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The Operator Algebra of Almost Toeplitz Matrices and the Optimal Control of Large–Scale Systems

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Abstract— **We propose a definition of almost Toeplitz matrices as matrices with off-diagonal decay that are close to begin Toeplitz in their center columns and decrease in Toeplitzness toward their first and last columns. We prove that such matrices form an operator algebra under matrix addition and multiplication. We use this framework to show that algebraic Riccati equations with almost Toeplitz coefficient matrices have almost Toeplitz solutions.**

I. Introduction

Toeplitz matrices arise in many problems of engineering and applied mathematics. For example, large-scale systems composed of similar components are described by Toeplitz matrices, as are finite-difference approximations to partial differential operators.

It is well-known that the product of two biinfinite Toeplitz matrices and the inverse of a biinfinite Toeplitz matrix are again Toeplitz matrices. This is *not* true, however, in the case of finite-dimensional Toeplitz matrices. Nonetheless, for certain types of finite-dimensional Toeplitz matrices, their products and inverses remain "very close to being Toeplitz" or "almost Toeplitz" in their interior, i.e., if the first few and last few columns of an almost Toeplitz matrix are ignored the remaining columns resemble closely those of a Toeplitz matrix.

An important motivation for this work has been to show that the solution of a wide range of algebraic matrix equations, whose coefficients are Toeplitz matrices, are almost Toeplitz. Of particular interest are the algebraic Lyapunov and Riccati equations, which arise in the analysis and synthesis of linear systems. In this paper we show that if the coefficient matrices A and Q are Toeplitz then the solution P of the algebraic Riccati equation $A^*P + PA + Q - P^2 = 0$ is almost Toeplitz.

Almost Toeplitz matrices have been the subject of studies in the past; see [1], [2], [3], [4] and references therein. References [1],[4, p. 137] characterize the "distance from Toeplitzness" of an $N \times N$ matrix P by the integer $d_P = \operatorname{rank}(PS - SP)$, where S is the shift matrix. If d_P is small compared to N then P is almost Toeplitz. Unfortunately this characterization is too rigid for the type of applications of interest to us. For example, d increases under matrix multiplication; d_{AB} can be as large as 4 even though $d_A = d_B = 2$ for any two Toeplitz matrices A, B that are not diagonal or zero. More importantly, if P is the solution of the algebraic Riccati equation $A^*P + PA + Q - P^2 = 0$ with Toeplitz coefficients A, Q ($d_A = d_Q = 2$) then d_P can be of the same order as N. Thus, based on the definition of almost Toeplitzness proposed by [1], [4], one would conclude that the solution P is far from being Toeplitz. However, simple numerical examples show that for a wide class of Toeplitz matrices A, Q the solution P is indeed very Toeplitz-like in its interior.

The definition of almost Toeplitzness that we propose in this paper is based on selecting all possible pairs of columns (P_r, P_s) of a matrix P, appropriately shifting them to form the pairs $(\widetilde{P}_r, \widetilde{P}_s)$, and placing bounds on the norms $\|\widetilde{P}_r - \widetilde{P}_s\|$. The vectors $\widetilde{P}_r, \widetilde{P}_s$ are formed by shifting the elements of P_r, P_s in such a way that the element of P_r that belongs on the diagonal of P becomes aligned with the element of P_s that belongs on the diagonal of P. Clearly if P is a biinfinite Toeplitz matrix then $\|\widetilde{P}_r - \widetilde{P}_s\| = 0$ for all pairs of indices (r, s). If P is a finite-dimensional almost Toeplitz matrix then $\|\widetilde{P}_r - \widetilde{P}_s\|$ is large when $|r - s|$ is large, and $\|\widetilde{P}_r - \widetilde{P}_s\|$ is small when both (i) $|r - s|$ is small and (ii) $1 \ll r \ll N$, $1 \ll s \ll N$ (i.e., when neither P_r nor P_s correspond to the first few or last few columns of P).

We show that, under some additional conditions on the off-diagonal decay rate of matrix elements, almost Toeplitz matrices form an operator algebra $\mathscr{A}$; if $A, B, C_n, n \geq 0$ belong to $\mathscr{A}$ then so do $A+B$, AB, and C, if $C_n \to C$. This gives us a powerful framework in which to analyze the solution of matrix equations with almost Toeplitz coefficients. In particular, we show how this framework can be utilized to prove that the solution of an algebraic Riccati equation with Toeplitz coefficients is almost Toeplitz. Furthermore, we describe how this justifies the approximation of a large-scale system with a spatially invariant one, for which Fourier methods can be used to significantly simplify the problem of optimal controller design.

