|-1) \quad (13)$$

B. Heterogeneous NDS $\mathcal{H}_2$ Norm

In the heterogeneous case, the NDS $\mathcal{H}_2$ norm can be derived by using (9) as,

$$\|T_{het}^{w\mapsto\mathcal{G}}\|_2^2 = \mathbf{tr}\big\{(E(\mathcal{G})^T\otimes I)\mathbf{C}^y\mathbf{X}_c(\mathbf{C}^y)^T(E(\mathcal{G})\otimes I)\big\}, \quad (14)$$

where $\mathbf{X}_c$ denotes the block diagonal aggregation of each agent's controllability grammian, as definied in (8). First, note that $\mathbf{tr}\big\{\mathbf{C}^y\mathbf{X}_c(\mathbf{C}^y)^T\big\} = \sum_{i=1}^{|\mathcal{V}|}\|T_i^{w\mapsto y}\|_2^2$. Using the cycle property of the trace operator and exploiting the block diagonal structure of the argument leads to the following identity simplification,

$$\mathbf{tr}\left\{\mathbf{C}^y\mathbf{X}_c(\mathbf{C}^y)^T((\Delta(\mathcal{G})-A(\mathcal{G}))\otimes I)\right\} = \sum_i \mathbf{tr}\left\{C_i^y X_c^{(i)}(C_i^y)^T(d_i\otimes I)\right\} = \sum_i d_i \,\|T_i^{w\mapsto y}\|_2^2,$$

where d_i is the degree of the i-th agent in the graph.

This can now be used to obtain the following expression for the $\mathcal{H}_2$ norm of the system,

$$\|T_{het}^{w\mapsto\mathcal{G}}\|_2 \quad = \quad \left(\sum_i d_i \,\|T_i^{w\mapsto y}\|_2^2\right)^{1/2}. \tag{15}$$

An even further examination of the above term reveals that it can be written as the Frobenius norm of a node-weighted incidence matrix,

$$\|T_{het}^{w\mapsto\mathcal{G}}\|_2 \quad = \quad \| \begin{bmatrix} \|T_1^{w\mapsto y}\|_2 & & \\ & \ddots & \\ & & \|T_{|\mathcal{V}|}^{w\mapsto y}\|_2 \end{bmatrix} E(\mathcal{G})\|_F. \tag{16}$$

When each agent has the same dynamics, (16) reduces to the expression in (11). This characterization paints a very clear picture of how the placement of an agent within a certain topology affects the overall system gain. In order to minimize the gain, it is beneficial to keep systems with high norm in locations with minimum degree.

V. SYNTHESIS OF NDS

The results of §IV can be used to develop a performance metric for the synthesis of NDS. The objective is to design a local controller K_i for each agent in the ensemble that minimizes some local performance objective, $\|T_i^{w\mapsto z}\|_2$ while additionally minimizing the global NDS objective, $\|T_{het}^{w\mapsto\mathcal{G}}\|_2$. This is visualized in the block diagram in Figure 2. It should be noted that this problem does not consider the design of feedback controllers to achieve higher level objectives for the network, such as formation control. Rather, this procedure is analogous to an inner-loop control design, whereas additional performance involving the network would be likened to the outer-loop design of the system.

In this setting, we propose two scenarios for the synthesis of NDS. In the first case, we consider designing the local control for each agent when the underlying topology and the placement of agents within that topology is given and fixed. A semi-definite programming solution is developed to solve this problem.

The second case examines how to design the optimal topology and placement of agents within the topology, assuming that each agent already has a local controller designed. We cite a result from combinatorial optimization and describe how it can be applied to this problem.

A. Local Agent Control for Fixed Topology

For this problem we will consider a heterogeneous NDS with a given and fixed topology, $E(\mathcal{G})$. Each agent, Σ_i, is also assigned a fixed location within the network. From a synthesis point of view, each agent behaves independently and does not use information from the NDS for its control.

