

Scheduling Kalman Filters in Continuous Time

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Abstract—A set of N independent Gaussian linear time invariant systems is observed by M sensors whose task is to provide the best possible steady-state causal minimum mean square estimate of the state of the systems, in addition to minimizing a steady-state measurement cost. The sensors can switch between systems instantaneously, and there are additional resource constraints, for example on the number of sensors which can observe a given system simultaneously. We first derive a tractable relaxation of the problem, which provides a bound on the achievable performance. This bound can be computed by solving a convex program involving linear matrix inequalities. Exploiting the additional structure of the sites evolving independently, we can decompose this program into coupled smaller dimensional problems. In the scalar case with identical sensors, we give an analytical expression for an index policy proposed in a more general context by Whittle. In the general case, we develop open-loop periodic switching policies whose performance matches the bound arbitrarily closely.

I. INTRODUCTION

Advances in sensor networks and the development of unmanned vehicle systems for intelligence, reconnaissance and surveillance missions require the development of data fusion schemes that can handle measurements originating from a large number of sensors observing a large number of targets, see e.g. [1], [2]. These problems have a long history [3], and can be used to formulate static sensor scheduling problems as well as trajectory optimization problems for mobile sensors [4], [5].

In this paper, we consider M sensors tracking the state of N sites or targets in continuous time. We assume that the sites can be described by N plants with independent linear time invariant dynamics,

$$\dot{x}_i = A_i x_i + B_i u_i + w_i, \quad x_i(0) = x_{i,0}, \quad i = 1, \dots, N.$$

We assume that the plant controls $u_i(t)$ are deterministic and known for $t \geq 0$. Each driving noise $w_i(t)$ is a stationary white Gaussian noise process with zero mean and known power spectral density matrix W_i : $\text{Cov}(w_i(t), w_i(t')) = W_i \delta(t - t')$. The initial conditions are random variables with known mean $\bar{x}_{i,0}$ and covariance matrices $\Sigma_{i,0}$. By independent systems we mean that the noise processes of the different plants are independent, as well as the initial conditions $x_{i,0}$. Moreover the initial conditions are assumed independent of the noise processes. We shall also assume that

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Assumption 1: The matrices $\Sigma_{i,0}$ are positive definite for all $i \in \{1, \dots, N\}$.

This can be achieved by adding an arbitrarily small perturbation to a potentially non invertible matrix $\Sigma_{i,0}$. We have M sensors to observe the N plants. If sensor j is used to observe plant i , we obtain measurements

$$y_{ij} = C_{ij} x_i + v_{ij}.$$

Here v_{ij} is a stationary white Gaussian noise process with power spectral density matrix V_{ij} , assumed positive definite. Also, v_{ij} is independent of the other measurement noises, process noises, and initial states. Finally, to guarantee convergence of the filters later on, we assume throughout that

Assumption 2: For all $i \in \{1, \dots, N\}$, there exists a set of indices $j_1, j_2, \dots, j_{n_i} \in \{1, \dots, M\}$ such that the pair (A_i, \tilde{C}_i) is detectable, where $\tilde{C}_i = [C_{ij_1}^T, \dots, C_{ij_{n_i}}^T]^T$.

Assumption 3: The pairs $(A_i, W_i^{1/2})$ are controllable, for all $i \in \{1, \dots, N\}$.

Let us define $\pi_{ij}(t) = 1$ if plant i is observed at time t by sensor j , and $\pi_{ij}(t) = 0$ otherwise. We assume that each sensor can observe at most one system at each period, hence we have the constraint

$$\sum_{i=1}^N \pi_{ij}(t) \leq 1, \quad \forall t, \quad j = 1, \dots, M. \quad (1)$$

If instead sensor j is required to be always operated, constraint (1) should simply be changed to

$$\sum_{i=1}^N \pi_{ij}(t) = 1. \quad (2)$$

We also add the following constraint, similar to the one used by Athans [6]. We suppose that each system can be observed by at most one sensor at each time, so we have

$$\sum_{j=1}^M \pi_{ij}(t) \leq 1, \quad \forall t, \quad i = 1, \dots, N. \quad (3)$$

Similarly if system i must always be observed by some sensor, constraint (3) can be changed to an equality constraint

$$\sum_{j=1}^M \pi_{ij}(t) = 1. \quad (4)$$

Note that a sensor in our discussion can correspond to a combination of several physical sensors, and so the constraints above can capture seemingly more general problems where we allow for example more than one simultaneous measurements per system.

We consider an infinite-horizon average cost problem. The parameters of the model are assumed known. We wish

to design an observation policy $\pi(t) = \{\pi_{ij}(t)\}$ satisfying the constraints (1), (3), or their equality versions, and an estimator \hat{x}_π of x , depending at each instant only on the past and current observations produced by the observation policy, such that the average error variance is minimized, in addition to some observation costs. The policy π itself can also only depend on the past observations. More precisely, we wish to minimize, subject to the constraints (1), (3),

$$\gamma = \min_{\pi, \hat{x}_\pi} \limsup_{T \rightarrow \infty} \frac{1}{T} E \left[\int_0^T \sum_{i=1}^N ((x_i - \hat{x}_{\pi,i})' T_i (x_i - \hat{x}_{\pi,i}) + \sum_{j=1}^M \kappa_{ij} \pi_{ij}(t)) dt \right], \quad (5)$$

where the constants κ_{ij} are a cost paid per unit of time when plant i is observed by sensor j . The T_i 's are positive semidefinite weighting matrices.