II. The Operator Algebra of Almost Toeplitz Matrices

In this paper all vectors belong to $\mathbb{C}^{(2N+1)}$ and all matrices belong to $\mathbb{C}^{(2N+1)\times(2N+1)}$. We use the standard Euclidean 2-norm for vectors and the induced 2-norm for matrices. Omitted entries in sparse matrices are zero.

Definition 1 (Spatially Decaying Matrices): The matrix A belongs to the space $\mathscr{Q}_\alpha$ of spatially decaying matrices

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with decay rate α if

$$|A_{ij}| \;\le\; \kappa\,(1+|i-j|)^{-\alpha} \quad \text{for all} \quad \begin{array}{c} 1 \le i \le 2N+1 \\ 1 \le j \le 2N+1 \end{array} \qquad (1)$$

for $\alpha > 1$ and some constant $\kappa > 0$.

It is shown in [5], [6] that $\mathscr{Q}_\alpha$ forms an operator algebra under matrix addition and multiplication.

Definition 2: The shift matrices S_n, $-N \le n \le N$ are defined as matrices with 1s on their nth subdigonal and zeros elsewhere; we assume $n > 0$ corresponds to the lower subdiagonals and that S_0 is the identity matrix. The basis vectors ϕ_m, $-N \le m \le N$ are defined as the column vectors

$$\phi_m \;=\; \mathrm{col}\,\{\delta_{N+1+m,j}\}, \quad 1 \le j \le 2N+1.$$

Note that with the indexing convention used in the definition of ϕ_m we have $\phi_0 = \begin{bmatrix} \cdots & 0 & 1 & 0 & \cdots \end{bmatrix}^T$, where the 1 appears in the $(N+1)$st location, and

$$\phi_{-N} = \begin{bmatrix} 1 & 0 & 0 & \cdots \end{bmatrix}^T, \qquad \phi_N = \begin{bmatrix} \cdots & 0 & 0 & 1 \end{bmatrix}^T.$$

Definition 3 (Almost Toeplitz Matrices $\mathscr{T}_{\delta,\rho,\alpha}$): The matrix A belongs to the space $\mathscr{T}_{\delta,\rho,\alpha}$ of almost Toeplitz matrices if the following two conditions hold.

(i) A satisfies the inequality

$$\|(AS_n - S_nA)\,\phi_m\| \;\le\; f_{\delta,\rho}(m,n) \qquad (2)$$
$$\text{for all} \quad \begin{array}{c} -N \le m \le N \\ -2N \le n \le 2N \end{array}$$

where

$$\begin{aligned} &f_{\delta,\rho}(m,n) \\ &\;=\; \mu\left((1+N+m)^{-\delta} + (1+N-m)^{-\delta}\right) \\ &\quad+\; \eta\left((1+|N+m+n|)^{-\rho} + (1+|N-m-n|)^{-\rho}\right) \end{aligned}$$

for $\delta, \rho > 1$, and some constants $\mu, \eta > 0$;

(ii) $A \in \mathscr{Q}_\alpha$ for some α that satisfies $\alpha > \delta+1$, $\alpha > \rho+1$.

It is important to note that μ and η are uniform constants independent of N.

Let us elaborate on (2). The expression $\|(AS_n - S_nA)\,\phi_m\|$ for $n \ge 0$ (respectively $n \le 0$) takes the column of A that is m columns removed from the center column, shifts its entries down (respectively up) by n places, and compares the resulting vector with the column of A that is $m+n$ columns removed from the center column. For example, let

$$A \;=\; \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}.$$

Then, for $m = 0$ and $n = 1$ we have

$$\phi_0 \;=\; \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad S_1\phi_0 \;=\; \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad AS_1\phi_0 \;=\; \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix},$$

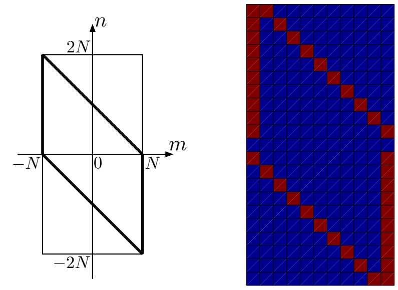

Fig. 1. Left: The rectangle encloses the domain in which m, n vary. The bold lines highlight the neighborhood where $\|(AS_n - S_nA)\,\phi_m\|$ is largest. Right: The values taken by $\|(AS_n - S_nA)\,\phi_m\|$ where A is the 11×11 ($N = 5$) version of the matrix A. The colors red and blue represent the values 1 and 0, respectively.

$$A\phi_0 \;=\; \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad S_1A\phi_0 \;=\; \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix},$$

and thus $\|(AS_1 - S_1A)\,\phi_0\| = 0$. However, for $m = 2$ and $n = -1$ we have

$$\phi_2 \;=\; \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad S_{-1}\phi_2 \;=\; \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad AS_{-1}\phi_2 \;=\; \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix},$$

$$A\phi_2 \;=\; \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \quad S_{-1}A\phi_2 \;=\; \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 0 \end{bmatrix},$$

and thus $\|(AS_{-1} - S_{-1}A)\,\phi_2\| = 1$.