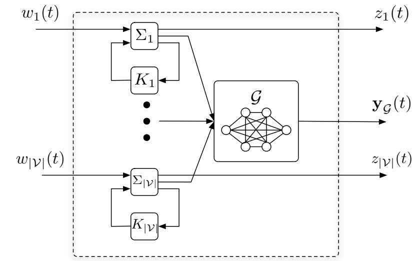

Fig. 2. $\mathcal{H}_2$ Synthesis of NDS

To simplify this discussion, we will assume that each agent has full-state feedback available for its controller ($C_i^y = I$). For this example, we also assume that the global NDS output corresponds to a relative position measurement. Therefore, the NDS output $\mathbf{y}_{\mathcal{G}}(t)$ will be described as

$$\mathbf{y}_{\mathcal{G}}(t) = E(\mathcal{G})^T \otimes \begin{bmatrix} \mathbf{1}^T & 0 & \cdots & 0 \end{bmatrix} = E(\mathcal{G})^T \otimes C_p\,; \tag{17}$$

here we have assumed the states corresponding to position are the first p states of the state vector $x(t)$.

The state-feedback optimal $\mathcal{H}_2$ control problem for a single agent without considering the global NDS layer can be formulated as an SDP [11].

$$\min_{W_i,X_i,Z_i} \quad \mathbf{tr}[W_i] \tag{18}$$

s.t.

$$\begin{bmatrix} A_i & B_i \end{bmatrix}\begin{bmatrix} X_i \\ Z_i \end{bmatrix} + \begin{bmatrix} X_i & Z_i^T \end{bmatrix}\begin{bmatrix} A_i^T \\ B_i^T \end{bmatrix} + \Gamma_i\Gamma_i^T < 0$$
$$\begin{bmatrix} X_i & (C_i^z X_i + D_i^z Z_i)^T \\ (C_i^z X_i + D_i^z Z_i) & W_i \end{bmatrix} > 0;$$

the control gain can then be reconstructed as $K_i = Z_i X_i^{-1}$.

From the above SDP, we have that $\|T_i^{w\mapsto z}\|_2^2 = \mathbf{tr}(W_i)$. Here, we note that in the above SDP, the matrix X_i actually corresponds to the controllability grammian of the closed-loop system for agent i. That is, it is the controllability grammian for a realization of the system $T_i^{w\mapsto z}$.

The SDP in (18), however, does not incorporate the global NDS performance objective into the problem. While each agent can generate a solution to (18) independently of each other, the addition of the global NDS layer couples the design of each agent's controller. To illustrate this, we should examine the map $T_{het}^{w\mapsto\mathcal{G}}$ in the context of Figure 2. This is easily accomplished by considering the system in (6). We will treat the NDS output $\mathbf{y}_{\mathcal{G}}(t)$ as an additional performance variable, and rewrite the system as

$$\begin{aligned} \dot{\mathbf{x}}(t) \quad &= \quad \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{\Gamma}\mathbf{w}(t) \\ \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{y}_{\mathcal{G}}(t) \end{bmatrix} \quad &= \quad \begin{bmatrix} \mathbf{C}^z \\ E(\mathcal{G})^T \otimes C_p \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} \mathbf{D}^z \\ 0 \end{bmatrix}\mathbf{u}(t) \\ \mathbf{y}(t) \quad &= \quad I\mathbf{x}(t). \end{aligned} \tag{19}$$

Using the augmented state-space description in (19) we have the following result for the synthesis of controllers for each agent while incorporating the global NDS objecitve.