Literature Review and Contributions of this paper. The sensor scheduling problem presented above, except for minor variations, is an infinite horizon version of the problem considered by Athans in [6]. See also Meier et al. [3] for the discrete-time version. Athans considered the observation of only one plant. We include here several plants to show how their independent evolution property can be leveraged in the computations, using the dual decomposition method from optimization. Discrete-time versions of this sensor selection problem have received a significant amount of attention, see e.g. [7], [8], [9], [4], [10], [11], [12]. All algorithms proposed so far, except for the greedy policy of [11] in the completely symmetric case, either run in exponential time or consist of heuristics with no performance guarantee. However, to the authors' knowledge, the problem has not been shown to be intractable. We do not consider the discrete-time problem in this paper. Finite-horizon continuous-time versions of the problem, besides the presentation of Athans [6], have also been the subject of several papers [13], [14], [15], [16]. For these problems the solutions proposed, usually based on optimal control techniques, also involve computational procedures that scale poorly with the dimension of the problem.

Somewhat surprisingly however, and with the exception of [17], it seems that the infinite-horizon continuous time version of the Kalman filter scheduling problem has not been considered previously. Mourikis and Roumeliotis [17] consider initially also a discrete time version of the problem for a particular robotic application. However, their discrete model originates from the sampling at high rate of a continuous time system. To cope with the difficulty of determining a sensor schedule, they assume instead a model where each sensor can independently process each of the available measurements at a constant frequency, and seek the optimal measurement frequencies. In fact, they obtain these frequencies by introducing heuristically a continuous time Riccati equation, and show that the frequencies can then be computed by solving a semidefinite program. In contrast, we consider the more standard schedule-based version of the problem in continuous time, which is a priori more constraining. We show that essentially the same convex program provides in fact

a *lower bound* on the cost achievable by any measurement policy. In addition, we provide additional insight into the decomposition of the computations of this program, which can be useful in the framework of [17] as well.

The rest of the chapter is organized as follows. Section II briefly recalls that for a fixed policy $\pi(t)$, the optimal estimator is obtained by a type of Kalman-Bucy filter. The properties of the Kalman filter (independence of the error covariance matrix with respect to measurements) imply that the remaining problem of finding the optimal scheduling policy π is a deterministic control problem. In section III we treat a simplified scalar version of the problem with identical sensors as a special case of the classical ‘‘Restless Bandit Problem’’ (RBP) [18], and provide analytical expressions for an index policy and for the elements necessary to compute efficiently a lower bound on performance, both of which were proposed in the general setting of the RBP by Whittle. Then, for the multidimensional case treated in full generality in section IV, we show that the lower bound on performance can be computed as a convex program involving linear matrix inequalities. This lower bound can be approached arbitrarily closely by a family of new periodically switching policies described in section IV-C. Approaching the bound with these policies is limited only by the frequency with which the sensors can actually switch between the systems. In general, our solution has much more attractive computational properties than the solutions proposed so far for the finite-horizon problem. Additional details and some proofs omitted in this paper can be found in [19], [20].

II. OPTIMAL ESTIMATOR

For a given observation policy $\pi(t) = \{\pi_{ij}(t)\}_{i,j}$, the minimum variance filter $\hat{x}_\pi(t)$ is given by the Kalman-Bucy filter [21], see [6]. The resulting estimator is unbiased and the error covariance matrix $\Sigma_{\pi,i}(t)$ for site i verifies the matrix Riccati differential equation

$$\dot{\Sigma}_{\pi,i}(t) = A_i \Sigma_{\pi,i} + \Sigma_{\pi,i} A_i^T + W_i - \Sigma_{\pi,i} \left(\sum_{j=1}^M \pi_{ij}(t) C_{ij}^T V_{ij}^{-1} C_{ij} \right) \Sigma_{\pi,i}, \quad (6)$$

with $\Sigma_{\pi,i}(0) = \Sigma_{i,0}$. We can then reformulate the optimization of the observation policy as a *deterministic* optimal control problem. Rewriting $E((x_i - \hat{x}_i)' T_i (x_i - \hat{x}_i)) = \text{Tr}(T_i \Sigma_i)$, the problem is to minimize

$$\gamma = \min_{\pi} \limsup_{T \rightarrow \infty} \frac{1}{T} \left[\int_0^T \sum_{i=1}^N \left(\text{Tr}(T_i \Sigma_{\pi,i}(t)) + \sum_{j=1}^M \kappa_{ij} \pi_{ij}(t) \right) dt \right], \quad (7)$$

subject to the constraints (1), (3), or their equality versions, and the dynamics (6).

III. SCALAR SYSTEMS AND IDENTICAL SENSORS

We assume in this section that

- 1) the sites or targets have one-dimensional dynamics, i.e., $x_i \in \mathbb{R}$, $i = 1, \dots, N$.
- 2) all the sensors are identical, i.e., $C_{ij} = C_i$, $V_{ij} = V_i$, $\kappa_{ij} = \kappa_i$, $j = 1, \dots, M$.