If the interior columns of a matrix resemble those of a Toeplitz matrix, then the value of $\|(AS_n - S_nA)\,\phi_m\|$ is small when *both* $|m|$ and $|n|$ are small and increases as *either* $|m|$ or $|m+n|$ approaches N; see Figure 1. This can be explained as follows: If A is Toeplitz-like in its interior, then for small values of $|n|$ the matrix $AS_n - S_nA$ has small entries on its interior columns, which means that $(AS_n - S_nA)\,\phi_m$ has small norm for small values of $|m|$. As $|n|$ increases, large entries appear on the outer columns of $AS_n - S_nA$, which means that $(AS_n - S_nA)\,\phi_m$ has large norm for large values of $|m|$. Therefore, the bound $f_{\delta,\rho}(\cdot,\cdot)$ is chosen to have the same behavior as a function of m, n as that just described.

Notice that in the example given above even though A is a Toeplitz matrix there exist values of m, n for which $\|(AS_n - S_nA)\,\phi_m\| \ne 0$. In fact, according to Definition 3 only diagonal and biinfinite Toeplitz matrices are *exactly* Toeplitz in the sense that $\|(AS_n - S_nA)\,\phi_m\| = 0$ for all m, n.

The following theorem constitutes the main result of this paper.

Theorem 1 (Operator Algebra $\mathscr{T}_{\delta,\rho,\alpha}$): The space $\mathscr{T}_{\delta,\rho,\alpha}$ of almost Toeplitz matrices forms an operator algebra under matrix addition and multiplication.

Proof: We have to show that

(a) $A + B \in \mathscr{T}_{\delta,\rho,\alpha}$ for every $A, B \in \mathscr{T}_{\delta,\rho,\alpha}$;
(b) $AB \in \mathscr{T}_{\delta,\rho,\alpha}$ for every $A, B \in \mathscr{T}_{\delta,\rho,\alpha}$;
(c) $\mathscr{T}_{\delta,\rho,\alpha}$ is closed.

Proof of (a): Assume that $A, B \in \mathscr{T}_{\delta,\rho,\alpha}$. We have

$$\begin{aligned}
&\|([A+B]S_n - S_n[A+B])\,\phi_m\| \\
&\le \|(AS_n - S_nA)\,\phi_m\| + \|(BS_n - S_nB)\,\phi_m\| \\
&\le (\mu_A + \mu_B)\big((1{+}N{+}m)^{-\delta} + (1{+}N{-}m)^{-\delta}\big) \\
&\quad + (\eta_A + \eta_B)\big((1{+}|N{+}m{+}n|)^{-\rho} + (1{+}|N{-}m{-}n|)^{-\rho}\big).
\end{aligned}$$

Furthermore, since $A, B \in \mathscr{Q}_\alpha$ and $\mathscr{Q}_\alpha$ is an operator algebra then $A + B \in \mathscr{Q}_\alpha$, [5], [6]. Thus $A + B \in \mathscr{T}_{\delta,\rho,\alpha}$.

Proof of (b): Assume that $A, B \in \mathscr{T}_{\delta,\rho,\alpha}$. We have

$$\begin{aligned}
&\|(ABS_n - S_nAB)\,\phi_m\| \\
&= \|(ABS_n - AS_nB)\,\phi_m + (AS_nB - S_nAB)\,\phi_m\| \\
&\le \|A\,(BS_n - S_nB)\,\phi_m\| + \|(AS_n - S_nA)\,B\,\phi_m\|. \quad (3)
\end{aligned}$$

We simplify each of the terms on the right of inequality (3) separately. For the first term we have

$$\begin{aligned}
&\|A\,(BS_n - S_nB)\,\phi_m\| \\
&\le \|A\|\,\mu_B\big((1{+}N{+}m)^{-\delta} + (1{+}N{-}m)^{-\delta}\big) \\
&\quad + \|A\|\,\eta_B\big((1{+}|N{+}m{+}n|)^{-\rho} + (1{+}|N{-}m{-}n|)^{-\rho}\big).
\end{aligned}$$