Theorem 5.1: Given the NDS system described in (19), a local state-feedback controller of the form $u_i(t) = K_i x_i(t)$ that minimizes local performance objectives in addition to the global NDS performance objective can be found by solving

$$\min_{W_i,X_i,Z_i,V_i} \quad \sum_i^{|\mathcal{V}|} \mathbf{tr}[W_i] + \mathbf{tr}[V_i] \qquad (20)$$

s.t.

$$\begin{bmatrix} A_i & B_i \end{bmatrix} \begin{bmatrix} X_i \\ Z_i \end{bmatrix} + \begin{bmatrix} X_i & Z_i^T \end{bmatrix} \begin{bmatrix} A_i^T \\ B_i^T \end{bmatrix} + \Gamma_i \Gamma_i^T \leq 0 \quad (21)$$

$$\begin{bmatrix} X_i & (C_i^z X_i + D_i^z Z_i)^T \\ (C_i^z X_i + D_i^z Z_i) & W_i \end{bmatrix} > 0 \quad (22)$$

$$\begin{bmatrix} X_i & (C_p X_i)^T \\ C_p X_i & \frac{1}{d_i} V_i \end{bmatrix} > 0 \qquad (23)$$

where $K_i = Z_i X_i^{-1}$.

Proof: Consider the system in (19) with a control $\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)$ implemented, where $\mathbf{K} = \mathbf{diag}(K_1, \ldots, K_{|\mathcal{V}|})$. The closed-loop system becomes

$$\Sigma_{cl} \left\{ \begin{array}{rcl} \dot{\mathbf{x}}(t) & = & (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{\Gamma}\mathbf{w}(t) \\ \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{y}_{\mathcal{G}}(t) \end{bmatrix} & = & \begin{bmatrix} \mathbf{C}^z + \mathbf{D}^z\mathbf{K} \\ E(\mathcal{G})^T \otimes C_p \end{bmatrix} \mathbf{x}(t). \end{array} \right. \qquad (24)$$

To guarantee the stability of the closed loop system, we require that $\mathbf{A} + \mathbf{B}\mathbf{K}$ be Hurwitz. This is guaranteed by the LMI given in (21) by noting the block diagonal structure of the matrix, and defining $Z_i = K_i X_i$. In fact, when the constraint (21) is satisfied at equality, we note that X_i is the controllability grammian for the system in (24).

The $\mathcal{H}_2$ norm of (24) can be calculated as

$$\begin{aligned} \|\Sigma_{cl}\|_2^2 &= \mathbf{tr}\left\{ \begin{bmatrix} \mathbf{C}^z + \mathbf{D}^z\mathbf{K} \\ E(\mathcal{G})^T \otimes C_p \end{bmatrix} \mathbf{X} \begin{bmatrix} \mathbf{C}^z + \mathbf{D}^z\mathbf{K} \\ E(\mathcal{G})^T \otimes C_p \end{bmatrix}^T \right\} \\ &= \mathbf{tr}\{(\mathbf{C}^z + \mathbf{D}^z\mathbf{K})\mathbf{X}(\mathbf{C}^z + \mathbf{D}^z\mathbf{K})^T\} + \\ &\quad \mathbf{tr}\{(E(\mathcal{G})^T \otimes C_p)\mathbf{X}(E(\mathcal{G})^T \otimes C_p)^T\}, \qquad (25) \end{aligned}$$

where $\mathbf{X} = \mathbf{diag}(X_1, \ldots, X_{|\mathcal{V}|})$.

The first term on the right hand side corresponds precisely to the $\mathcal{H}_2$ norm of the system in (3) with the feedback law $\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)$ implemented. The second term is the $\mathcal{H}_2$ norm of $T_{het}^{w \mapsto \mathcal{G}}$. Using the results from §IV we can express the performance as

$$\begin{aligned} \|T_{het}^{w\mapsto\mathcal{G}}\|_2^2 &= \mathbf{tr}\{(E(\mathcal{G})^T \otimes C_p)\mathbf{X}(E(\mathcal{G}) \otimes C_p)\} \\ &= \sum_i^{|\mathcal{V}|} d_i \mathbf{tr}\{C_p X_i C_p^T\}. \qquad (26) \end{aligned}$$

The objective is to minimize $\|\Sigma_{cl}\|_2$, which can be accomplished by minimizing both terms in the right-hand side of (25). Using the matrix Schur-complement, we note that $d_i C_p X_i C_p^T < V_i$ is equivalent to

$$\begin{bmatrix} X_i & (C_p X_i)^T \\ C_p X_i & \frac{1}{d_i} V_i \end{bmatrix} > 0. \qquad (27)$$

We now note that if $d_i C_p X_i C_p^T < V_i$, then $d_i \mathbf{tr}\{C_p X_i C_p^T\} < \mathbf{tr}\{V_i\}$.