Because of condition 2, we can simplify the problem formulation introduced above so that it corresponds exactly to a

special case of the RBP introduced by Whittle in [18]. We define

$$\pi_i(t) = \begin{cases} 1 & \text{if plant } i \text{ is observed at time } t \text{ by a sensor} \\ 0 & \text{otherwise.} \end{cases}$$

Note that a constraint (4) for some system i can be eliminated, by removing one available sensor, which is always measuring the system i . Constraints (2) and (3) can then be replaced by the single constraint $\sum_{i=1}^N \pi_i(t) = M, \forall t$. This constraint means that at each period, exactly M of the N sites are observed. We treat this case in the following, but again the equality sign can be changed to an inequality. To obtain a lower bound on the achievable performance, we relax the constraint to enforce it only on average

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sum_{i=1}^N \pi_i(t) dt = M. \quad (8)$$

Then we adjoin this constraint using a (scalar) Lagrange multiplier λ to form the Lagrangian

$$L(\pi, \lambda) = \limsup_{T \rightarrow \infty} \frac{1}{T} \left[\int_0^T \sum_{i=1}^N [\text{Tr}(T_i \Sigma_{\pi,i}(t)) + (\kappa_i + \lambda) \pi_i(t)] dt - \lambda M \right].$$

Here κ_i is the cost per time unit for observing site i . The dynamics of $\Sigma_{\pi,i}$ are now given by

$$\frac{d}{dt} \Sigma_{\pi,i}(t) = A_i \Sigma_{\pi,i} + \Sigma_{\pi,i} A_i^T + W_i - \pi_i(t) \Sigma_{\pi,i} C_i^T V_i^{-1} C_i \Sigma_{\pi,i}. \quad (9)$$

Then the original optimization problem (7) with the relaxed constraint (8) can be expressed as

$$\gamma = \inf_{\pi} \sup_{\lambda} L(\pi, \lambda) = \sup_{\lambda} \inf_{\pi} L(\pi, \lambda),$$

where the exchange of the supremum and the infimum can be justified using a minimax theorem for constrained dynamic programming [22]. Hence we are led to consider the computation of the dual function

$$\gamma(\lambda) = \min_{\pi} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sum_{i=1}^N [\text{Tr}(T_i \Sigma_{\pi,i}(t)) + (\kappa_i + \lambda) \pi_i(t)] dt,$$

which has the important property of *being separable by site*, i.e., $\gamma(\lambda) = \sum_{i=1}^N \gamma^i(\lambda)$, where for each site i we have

$$\gamma^i(\lambda) = \min_{\pi_i} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{Tr}(T_i \Sigma_{\pi,i}(t)) + (\kappa_i + \lambda) \pi_i(t) dt. \quad (10)$$

When the dynamics of the sites are one dimensional, i.e., $\Sigma_i \in \mathbb{R}$, we can obtain an analytical expression of this dual function, which provides a lower bound on the cost for each λ . The results are presented in paragraph (III-B). First, we explain how these computations also provide the elements necessary to design a scheduling policy.

A. Restless Bandits

The Restless Bandit Problem (RBP) was introduced by Whittle in [18] as a generalization of the classical Multi-Armed Bandit Problem (MABP), which was first solved by Gittins [23]. In the RBP, we have N projects evolving independently, M of which can be activated at each time. Projects that are active can evolve according to different

dynamics than projects that remain passive. In our context, the projects correspond to the systems and the active action corresponds to taking a measurement. Whittle [18] proposed an index policy which, although suboptimal for a general RBP, generalizes Gittins' optimal policy for the MABP.

Consider the objective (10) for system i . The Lagrange multiplier λ can be interpreted as a tax penalizing measurements of the system. As λ increases, the passive action (i.e., not measuring) should become more attractive. Let us denote $\mathcal{P}^i(\lambda)$ the set of covariance matrices Σ^i where the passive action is optimal. Let \mathbb{S}_+^n be the set of symmetric positive semidefinite matrices. Then we say that

Definition 4: System i is indexable if and only if $\mathcal{P}^i(\lambda)$ is monotonically increasing from \emptyset to \mathbb{S}_+^n as λ increases from $-\infty$ to $+\infty$. If system i is indexable, we define its *Whittle index* by $\lambda^i(\Sigma_i) = \inf\{\lambda \in \mathbb{R} : \Sigma^i \in \mathcal{P}^i(\lambda)\}$.

However natural the indexability requirement may be, Whittle provided an example of a RBP where it is not verified. We will see in the next paragraph however that for our particular problem, at least in the scalar case, indexability of the systems is guaranteed. The idea behind the definition of the Whittle index consists in defining an intrinsic "value" for measuring system i , taking into account both the immediate and future gains. If the covariance of system i is Σ^i , the Whittle index defines this value as the tax that should be paid to make the controller indifferent between measuring and not measuring the system. Finally, if all the systems are indexable, *the Whittle policy chooses at each instant to measure the M systems with highest Whittle indices*. There is significant experimental data and some theoretical evidence indicating that when the Whittle policy is well-defined for an RBP, its performance is often very close to optimal, see e.g. [24], [25], [26].