For the second term we have

$$\begin{aligned}
&\|(AS_n - S_nA)\,B\,\phi_m\| \\
&\le \sum_{k=-N-m}^{N-m}\Big[\|(AS_n - S_nA)\,\phi_{m+k}\|\,|B_{(N+1+m+k)(N+1+m)}|\Big] \\
&\le \sum_{k=-N-m}^{N-m}\Big[\mu_A\big((1{+}N{+}m{+}k)^{-\delta} \quad (4)\\
&\qquad + (1{+}N{-}m{-}k)^{-\delta}\big)\,\kappa_B\,(1{+}|k|)^{-\alpha} \\
&\qquad + \eta_A\big((1{+}|N{+}m{+}k{+}n|)^{-\rho} \\
&\qquad + (1{+}|N{-}m{-}k{-}n|)^{-\rho}\big)\,\kappa_B\,(1{+}|k|)^{-\alpha}\Big] \\
&\le \mu'_A\big((1{+}N{+}m)^{-\delta} + (1{+}N{-}m)^{-\delta}\big) \\
&\quad + \eta'_A\big((1{+}|N{+}m{+}n|)^{-\rho} + (1{+}|N{-}m{-}n|)^{-\rho}\big)
\end{aligned}$$

where in the second inequality we have used the fact that $B \in \mathscr{Q}_\alpha$. μ'_A and η'_A are uniform constants independent of N; the expressions for these constants are derived in the appendix. We have thus shown that

$$\begin{aligned}
&\|(ABS_n - S_nAB)\,\phi_m\| \\
&\le (\|A\|\,\mu_B + \mu'_A)\big((1{+}N{+}m)^{-\delta} + (1{+}N{-}m)^{-\delta}\big) \\
&\quad + (\|A\|\,\eta_B + \eta'_A)\big((1{+}|N{+}m{+}n|)^{-\rho} \\
&\qquad + (1{+}|N{-}m{-}n|)^{-\rho}\big).
\end{aligned}$$

Finally, since $A, B \in \mathscr{Q}_\alpha$ and $\mathscr{Q}_\alpha$ is an operator algebra then $AB \in \mathscr{Q}_\alpha$, [5], [6]. Thus $AB \in \mathscr{T}_{\delta,\rho,\alpha}$.

Proof of (c): We consider a sequence of matrices $\{A_q\}_{q\in\mathbb{N}}$ such that $A_q \in \mathscr{T}_{\delta,\rho,\alpha}$ for all $q \ge 0$ and $A_q \to A$ as $q \to \infty$. Let

$$\begin{aligned}
&\|(A_qS_n - S_nA_q)\,\phi_m\| \\
&\le \mu_q\big((1{+}N{+}m)^{-\delta} + (1{+}N{-}m)^{-\delta}\big) \\
&\quad + \eta_q\big((1{+}|N{+}m{+}n|)^{-\rho} + (1{+}|N{-}m{-}n|)^{-\rho}\big),
\end{aligned}$$

where $\{\mu_q\}_{q\in\mathbb{N}}$ and $\{\eta_q\}_{q\in\mathbb{N}}$ are convergent sequences of real numbers. Then, from continuity of the norm and the convergence assumption on the sequence $\{A_q\}_{q\in\mathbb{N}}$, it follows that

$$\begin{aligned}
&\|(AS_n - S_nA)\,\phi_m\| \\
&\le \mu\big((1{+}N{+}m)^{-\delta} + (1{+}N{-}m)^{-\delta}\big) \\
&\quad + \eta\big((1{+}|N{+}m{+}n|)^{-\rho} + (1{+}|N{-}m{-}n|)^{-\rho}\big),
\end{aligned}$$

Furthermore, since $A_q \in \mathscr{Q}_\alpha$ and $\mathscr{Q}_\alpha$ is an operator algebra then $A \in \mathscr{Q}_\alpha$, [5], [6]. Thus $\mathscr{T}_{\delta,\rho,\alpha}$ is closed. ■

It is important to interpret the constants involved in Definition 3 and those in the above theorem correctly. For example, let A and B be Toeplitz matrices. Then one can think of A and B as $(2N+1)\times(2N+1)$ truncations of corresponding biinfinite Toeplitz matrices $\mathcal{A}$ and $\mathcal{B}$. If now N is increased, the independence of the constants from N implies that $\|([A+B]S_n - S_n[A+B])\,\phi_m\|$ and $\|(ABS_n - S_nAB)\,\phi_m\|$ get smaller for fixed values of m and n, which means that as N increases the matrices $A+B$ and AB increasingly resemble Toeplitz matrices in their interior. Furthermore, the same decay properties continue to hold as N increases. This is in the same spirit as the operator algebra of spatially decaying matrices [7, Sec. 2].

III. Application: Optimal Control of Large-Scale Systems

The algebraic Riccati equation

$$A^*P + PA + Q - PR^{-1}P = 0 \qquad (5)$$

is of great importance in problems of optimal control. In this section we use the operator algebra framework developed above to prove that the solution P of an algebraic Riccati equation whose coefficients A, A^*, Q, R^{-1} belong to $\mathscr{T}_{\delta,\rho,\alpha}$ also belongs to $\mathscr{T}_{\delta,\rho,\alpha}$.