A similar derivation is used to arrive at the LMI in (22). ■

A striking feature of the SDP (20)-(23) is its structure. Although the global NDS layer couples each agent, we see that the coupling can be removed via the formulation of the $\mathcal{H}_2$ norm. The SDP is therefore separable across each of the agents which has implications for the parallelization of the computation and decision-making process.

B. Topology Design and Agent Placement

In this section, we consider how to design the underlying connection topology and where to place agents within that topology. Recall from §IV that in terms of the $\mathcal{H}_2$ norm objective, an optimal topology should always correspond to a spanning tree. The design problem, therefore, is to determine which spanning tree will achieve the smallest $\mathcal{H}_2$ norm for the NDS.

We assume in this case that each agent has already adopted a feedback controller for its operation. Using the same relative position sensing model, the NDS state-space description can be written as

$$\begin{array}{rcl} \dot{\mathbf{x}}(t) & = & \mathbf{A}\mathbf{x}(t) + \mathbf{\Gamma}\mathbf{w}(t) \\ \mathbf{y}_{\mathcal{G}}(t) & = & (E(\mathcal{G})^T \otimes C_p)\mathbf{x}(t). \end{array} \qquad (28)$$

The design of the topology reduces to the design of the incidence matrix, $E(\mathcal{G})$. This problem is combinatorial in nature, as there are only a finite number of graphs that can be constructed from a set of N nodes. As the number of agents in the NDS becomes large, solving this problem becomes prohibitively hard. However, we find that with an appropriate modification of the problem statement, results from combinatorial optimization can be used, leading to a polynomial-time algorithm.

Specifically, the *minimum spanning tree* (MST) problem solves this problem. The MST can be efficiently solved using Kruskal's algorithm in $\mathcal{O}(|\mathcal{E}| \log |\mathcal{V}|)$ time. The algorithm is given below and a proof of its correctness can be found in [12].

Algorithm 1: Kruskal's Algorithm

Data: A connected undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and weights $w : \mathcal{E} \mapsto \mathbb{R}$.
Result: A spanning tree $\mathcal{G}_t$ of minimum weight.
begin
  Sort the edges such that
  $w(e_1) \leq w(e_2) \leq \cdots \leq w(e_{|\mathcal{E}|})$, where $e_i \in \mathcal{E}$
  Set $\mathcal{G}_t := \mathcal{G}_t(\mathcal{V}, \emptyset)$
  for $i := 1$ **to** $|\mathcal{E}|$ **do**
    if $\mathcal{G}_t + e_i$ *contains no cycle* **then**
      set $\mathcal{G}_t := \mathcal{G}_t + e_i$
end

In order to apply the MST to the $\mathcal{H}_2$ synthesis problem we must reformulate the original problem statement. To begin, we first write the expression for the $\mathcal{H}_2$ norm of the system in (28) as

$$\|T_{het}^{w\mapsto\mathcal{G}}\|_2^2 = \sum_i^{|\mathcal{V}|} d_i \mathbf{tr}\{C_p X_i C_p^T\} = \sum_i^{|\mathcal{V}|} d_i \|T_i^{w\mapsto p}\|_2^2. \quad (29)$$

We re-emphasize here that the NDS norm description is related to the degree of each node in the network. Using the weighted incidence graph interpretation of the norm, as in (16), we see that the gain of each agent, $\|T_i^{w\mapsto p}\|_2^2$, acts as a weight on the nodes.

As each agent is assumed to have fixed dynamics, the problem of minimizing the NDS $\mathcal{H}_2$ norm reduces to finding the degree of each agent while ensuring the resulting topology is a spanning tree. This objective is related to properties of the nodes of the graph. To use the MST results, we must convert the objective from weights on the nodes to weights on the edges.

Consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with fixed weights w_i on each node ($i = 1, \ldots, |\mathcal{V}|$). The node-weighted Frobenius norm of the incidence matrix is

$$\|WE(\mathcal{G})\|_F^2 = \sum_i d_i w_i^2;\ W = \mathbf{diag}(w_1, \ldots, w_{|\mathcal{V}|}). \quad (30)$$

Next, consider the effect of adding an edge $\hat{e} = (i,j)$ to $\mathcal{E}$ in terms of the Frobenius norm of the augmented incidence matrix,

$$\|W \begin{bmatrix} E(\mathcal{G}) & \hat{e} \end{bmatrix}\|_F^2 = \left(\sum_k d_k w_k^2\right) + w_i^2 + w_j^2\ , \quad (31)$$

where d_k represents the degree of node k *before* adding the new edge $\hat{e}$. This shows that each edge $\hat{e} = (i,j)$ contributes $(w_i^2 + w_j^2)$ to the system norm. Therefore, weights on the edges can be constructed by adding the node weights corresponding to the nodes adjacent to each edge as

$$\mathbf{w}_e = |E(\mathcal{G})^T| \mathbf{w}_n^2. \quad (32)$$

This result can be used to generate an equivalent norm characterization to the one presented in (29)

$$\|T_{het}^{w\mapsto\mathcal{G}}\|_2^2 \quad = \quad \|\ |E(\mathcal{G})^T| \begin{bmatrix} \|T_1^{w\mapsto p}\|_2^2 \\ \vdots \\ \|T_{|\mathcal{V}|}^{w\mapsto p}\|_2^2 \end{bmatrix} \|_1, \quad (33)$$

where $\|x\|_1 = \sum_i |x_i|$.

Using the above transformation from node weights to edge weights, we arrive at the following result.

Theorem 5.2: The connection topology that minimizes the $\mathcal{H}_2$ norm of (28) can be found using Kruskal's MST algorithm with input data $\mathcal{G}$, and weights parameterized as in (32), with $\mathbf{w}_n$ corresponding to the $\mathcal{H}_2$ norm of each agent.

Proof: The proof follows from (29) and the transformation of node weights to edge weights described in (30)-(32). ■

Lemma 5.3: When the input graph in Theorem 5.2 is the complete graph, then the star graph with center node corresponding to the agent with minimum norm is the (non-unique) optimal topology.

Proof: The degree of the center node in a star graph is $N-1$, and all other nodes have degree one. Assume the node weights are sorted as $w_1 \leq \cdots \leq w_N$, then the $\mathcal{H}_2$ norm of the NDS is $\|T_{het}^{w\mapsto\mathcal{G}}\|_2^2 = (N-1)w_1 + \sum_{i=2}^N w_i$. Any other tree can be obtained by removing and adding a single edge, while ensuring connectivity. With each such operation, the cost is non-decreasing, as any new edge will increase the degree of node $i > 1$ and by assumption $w_1 \leq w_i$. ■

Theorem 5.2 is especially useful if there are certain communication or sensing constraints between agents. For example, one may consider a scenario where agents are initially randomly distributed and can only sense neighboring agents within a specified range. The results of Theorem 5.2 can be used to determine the optimal spanning tree for that initial network.

VI. Concluding Remarks

This paper focused on the analysis and synthesis of a class of linear NDS based on a relative sensing model. The results of this paper highlight an important connection between certain graph-theoretic concepts and systems-theoretic properties. Perhaps the most salient feature of this work pertains to the application of the celebrated MST algorithm from combinatorial optimization for designing interconnection topology for overall optimal $\mathcal{H}_2$ performance. This work also suggests a close relationship between systems-theoretic properties and graph properties in NDS which can be examined further in the systems community. In fact, we believe that developing efficient solution methods for the design of such systems will involve connecting and interpreting results from graph theory and combinatorial optimization in a systems-theoretic context.

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