B. Solution of the Scalar Optimal Control Problem

For lack of space, we only present the result of the computations, which can be found in [19], [20]. We define $x_{1,i}$ and $x_{2,i}$ to be the negative and positive solutions of the algebraic Riccati equation (ARE)

$$2A_i x + W_i - \frac{C_i^2}{V_i} x^2 = 0,$$

$x_{e,i} = -W_i/(2A_i)$ if $A_i < 0$, $x_{e,i} = +\infty$ if $A_i \geq 0$ and $\Sigma_{th,i}$ to be the unique positive solution of the cubic equation

$$X^3 - \frac{2V_i(\lambda + \kappa_i)}{T_i C_i^2} A_i X - \frac{2V_i(\lambda + \kappa_i)}{T_i C_i^2} W_i = 0. \quad (11)$$

Theorem 1: For the scalar Kalman filter scheduling problem with identical sensors, the systems are indexable. For system i , the Whittle index $\lambda_i(\Sigma_i)$ is given as follows.

$$\lambda_i(\Sigma_i) = \begin{cases} -\kappa_i + \frac{T_i \Sigma_i^2}{\Sigma_i - x_{1,i}} & \text{if } \Sigma_i \leq x_{2,i}, \\ -\kappa_i + \frac{C_i^2}{2V_i} \frac{T_i \Sigma_i^3}{A_i \Sigma_i + W_i} & \text{if } x_{2,i} < \Sigma_i < x_{e,i}, \\ -\kappa_i + \frac{C_i^2 \Sigma_i^2}{2|A_i| V_i} & \text{if } x_{e,i} \leq \Sigma_i, \end{cases}$$

with the convention $x_{e,i} = +\infty$ if $A_i \geq 0$. A lower bound on the achievable performance is obtained by maximizing the

concave function $\gamma(\lambda) = \sum_{i=1}^N \gamma^i(\lambda) - \lambda M$ over λ , where the term $\gamma^i(\lambda)$ is given by

$$\begin{cases} T_i x_{2,i} + \kappa_i + \lambda & \text{if } \lambda \leq \lambda_i(x_{2,i}), \\ T_i \Sigma_{th,i}(\lambda) + \frac{V_i(\kappa_i + \lambda)(2A_i \Sigma_{th,i}(\lambda) + W_i)}{C_i^2(\Sigma_{th,i}(\lambda))^2}, & \lambda_i(x_{2,i}) < \lambda < \lambda_i(x_{e,i}), \\ T_i x_{e,i} & \text{if } \lambda_i(x_{e,i}) \leq \lambda. \end{cases}$$

IV. MULTIDIMENSIONAL SYSTEMS

Generalizing the computations of the previous section to multidimensional systems requires solving the corresponding optimal control problem in higher dimensions, for which it is not clear that a closed form solution exist. Moreover we have considered in section III a particular case of the sensor scheduling problem where all sensors are identical. We now return to the general multidimensional problem and sensors with possibly distinct characteristics, as described in the introduction.

For the infinite-horizon average cost problem, we show that computing the value of a lower bound similar to the one presented in section III reduces to a convex optimization problem involving, at worst, Linear Matrix Inequalities (LMI) whose size grows polynomially with the problem essential parameters. Moreover, one can further decompose the computation of this convex program into N coupled subproblems as in the standard restless bandit case.

A. Performance Bound

We define the following quantities:

$$\tilde{\pi}_{ij}(T) = \frac{1}{T} \int_0^T \pi_{ij}(t) dt, \quad \forall T. \quad (12)$$

Since $\pi_{ij}(t) \in \{0, 1\}$ we must have $0 \leq \tilde{\pi}_{ij}(T) \leq 1$. Our first goal, following the idea exploited in the restless bandit problem, is to obtain a lower bound on the cost of the optimal control problem for horizon T in terms of the numbers $\tilde{\pi}_{ij}(T)$ instead of the functions $\pi_{ij}(t)$.

It will be easier to work with the information matrices $Q_i(t) = \Sigma_i^{-1}(t)$. Note that invertibility of $\Sigma_i(t)$ is guaranteed by our assumption 1, as a consequence of [27, theorem 21.1]. Hence we replace the dynamics (6) by the equivalent

$$\dot{Q}_i = -Q_i A_i - A_i^T Q_i - Q_i W_i Q_i + \sum_{j=1}^M \pi_{ij}(t) C_{ij}^T V_{ij}^{-1} C_{ij}, \quad (13)$$

for all $i \in \{1, \dots, N\}$. Let us define, for all T ,

$$\tilde{\Sigma}_i(T) := \frac{1}{T} \int_0^T \Sigma_i(t) dt, \quad \tilde{Q}_i(T) := \frac{1}{T} \int_0^T Q_i(t) dt.$$

By linearity of the trace operator, we can rewrite the objective function

$$\limsup_{T \rightarrow \infty} \sum_{i=1}^N \left\{ \text{Tr}(T_i \tilde{\Sigma}_i(T)) + \sum_{j=1}^M \kappa_{ij} \tilde{\pi}_{ij}(T) \right\}.$$

Let $\mathbb{S}^n, \mathbb{S}_+^n, \mathbb{S}_{++}^n$ denote the set of symmetric, symmetric positive semidefinite and symmetric positive definite matrices respectively. A function $f: \mathbb{R}^m \rightarrow \mathbb{S}^n$ is called *matrix convex* if and only if for all $x, y \in \mathbb{R}^m$ and $\alpha \in [0, 1]$, we have

$$f(\alpha x + (1 - \alpha)y) \preceq \alpha f(x) + (1 - \alpha)f(y),$$

where \preceq refers to the usual partial order on \mathbb{S}^n , i.e., $A \preceq B$ if and only if $B - A \in \mathbb{S}_+^n$. Equivalently, f is matrix convex if the scalar function $x \mapsto z^T f(x) z$ is convex for all vectors z . The following lemma will be useful

Lemma 5: The functions

$$\begin{array}{ll} \mathbb{S}_{++}^n \rightarrow \mathbb{S}_{++}^n & \mathbb{S}^n \rightarrow \mathbb{S}^n \\ X \mapsto X^{-1} & X \mapsto X W X \end{array}$$

for $W \in \mathbb{S}_{++}^n$, are matrix convex.