The following theorem is from [6] (see also [8], [9]).

Theorem 2: Let $\mathscr{A}$ be an operator algebra. If $A, A^*, Q, R^{-1} \in \mathscr{A}$ then the unique positive definite solution P of the Riccati equation (5) satisfies $P \in \mathscr{A}$.

The basic idea of the proof is to first replace (5) with the differential Riccati equation

$$\frac{d}{dt}X(t) = A^*X(t) + X(t)A + Q - X(t)R^{-1}X(t),$$

whose solution is assumed to converge to P as $t \to \infty$. We then approximate the dX/dt term with a finite difference to obtain

$$\begin{aligned}
X(t+\delta t) &= X(t) \\
&\quad + \delta t\big(A^*X(t) + X(t)A + Q - X(t)R^{-1}X(t)\big).
\end{aligned}$$

At $t = 0$ if $X(0) \in \mathscr{T}_{\delta,\rho,\alpha}$ then by closure of $\mathscr{T}_{\delta,\rho,\alpha}$ under multiplication and addition it is clear that $X(\delta t) \in \mathscr{T}_{\delta,\rho,\alpha}$. Replacing $X(\delta t)$ back into the above equation, we obtain that $X(2\,\delta t) \in \mathscr{T}_{\delta,\rho,\alpha}$, and so on. We finally use the property that $\mathscr{T}_{\delta,\rho,\alpha}$ is closed to conclude that the convergent sequence $X(n\,\delta) \in \mathscr{T}_{\delta,\rho,\alpha}$, $n \geq 0$ converges to $P \in \mathscr{T}_{\delta,\rho,\alpha}$. References [6], [8] make this argument rigorous by using the Banach fixed point theorem. We have thus shown that if $A, A^*, Q, R^{-1} \in \mathscr{T}_{\delta,\rho,\alpha}$ then the solution P of the Riccati equation (5) satisfies $P \in \mathscr{T}_{\delta,\rho,\alpha}$.

The importance of this result is that in certain cases it allows us to use the Fourier transform to readily compute an approximation $\mathcal{P}$ to the solution P of (5). We elaborate on this statement:

Let $\mathcal{A}, \mathcal{A}^*, \mathcal{Q}, \mathcal{R}^{-1}$ be spatially decaying biinfinite Toeplitz matrices with common decay rate $\alpha \gg 1$,[1] and let $\mathcal{P}$ be the solution of

$$\mathcal{A}^*\mathcal{P} \,+\, \mathcal{P}\mathcal{A} \,+\, \mathcal{Q} - \mathcal{P}\mathcal{R}^{-1}\mathcal{P} \;=\; 0.$$

Applying a Fourier transformation to these matrices yields their Fourier symbols $\hat{a}, \hat{a}^*, \hat{q}, \hat{r}^{-1}$, respectively. Thus the Fourier symbol $\hat{p}$ of $\mathcal{P}$ can be found from the parameterized family of scalar equations

$$\hat{a}(\theta)^*\,\hat{p}(\theta) \,+\, \hat{p}(\theta)\,\hat{a}(\theta) \,+\, \hat{q}(\theta) - \hat{p}(\theta)\,\hat{r}(\theta)^{-1}\,\hat{p}(\theta) \;=\; 0$$

where $\theta \in [0, 2\pi)$ is the Fourier parameter. An inverse Fourier transformation of $\hat{p}$ yields $\mathcal{P}$.

Now let A, A^*, Q, R^{-1} be the $(2N{+}1){\times}(2N{+}1)$ truncations of $\mathcal{A}, \mathcal{A}^*, \mathcal{Q}, \mathcal{R}^{-1}$, respectively. Then it is possible to find $0 < \delta < \alpha - 1$, $1 < \rho < \alpha - 1$ such that $A, A^*, Q, R^{-1} \in \mathscr{T}_{\delta,\rho,\alpha}$. Therefore $P \in \mathscr{T}_{\delta,\rho,\alpha}$ by Theorem 2. Furthermore, from (2) we conclude that for a given m and n the value of $f_{\delta,\rho}(m,n)$ becomes smaller as N increases. This means that the matrix P becomes more Toeplitz in its interior with increasing N. As a result, for large enough N one can use $\mathcal{P}$ as an approximation to P. This is beneficial since algebraic Riccati equations are computationally expensive to solve for large matrices P, whereas biinfinite Toeplitz solutions $\mathcal{P}$ are easy to find using Fourier methods, as illustrated above.

