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# Lyapunov Redesign of Adaptive Controllers for Polynomial Nonlinear Systems

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*Abstract*— **In this paper, we study adaptive control redesign problem of polynomial nonlinear systems with matching parametric uncertainties. By transforming the system into its corresponding error dynamics, we will develop an adaptive control scheme in attenuating the effect of the unknown parameters on the controlled output, which is composed of tracking errors and control efforts. To achieve better controlled performance, the Lyapunov functions will be relaxed from quadratic to higher order and the resulting controller gain is generalized from constant to parameter dependent. The synthesis conditions of adaptive control will be formulated as polynomial matrix inequalities and are solvable by recast the resulting conditions into a Sum of Squares (SOS) optimization problem, from which the adaptive control law as well as the parameter adaptation law are derived with zero tracking and parameter estimation errors. An example is provided to demonstrate effectiveness of the proposed adaptive control redesign approach.**

**Keywords:** Adaptive control; parametric uncertainties; higher-order Lyapunov function; SOS programming.

## I. Introduction

Adaptive control estimates unknown constant or slow varying parameters from on-line operating adaptive law and implements the controller with estimated parameters [18], [1], [9], [6]. Essentially, the adaptation of controller parameters is based on the performance error $y(t) - y_m(t)$ such that the closed-loop system adjusts itself towards an operating condition at which the desired system performance is achieved asymptotically, i.e., $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$.

Interest in adaptive control of nonlinear systems was stimulated by the inability of nonlinear feedback control to handle the presence of unknown parameters. The first series of adaptive nonlinear control schemes were restricted to systems satisfying the matching condition [2], and subsequently relaxed to the extended matching condition [11]. For a period of time, the extended matching condition could not be crossed by Lyapunov-based designs, which redirected researchers to estimation-based design. Nam and Arapostathis [10], Sastry and Isidori [19] combined feedback linearization with adaptation techniques from adaptive linear control. However, these schemes required the nonlinearities be restricted by linear growth conditions in order to achieve global stability. The only nonlinear estimation-based results which went beyond the linear growth constraints were obtained by using Lyapunov functions to characterize relationships between nonlinear growth constraints and controller stabilizing properties [13], [15], but still somehow restricted by matching conditions. The result presented in [5] finally broke the extended matching barrier, which was achieved with a new recursive design procedure called adaptive backstepping. Adaptive backstepping developed by [5] confluenced the adaptive estimation idea. Currently, there are two commonly used approaches for the design of nonlinear adaptive controllers: Lyapunov-based and estimation-based [6]. The distinction between Lyapunov-based and estimation-based schemes is dictated by the type of parameter update law and the corresponding proof of stability and parameter convergence [14], [7], [23]. With the development of backstepping techniques, significant progress has been made on Lyapunov-based designs. Nevertheless, estimation-based designs are more broadly applicable and allow a choice of parameter update laws from a wide repertoire of gradient and least-squares optimization techniques.

In spite of major development, adaptive control schemes have not yet become systematic engineering design tools. A potential drawback of adaptive control based on backstepping is lack of optimality and robustness. Moreover, its somewhat complicated design process, restrictive modeling assumption (such as linear parameterization), and poor transient performance often hinder its wide applications to practical problems and make it unfavorable from robustness consideration. To improve robustness property of adaptive control, recent studies in deterministic robust control (DRC) area [24], [20] and adaptive or robust adaptive control (RAC) area [4], [3] have been proposed to integrate robust and adaptive control techniques together for nonlinear systems. Adaptive robust control (ARC) was developed to preserve the theoretical performance results of both RAC and DRC, and prompt them to complement each other so that the well-known practical performance limitations of each were overcome [21], [22], [8]. Nevertheless, the developed robust adaptive techniques have only emphasized on synthesizing an adaptive control law and a parameter adaptation law based on some simple types of Lyapunov functions, whereas no optimality involved in. To overcome the limitation on existing Lyapunov functions, we propose an iterative algorithm to relax the Lyapunov functions from quadratic to higher-order and the resulting control gain is generalized from constant to parameter dependent. By employing higher-order Lyapunov functions and computationally effective optimization approach to redesign adaptive control laws, it would help provide more freedom on achieving a better performance level, as will be shown in the example.

The notation used in this paper is fairly standard. We

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denote $\mathbf{R}^{m\times n}$ as the set of real $m \times n$ matrices. In large symmetric matrix expressions, terms denoted by "$\star$" are to be induced by symmetry. Given a vector $x = [x_1 \quad \cdots \quad x_n]^T \in \mathbf{R}^n$, $\frac{\partial V}{\partial x} = \left[\frac{\partial V}{\partial x_1} \quad \frac{\partial V}{\partial x_2} \quad \dots \quad \frac{\partial V}{\partial x_n}\right]$ is the derivative of $V$ with respect to $x$. A multivariate polynomial $p(x)$ is a sum of squares (SOS) if there exist polynomials $p_1(x), \dots, p_\ell(x)$ such that $p(x) = \sum_{i=1}^{\ell} p_i^2(x)$.