Proof: See [28, p.76, p.110]. \blacksquare

A consequence of this lemma is that Jensen's inequality is valid for these functions. We use it first as follows

$$\forall T, \quad \left(\frac{1}{T} \int_0^T \Sigma_i(t) dt \right)^{-1} \preceq \frac{1}{T} \int_0^T Q_i(t) dt = \tilde{Q}_i(T),$$

hence $\forall T, \quad \tilde{\Sigma}_i(T) \succeq (\tilde{Q}_i(T))^{-1}$, and so $\text{Tr}(T_i \tilde{\Sigma}_i(T)) \geq \text{Tr}(T_i (\tilde{Q}_i(T))^{-1})$. Next, integrating (13), we have

$$\begin{aligned} \frac{1}{T} (Q_i(T) - Q_{i,0}) &= -\tilde{Q}_i(T) A_i - A_i^T \tilde{Q}_i(T) \\ &\quad - \frac{1}{T} \int_0^T Q_i(t) W_i Q_i(t) dt + \sum_{j=1}^M \tilde{\pi}_{ij}(T) C_{ij}^T V_{ij}^{-1} C_{ij}. \end{aligned}$$

Using Jensen's inequality and lemma 5 again, we have

$$\frac{1}{T} \int_0^T Q_i(t) W_i Q_i(t) dt \succeq \tilde{Q}_i(T) W_i \tilde{Q}_i(T),$$

and so we obtain, using also $Q_i(T) \succeq 0$,

$$\tilde{Q}_i(T) A_i + A_i^T \tilde{Q}_i(T) + \tilde{Q}_i(T) W_i \tilde{Q}_i(T) - \sum_{j=1}^M \tilde{\pi}_{ij}(T) C_{ij}^T V_{ij}^{-1} C_{ij} \preceq \frac{Q_{i,0}}{T}. \quad (14)$$

So we see that for a fixed policy π and any time T , the quantity

$$\sum_{i=1}^N \left\{ \text{Tr}(T_i \tilde{\Sigma}_i(T)) + \sum_{j=1}^M \kappa_{ij} \tilde{\pi}_{ij}(T) \right\} \quad (15)$$

is lower bounded by the quantity $\sum_{i=1}^N \left\{ \text{Tr}(T_i (\tilde{Q}_i(T))^{-1}) + \sum_{j=1}^M \kappa_{ij} \tilde{\pi}_{ij}(T) \right\}$, where the matrices $\tilde{Q}_i(T)$ and the number $\tilde{\pi}_{ij}(T)$ are subject to the constraints (14) as well as $0 \leq \tilde{\pi}_{ij}(T) \leq 1$, $\sum_{i=1}^N \tilde{\pi}_{ij}(T) \leq 1$, $j = 1, \dots, M$, $\sum_{j=1}^M \tilde{\pi}_{ij}(T) \leq 1$, $i = 1, \dots, N$, for the inequality version of the resource constraints. Hence for any T , the quantity $Z^*(T)$ defined below is a lower bound on the value of (15) for any choice of policy π

$$Z^*(T) = \min_{\tilde{Q}_i, p_{ij}} \sum_{i=1}^N \left\{ \text{Tr}(T_i \tilde{Q}_i^{-1}) + \sum_{j=1}^M \kappa_{ij} p_{ij} \right\}, \quad s.t. \quad (16)$$

$$\tilde{Q}_i A_i + A_i^T \tilde{Q}_i + \tilde{Q}_i W_i \tilde{Q}_i - \sum_{j=1}^M p_{ij} C_{ij}^T V_{ij}^{-1} C_{ij} \preceq \frac{Q_{i,0}}{T}, \quad i = 1, \dots, N, \quad (17)$$

$\tilde{Q}_i \succ 0$, $i = 1, \dots, N$; $0 \leq p_{ij} \leq 1$, $i = 1, \dots, N$, $j = 1, \dots, M$,

$$\sum_{i=1}^N p_{ij} \leq 1, \quad j = 1, \dots, M; \quad \sum_{j=1}^M p_{ij} \leq 1, \quad i = 1, \dots, N.$$