IV. Examples

We illustrate the results of this paper with a few examples. We choose a graphical way of demonstrating matrices, which communicates well their almost Toeplitz structure.

Let A be the 21×21 ($N = 10$) version of the following matrices

$$A_1 = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -1 \end{bmatrix},$$

[1]For example, if $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{R}$ are *banded* biinfinite Toeplitz matrices then α can take *any* positive value.

and let Q, R be the $21{\times}21$ version of the following matrices

$$Q = \begin{bmatrix} 1 & -1/4 & & \\ -1/4 & 1 & -1/4 & \\ & -1/4 & 1 & -1/4 \\ & & -1/4 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Let P_1, P_2 be the solutions of the algebraic Riccati equations corresponding to A_1, A_2, respectively. Figures 2, 3 show the follwoing: the matrices A_i; the values taken by $\|(A_i S_n - S_n A_i)\,\phi_m\|$ as m, n vary; the matrices P_i; the values taken by $\|(P_i S_n - S_n P_i)\,\phi_m\|$ as m, n vary. The figures demonstrate that the solutions P_i maintain the almost Toeplitz structure possessed by the coefficients A_i, Q, R^{-1}.

V. Appendix: Derivation of Expressions for μ'_A, η'_A

The right hand side of inequality (4) can be rewritten as

$$\frac{\mu_A\,\kappa_B}{(1{+}N{+}m)^\delta}\sum_{k=-N-m}^{N-m}\Big(\frac{1}{1{+}|k|}\Big)^\alpha\Big(\frac{1{+}N{+}m}{1{+}N{+}m{+}k}\Big)^\delta$$
$$+\frac{\mu_A\,\kappa_B}{(1{+}N{-}m)^\delta}\sum_{k=-N-m}^{N-m}\Big(\frac{1}{1{+}|k|}\Big)^\alpha\Big(\frac{1{+}N{-}m}{1{+}N{-}m{-}k}\Big)^\delta$$
$$+\frac{\mu_A\,\kappa_B}{(1{+}|N{+}m{+}n|)^\rho}\sum_{k=-N-m}^{N-m}\Big(\frac{1}{1{+}|k|}\Big)^\alpha\Big(\frac{1{+}|N{+}m{+}n|}{1{+}|N{+}m{+}k{+}n|}\Big)^\rho$$
$$+\frac{\mu_A\,\kappa_B}{(1{+}|N{-}m{-}n|)^\rho}\sum_{k=-N-m}^{N-m}\Big(\frac{1}{1{+}|k|}\Big)^\alpha\Big(\frac{1{+}|N{-}m{-}n|}{1{+}|N{-}m{-}k{-}n|}\Big)^\rho$$

We now find uniform (independent of N) bounds on each of the sums in the above expression. We carry out the computations in detail for the second and fourth sums; the computations for the first and third sums will be very similar and thus omitted.

For the second sum, we have

$$\begin{aligned}
&\sum_{k=-N-m}^{N-m}\Big(\frac{1}{1{+}|k|}\Big)^\alpha\Big(\frac{1{+}N{-}m}{1{+}N{-}m{-}k}\Big)^\delta \\
\leq\;& \sum_{k=-N-m}^{0}\Big(\frac{1}{1{-}k}\Big)^\alpha\Big(\frac{1{+}N{-}m}{1{+}N{-}m{-}k}\Big)^\delta \\
&\qquad + \sum_{k=0}^{N-m}\Big(\frac{1}{1{+}k}\Big)^\alpha\Big(\frac{1{+}N{-}m}{1{+}N{-}m{-}k}\Big)^\delta \\
=\;& \sum_{k=0}^{N+m}\Big(\frac{1}{1{+}k}\Big)^\alpha\Big(\frac{1{+}N{-}m}{1{+}N{-}m{+}k}\Big)^\delta \\
&\qquad + \sum_{k=0}^{N-m}\Big(\frac{1}{1{+}k}\Big)^\alpha\Big(\frac{1}{1{-}\frac{k}{1+N-m}}\Big)^\delta \\
\leq\;& \sum_{k=0}^{N+m}\Big(\frac{1}{1{+}k}\Big)^\alpha + \sum_{k=0}^{N-m}\Big(\frac{1}{1{+}k}\Big)^\alpha(1{+}k)^\delta \\
\leq\;& \sum_{k=0}^{\infty}\Big(\frac{1}{1{+}k}\Big)^\alpha + \sum_{k=0}^{\infty}\Big(\frac{1}{1{+}k}\Big)^{\alpha-\delta}.
\end{aligned}$$