Consider now the same program where the right-hand side of (17) is replaced by 0 to obtain the constraint:

$$\tilde{Q}_i A_i + A_i^T \tilde{Q}_i + \tilde{Q}_i W_i \tilde{Q}_i - \sum_{j=1}^M p_{ij} C_{ij}^T V_{ij}^{-1} C_{ij} \preceq 0, \quad i = 1, \dots, N, \quad (18)$$

and denote the corresponding solution Z^* . Defining $\delta := 1/T$, and rewriting by slight abuse of notation $Z^*(\delta)$ instead of $Z^*(T)$ for δ positive, we also define $Z^*(0) = Z^*$. Note that $Z^*(0)$ is finite, since we can find a feasible solution to the program defining Z^* , as follows. For each i , we choose a set of indices $J_i = \{j_1, \dots, j_{n_i}\} \subset \{1, \dots, M\}$ such that (A_i, \tilde{C}_i) is observable, as in assumption 2. Once a set J_i has been chosen for each i , we form the matrix \hat{P} with elements $\hat{p}_{ij} = 1\{j \in J_i\}$. Finally, we form a matrix P with elements p_{ij} satisfying the constraints and nonzero exactly where the \hat{p}_{ij} are nonzero. Such a matrix is easy to find if we consider the inequality constraints (1) and (3). If equality constraints are involved instead, such a matrix P exists as a consequence of Birkhoff theorem, see theorem 6. Now we consider the quadratic inequality (18) for some value of i . From the detectability assumption 2 and the choice of p_{ij} , we deduce that the pair (A_i, \hat{C}_i) , with

$$\hat{C}_i = \begin{bmatrix} \sqrt{p_{i1}} C_{i1}^T V_{i1}^{-1/2} & \dots & \sqrt{p_{iM}} C_{iM}^T V_{iM}^{-1/2} \end{bmatrix}^T \quad (19)$$

is detectable. Also note that $\hat{C}_i^T \hat{C}_i = \sum_{j=1}^M p_{ij} C_{ij}^T V_{ij}^{-1} C_{ij}$. Together with the controllability assumption 3, we then know that (18) has a positive definite solution \tilde{Q}_i [29, theorem 2.4.25]. Hence $Z^*(0)$ is finite.

We can also define $Z^*(\delta)$ for $\delta < 0$, by changing the right-hand side of (17) into $\delta Q_{i,0} = -|\delta|Q_{i,0}$. We have that $Z^*(\delta)$ is finite for $\delta < 0$ small enough. Indeed, passing the term $\delta Q_{i,0}$ on the left hand side, this can then be seen as a perturbation of the matrix \tilde{C}_i above, and for δ small enough, detectability, which is an open condition, is preserved.

We will see below that the programs with values $Z^*(\delta)$ are convex. It is then a standard result of perturbation analysis (see e.g. [28, p. 250]) that $Z^*(\delta)$ is a convex function of δ , hence continuous on the interior of its domain, in particular continuous at $\delta = 0$. So $\limsup_{T \rightarrow \infty} Z^*(T) = \lim_{T \rightarrow \infty} Z^*(T) = Z^*$. Finally, for any policy π , we obtain the following lower bound on the achievable cost

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sum_{i=1}^N \left\{ \text{Tr}(T_i \Sigma_i(t)) + \sum_{j=1}^M \kappa_{ij} \pi_{ij}(t) \right\} dt \geq Z^*.$$

We now show how to compute Z^* (or similarly, $Z^*(\delta)$) by solving a convex program involving linear matrix inequalities. For each i , introduce a new (slack) matrix variable R_i . Since $Q_i \succ 0, R_i \succ Q_i^{-1}$ is equivalent, by taking the Schur complement, to

$$\begin{bmatrix} R_i & I \\ I & Q_i \end{bmatrix} \succ 0,$$

and the Riccati inequality (18) can be rewritten

$$\begin{bmatrix} \tilde{Q}_i A_i + A_i^T \tilde{Q}_i - \sum_{j=1}^M p_{ij} C_{ij}^T V_{ij}^{-1} C_{ij} & \tilde{Q}_i W_i^{1/2} \\ W_i^{1/2} \tilde{Q}_i & -I \end{bmatrix} \preceq 0.$$

We finally obtain, dropping the tildes from the notation \tilde{Q}_i , the semidefinite program

$$Z^* = \min_{R_i, Q_i, p_{ij}} \sum_{i=1}^N \left\{ \text{Tr}(T_i R_i) + \sum_{j=1}^M \kappa_{ij} p_{ij} \right\}, \quad (20)$$

$$\text{s.t.} \quad \begin{bmatrix} R_i & I \\ I & Q_i \end{bmatrix} \succ 0, \quad i = 1, \dots, N,$$

$$\begin{bmatrix} Q_i A_i + A_i^T Q_i - \sum_{j=1}^M p_{ij} C_{ij}^T V_{ij}^{-1} C_{ij} & Q_i W_i^{1/2} \\ W_i^{1/2} Q_i & -I \end{bmatrix} \preceq 0, \quad \forall i$$

$$0 \leq p_{ij} \leq 1, \quad i = 1, \dots, N, \quad j = 1, \dots, M,$$

$$\sum_{i=1}^N p_{ij} \leq 1, \quad j = 1, \dots, M, \quad (21)$$

$$\sum_{j=1}^M p_{ij} \leq 1, \quad i = 1, \dots, N.$$

Hence solving the program (20) provides a lower bound on the achievable cost for the original optimal control problem.

B. Problem Decomposition

It is well-know that efficient methods exist to solve (20) in polynomial time, which implies a computation time polynomial in the number of variables of the original problem. Still, as the number of targets increases, the large LMI (20) becomes difficult to solve. Note however that it can be decomposed into N small coupled LMIs, following the standard dual decomposition approach already used for the restless bandit problem. This decomposition is very useful to solve large scale programs with a large number of systems. For completeness, we repeat the argument below.

We first note that (21) is the only constraint which links the N subproblems together. So we form the Lagrangian

$$L(R, Q, p; \lambda) = \sum_{i=1}^N \left\{ \text{Tr}(T_i R_i) + \sum_{j=1}^M (\kappa_{ij} + \lambda_j) p_{ij} \right\} - \sum_{j=1}^M \lambda_j,$$

where $\lambda \in \mathbb{R}_+^M$ is a vector of Lagrange multipliers. We would take $\lambda \in \mathbb{R}^M$ if we had the constraint (2) instead of (1). Now the dual function is $G(\lambda) = \sum_{i=1}^N G_i(\lambda) - \sum_{j=1}^M \lambda_j$, where for each i

$$G_i(\lambda) = \min_{R_i, Q_i, \{p_{ij}\}_{1 \leq j \leq M}} \text{Tr}(T_i R_i) + \sum_{j=1}^M (\kappa_{ij} + \lambda_j) p_{ij}, \quad (22)$$

$$\text{s.t.} \quad \begin{bmatrix} R_i & I \\ I & Q_i \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} Q_i A_i + A_i^T Q_i - \sum_{j=1}^M p_{ij} C_{ij}^T V_{ij}^{-1} C_{ij} & Q_i W_i^{1/2} \\ W_i^{1/2} Q_i & -I \end{bmatrix} \preceq 0,$$

$$0 \leq p_{ij} \leq 1, \quad j = 1, \dots, M; \quad \sum_{j=1}^M p_{ij} \leq 1.$$

The optimization algorithm proceeds then as follows. We choose an initial value $\lambda^1 \geq 0$ and set $k = 1$.

1) For $i = 1, \dots, N$, compute $R_i^k, Q_i^k, \{p_{ij}^k\}_{1 \leq j \leq M}$ optimal solution of (22), and the value $G_i(\lambda^k)$.

- 2) We obtain the value $G(\lambda^k)$ of the dual function at λ^k . A supergradient of $G(\lambda^k)$ at λ^k is given by

$$\left[\sum_{i=1}^N p_{i1}^k - 1, \dots, \sum_{i=1}^N p_{iM}^k - 1 \right].$$

- 3) Compute λ^{k+1} in order to maximize $G(\lambda)$. For example, we can do this by using a supergradient algorithm, or any preferred nonsmooth optimization algorithm. Let $k:=k+1$ and go to step 1, or stop if convergence is satisfying.

Because the initial program (20) is convex, we know that the optimal value of the dual optimization problem is equal to the optimal value of the primal. Moreover, the optimal variables of the primal are obtained at step 1 of the algorithm above once convergence has been reached.

C. Open-loop Periodic Policies Achieving the Performance Bound

In this section we describe a family of open-loop policies that can approach arbitrarily closely the lower bound computed by (20), thus proving that this bound is tight. These policies are periodic switching strategies using a schedule obtained from the optimal parameters p_{ij} . Assuming no switching times or costs, their performance approaches the bound as the length of the switching cycle decreases toward 0.

Let $P = [p_{ij}]_{1 \leq i \leq N, 1 \leq j \leq M}$ be the matrix of optimal parameters obtained in the solution of (20). We assume here that constraints (1) and (3) were enforced, which is the most general case for the discussion in this section. Hence P verifies

$$0 \leq p_{ij} \leq 1, \quad i = 1, \dots, N, \quad j = 1, \dots, M, \\ \sum_{i=1}^N p_{ij} \leq 1, \quad j = 1, \dots, M, \quad \text{and} \quad \sum_{j=1}^M p_{ij} \leq 1, \quad i = 1, \dots, N.$$

Thus if $M = N$, P is a *doubly substochastic matrix*. Else if $M < N$ or $N < M$, we can add $N - M$ columns of zeros or $M - N$ rows of zeros to P respectively to obtain a doubly substochastic matrix. In any case, we call the resulting doubly substochastic matrix $\tilde{P} = [\tilde{p}_{ij}]$. If rows have been added, this is equivalent to the initial problem with additional “dummy systems”. If columns are added, these correspond to using “dummy sensors”. Dummy systems are not included in the objective function, and a dummy sensor $j > M$ is associated formally to the measurement noise covariance matrix $V_{ij}^{-1} = 0$ for all i , in effect producing no measurement. In the following we assume that \tilde{P} is an $N \times N$ doubly substochastic matrix, but the discussion in the $M \times M$ case is identical. First, we need the following theorem.

Theorem 6 ([30]): The set of $N \times N$ doubly substochastic matrices is the convex hull of the set \mathcal{P}_0 of $N \times N$ matrices which have a most one unit in each row and each column, and all other entries are zero.

Hence for the doubly substochastic matrix \tilde{P} , there exists a set of positive numbers ϕ_k and matrices $P_k \in \mathcal{P}_0$ such that

$$\tilde{P} = \sum_{k=1}^K \phi_k P_k, \quad \text{with} \quad \sum_{k=1}^K \phi_k = 1, \quad \text{for some integer } K. \quad (23)$$

An algorithm to obtain this decomposition is described in [31] and [32] for example, and runs in time $O(N^{4.5})$. See also [20]. Note that any matrix $A \in \mathcal{P}_0$ represents a valid sensor/system assignment (for the system with additional dummy systems or sensors), where sensor j is measuring system i if and only if $a_{ij} = 1$.