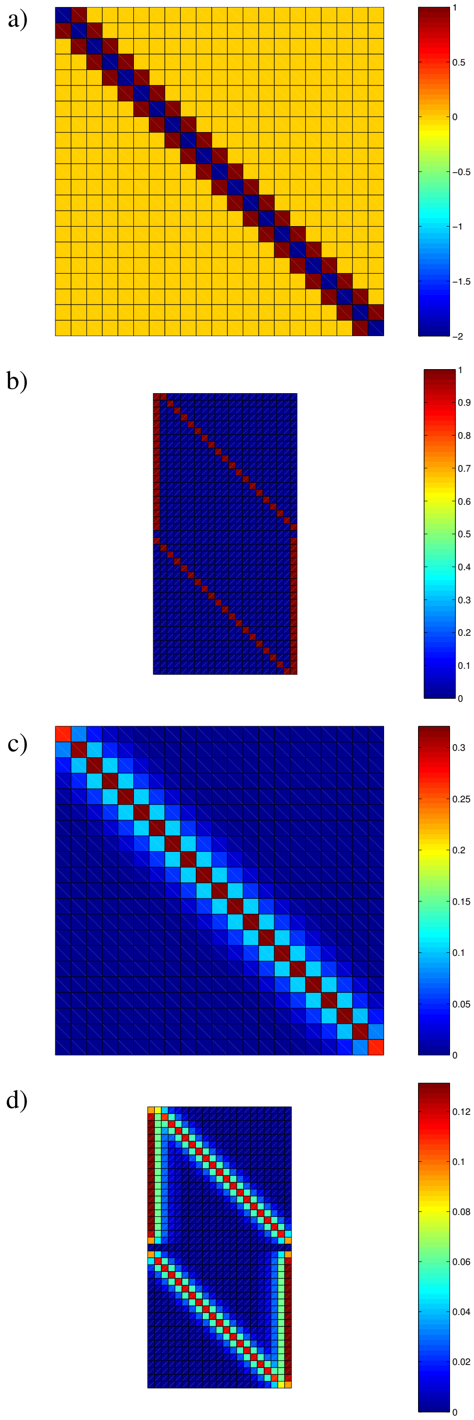

Fig. 2. a) The matrix A_1. b) The values taken by $\|(A_1 S_n - S_n A_1)\,\phi_m\|$ as m, n vary in their respective domains. c) The solution P_1 of the algebraic Riccati equation corresponding to A_1. d) The values taken by $\|(P_1 S_n - S_n P_1)\,\phi_m\|$.

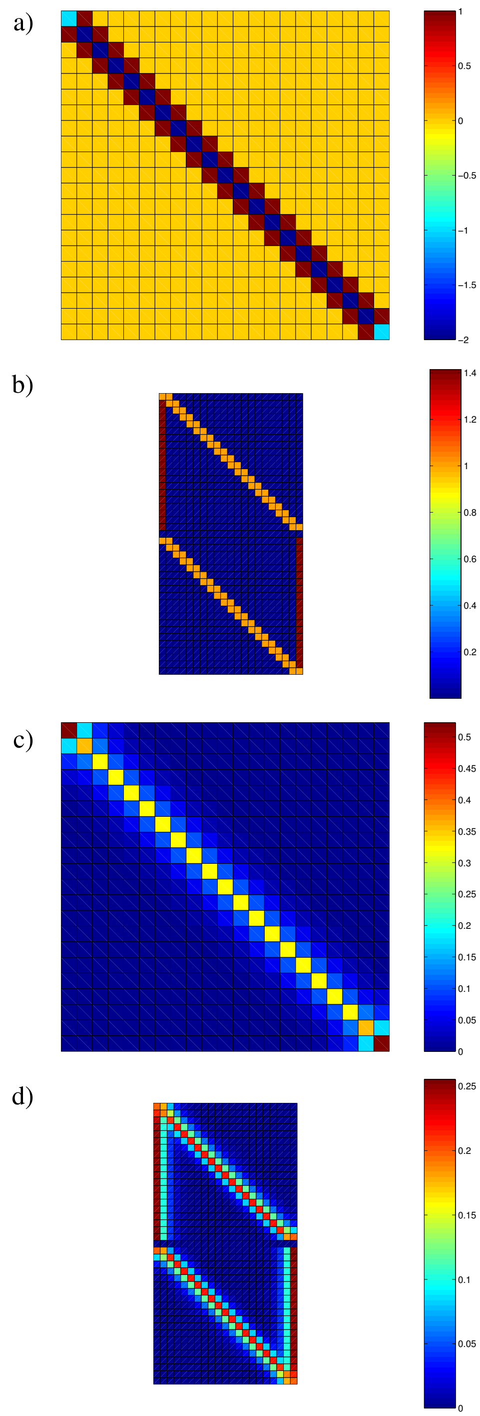

Fig. 3. a) The matrix A_2. b) The values taken by $\|(A_2 S_n - S_n A_2)\,\phi_m\|$ as m, n vary in their respective domains. c) The solution P_2 of the algebraic Riccati equation corresponding to A_2. d) The values taken by $\|(P_2 S_n - S_n P_2)\,\phi_m\|$.