The family of periodic switching policies considered is parametrized by a positive number ε representing a time interval over which the switching schedule is executed completely. Fixing ε , a policy is defined as follows:

- 1) At time $t = l\varepsilon, l \in \mathbb{N}$, associate sensor j to system i as specified by the matrix P_1 of the representation (23). Run the corresponding continuous-time Kalman filters, keeping this sensor/system association for a duration $\phi_1\varepsilon$.
- 2) At time $t = (l + \phi_1)\varepsilon$, switch to the assignment specified by P_2 . Run the corresponding continuous time Kalman filters until $t = (l + \phi_1 + \phi_2)\varepsilon$.
- 3) Repeat the switching procedure, switching to matrix P_{i+1} at time $t = l + \phi_1 + \dots + \phi_i$, for $i = 1, \dots, K - 1$.
- 4) At time $t = (l + \phi_1 + \dots + \phi_K)\varepsilon = (l + 1)\varepsilon$, start the switching sequence again at step 1 with P_1 and repeat the steps above.

1) Performance of the Periodic Switching Policies: For lack of space, the proofs of the results of this section are not reproduced here, but can be found in [19], [20]. Let us fix $\varepsilon > 0$ in the definition of the switching policy, and consider now the evolution of the covariance matrix $\Sigma_i^\varepsilon(t)$ for the estimate of the state of system i produced by this policy. The superscript indicates the dependence on the period ε of the policy. First we have

Lemma 7: For all $i \in \{1, \dots, N\}$, the estimate covariance $\Sigma_i^\varepsilon(t)$ converges as $t \rightarrow \infty$ to a periodic solution $\tilde{\Sigma}_i^\varepsilon(t)$ of period ε .

Next, denote by $\tilde{\Sigma}_i(t)$ the solution to the following Riccati differential equation (RDE), with initial condition $\Sigma_{i,0}$:

$$\dot{\Sigma}_i = A_i \Sigma_i + \Sigma_i A_i^T + W_i - \Sigma_i \left(\sum_{j=1}^M p_{ij} C_{ij}^T V_{ij}^{-1} C_{ij} \right) \Sigma_i. \quad (24)$$

Assumptions 2 and 3 guarantee that $\tilde{\Sigma}_i(t)$ converges to a positive definite limit denoted Σ_i^* . Moreover, Σ_i^* is the unique positive definite solution to the algebraic Riccati equation (ARE):

$$A_i \Sigma_i + \Sigma_i A_i^T + W_i - \Sigma_i \left(\sum_{j=1}^M p_{ij} C_{ij}^T V_{ij}^{-1} C_{ij} \right) \Sigma_i = 0. \quad (25)$$

The next lemma says that the periodic function $\tilde{\Sigma}_i^\varepsilon(t)$ oscillates in a neighborhood of Σ_i^* .

Lemma 8: For all $t \in \mathbb{R}_+$, we have $\tilde{\Sigma}_i^\varepsilon(t) - \Sigma_i^* = O(\varepsilon)$ as $\varepsilon \rightarrow 0$.

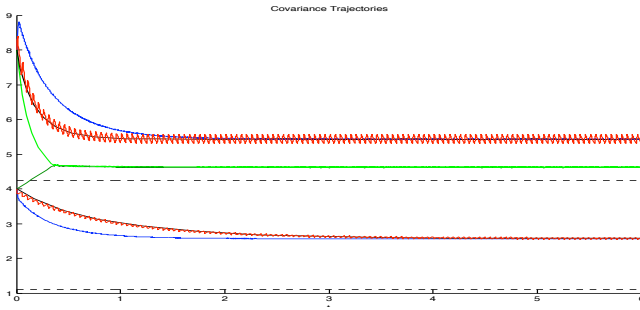


Fig. 1. Comparison of the variance trajectories under the periodic switching policy (oscillating red curves), Whittle's index policy (solid blue curves merging with the oscillating curves), and the greedy policy (converging green curves). Here a single sensor switches between two scalar systems. The period ε was chosen to be 0.05. The system parameters are $A_1 = 0.1, A_2 = 2, C_i = W_i = V_i = 1, \kappa_i = 0$. The dashed lines are the steady-state values that could be achieved with two identical sensors, each measuring one system. The performance of the Whittle policy is 7.98, which is optimal (i.e., matches the bound). The performance of the greedy policy is 9.2. Note that the greedy policy makes the variances converge, while the Whittle policy makes the Whittle indices (not shown) converge. The switching policy spends 23% of its time measuring system 1 and 77% of its time measuring system 2.

The following theorem then follows from the previous two lemmas, see [19], [20].

Theorem 9: Let Z^ε denote the performance of the periodic switching policy with period ε . Then $Z^\varepsilon - Z^* = O(\varepsilon)$ as $\varepsilon \rightarrow 0$, where Z^* is the performance bound (20). Hence the switching policies approach the lower bound arbitrarily closely as the period ε tends to 0.

Remark 10: It can be shown that the trajectories of $\Sigma_i^\varepsilon(t)$ in fact remains close to $\tilde{\Sigma}_i(t)$ for all t , characterizing the transient behavior of the switching policies, see [19].

2) **Numerical Simulation:** Figure 1 compares the covariance trajectories for the Whittle index policy, the periodic switching policy and the greedy policy (always measuring the system with highest mean square error) for a simple problem with one sensor switching between two scalar systems. Significant improvements over the greedy policy can be obtained in general.

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