Therefore

$$\sum_{k=-N-m}^{N-m} \big(\frac{1}{1+|k|}\big)^{\alpha} \big(\frac{1+N-m}{1+N-m-k}\big)^{\delta} \leq \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha} + \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha-\delta}.$$

Similarly it is possible to show that

$$\sum_{k=-N-m}^{N-m} \big(\frac{1}{1+|k|}\big)^{\alpha} \big(\frac{1+N+m}{1+N+m+k}\big)^{\delta} \leq \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha} + \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha-\delta}.$$

For the fourth sum, we consider two scenarios based on the sign of $N-m-n$.

Assume $N-m-n \geq 0$, and define $l := N-m-n$. Then

$$\begin{aligned}
&\sum_{k=-N-m}^{N-m} \big(\frac{1}{1+|k|}\big)^{\alpha} \big(\frac{1+|N-m-n|}{1+|N-m-k-n|}\big)^{\rho} \\
\leq\ & \sum_{k=-N-m}^{0} \big(\frac{1}{1-k}\big)^{\alpha} \big(\frac{1+l}{1+|l-k|}\big)^{\rho} + \sum_{k=0}^{N-m} \big(\frac{1}{1+k}\big)^{\alpha} \big(\frac{1+l}{1+|l-k|}\big)^{\rho} \\
\leq\ & \sum_{k=0}^{N+m} \big(\frac{1}{1+k}\big)^{\alpha} \big(\frac{1+l}{1+l+k}\big)^{\rho} + \sum_{k=0}^{N-m} \big(\frac{1}{1+k}\big)^{\alpha} (1+k)^{\rho} \\
\leq\ & \sum_{k=0}^{N+m} \big(\frac{1}{1+k}\big)^{\alpha} + \sum_{k=0}^{N-m} \big(\frac{1}{1+k}\big)^{\alpha-\rho} \\
\leq\ & \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha} + \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha-\rho}.
\end{aligned}$$

Assume $N-m-n \leq 0$, and define $l := -(N-m-n)$. Then

$$\begin{aligned}
&\sum_{k=-N-m}^{N-m} \big(\frac{1}{1+|k|}\big)^{\alpha} \big(\frac{1+|N-m-n|}{1+|N-m-k-n|}\big)^{\rho} \\
\leq\ & \sum_{k=-N-m}^{0} \big(\frac{1}{1-k}\big)^{\alpha} \big(\frac{1+l}{1+|l+k|}\big)^{\rho} + \sum_{k=0}^{N-m} \big(\frac{1}{1+k}\big)^{\alpha} \big(\frac{1+l}{1+|l+k|}\big)^{\rho} \\
=\ & \sum_{k=0}^{N+m} \big(\frac{1}{1+k}\big)^{\alpha} (1+k)^{\rho} + \sum_{k=0}^{N-m} \big(\frac{1}{1+k}\big)^{\alpha} \big(\frac{1+l}{1+l+k}\big)^{\rho} \\
\leq\ & \sum_{k=0}^{N-m} \big(\frac{1}{1+k}\big)^{\alpha-\rho} + \sum_{k=0}^{N+m} \big(\frac{1}{1+k}\big)^{\alpha} \\
\leq\ & \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha-\rho} + \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha}.
\end{aligned}$$

Therefore

$$\sum_{k=-N-m}^{N-m} \big(\frac{1}{1+|k|}\big)^{\alpha} \big(\frac{1+|N-m-n|}{1+|N-m-k-n|}\big)^{\delta} \leq \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha-\rho} + \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha}.$$

Similarly it is possible to show that

$$\sum_{k=-N-m}^{N-m} \big(\frac{1}{1+|k|}\big)^{\alpha} \big(\frac{1+|N+m+n|}{1+|N+m+k+n|}\big)^{\delta} \leq \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha-\rho} + \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha}.$$

Thus we choose

$$\begin{aligned}
\mu'_A &= \mu_A\, \kappa_B \big(\sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha} + \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha-\delta}\big), \\
\eta'_A &= \mu_A\, \kappa_B \big(\sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha} + \sum_{k=0}^{\infty} \big(\frac{1}{1+k}\big)^{\alpha-\rho}\big).
\end{aligned}$$

Note that μ'_A and η'_A are uniform constants independent of N. The assumptions $\alpha > \delta+1$, $\alpha > \rho+1$ are necessary for the convergence of the infinite sums.

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