## II. Standard Adaptive Control Scheme for Polynomial Nonlinear Systems

Consider a polynomial nonlinear system with matching uncertainties as

$$\dot{x} = A(x)x + \phi^T(x)\theta + u, \tag{1}$$

where the system state and control input $x, u \in \mathbf{R}^{n_x}$. $\phi^T(x)\theta$ describes the parametric uncertainties, which is composed of given basis functions $\phi^T(x) \in \mathbf{R}^{n_x \times \theta}$ and unknown weights $\theta \in \mathbf{R}^{\theta}$. In addition, $A(x)$ and $\phi^T(x)$ are restricted as polynomial functions of state $x$. Let $x_d(t)$ be the command output trajectory and $z = x - x_d(t)$ be the tracking error. The desired trajectory $x_d(t)$ is assumed as continuous function of time $t$. The actual value of the unknown parameter $\theta$ is a constant. For standard adaptive control design, the goal is to make $z$ as small as possible. Therefore, it is important to look at how the control input influences $z$:

$$\dot{z} = \dot{x} - \dot{x}_d(t) = A(x)x + \phi^T(x)\theta + u - \dot{x}_d(t).$$

To keep the transformed system in a clean form, we separate the polynomial and non-polynomial terms in the state equation. As a result, the transformed system is described by

$$\dot{z} = \widetilde{A}(z, x_d)z + \widetilde{\phi}^T(z, x_d)\theta + u + f(z, x_d, \dot{x}_d), \tag{2}$$

where $\widetilde{A}(z, x_d)$ and $\widetilde{\phi}^T(z, x_d)$ are polynomial functions of $z$ and $x_d$. $f(z, x_d, \dot{x}_d)$ represents the nonlinear terms produced by the introduction of $x_d$ through the transformation $x = z + x_d$.

The certainty equivalence principle assumes that the control law can be synthesized as if the system does not have any parameter uncertainties when the online parameter estimate $\hat{\theta}$ is used. Motivated by model reference adaptive control (MRAC) design [12], we will have the adaptive control scheme as following

$$u = u_m + u_s \tag{3}$$

$$u_m = -(\widetilde{A}(z, x_d)z + \widetilde{\phi}^T(z, x_d)\hat{\theta} + f(z, x_d, \dot{x}_d)) \tag{4}$$

$$u_s = -Kz, \quad K > 0 \tag{5}$$

$$\dot{\hat{\theta}} = \Gamma\widetilde{\phi}(z, x_d)z \tag{6}$$

with a quadratic Lyapunov function

$$V(z, \widetilde{\theta}) = z^T z + \widetilde{\theta}^T \Gamma^{-1} \widetilde{\theta}, \tag{7}$$

where $\widetilde{\theta} = \theta - \hat{\theta}$ denotes the estimation error between $\hat{\theta}$ and its true value. Note that the control law consists of two parts. The first part $u_m$ is used to cancel out redundant nonlinearities and approximate the unknown parameter $\theta$ by its estimation $\hat{\theta}$. $\hat{\theta}$ is governed by its own dynamics, which is the adaptation law as shown in eqn. (6). When $\hat{\theta}$ perfectly matches with $\theta$, $\dot{z}$ is driven to zero with the control force $u_m$. To compensate non-zero tracking error ($z \neq 0$) and stabilize the transformed system, we need to include the second part of control law $u_s$.

It has been proven that with the adaptive control law (3)-(5) and the parameter adaptation law (6), all signals in the system are bounded and the tracking error asymptotically converges to zero [21]. However, the parameter estimation error can only be guaranteed as bounded unless the persistent exciting (PE) condition is satisfied. When the system is very sensitive to the variation of the parameter, or in the situation that the parameter is poorly estimated and performs bad so that it has very large deviation from its true value, the operational accuracy of the entire system may be compromised.

## III. Adaptive Control Redesign for Performance Improvement

In this section, we will address the Lyapunov redesign of adaptive control problem for improving system performance with respect to certain optimality criteria. Specifically, we will focus on how to minimize the effect of imperfect parameter estimation on the controlled output. To this end, the Lyapunov function will be not restricted to the simple quadratic form as in (7), but relaxed to higher-order form, which helps provide more freedom to synthesize a better Lyapunov function. The direct consequences of this new Lyapunov redesign strategy is that the optimization of the performance level, the synthesis of adaptive controller gain and adaptation law, the determination of the Lyapunov form and its coefficients are all integrated into a simple but systematic design procedure. We will firstly formulate the optimization problem, then an iterative algorithm will be proposed to recursively improve the performance level and extend the Lyapunov function form.

Resort the system dynamics and define the performance output as

$$\dot{z} = \widetilde{\phi}^T(z, x_d)\widetilde{\theta} + u_s \tag{8}$$

$$e = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix} u_s \tag{9}$$

where $Q \geq 0, R > 0$ are specified penalties on $z$ and $u_s$. In this formulation, $\widetilde{\theta}$ is treated as a system disturbance. This disturbance term is special since it implicitly depends on state $z$ through its own dynamics. Our design objective is to synthesize an adaptive control law (3) together with an online parameter adaptation mechanism (6) so that the effect of $\widetilde{\theta}$ is as small as possible on the controlled output $e$ under a suitable Lyapunov function $V(z, \widetilde{\theta})$, i.e.

$$\begin{aligned} &\min \ \gamma \\ &\text{s.t.} \quad \dot{V}(z, \widetilde{\theta}) + (z^T Q z + u_s^T R u_s) - \gamma^2 \widetilde{\theta}^T \widetilde{\theta} < 0 \end{aligned} \tag{10}$$

In the following, we will provide an adaptive control synthesis condition for the disturbance attenuation problem mentioned above.

*Theorem 1:* For the system (8)-(9), if there exist a polynomial function $V(z,\widetilde{\theta}) = \mathcal{M}^T(z)P_1\mathcal{M}(z) + \widetilde{\theta}^T P_2 \widetilde{\theta}, P_1 > 0, P_2 > 0$, a matrix function $K(z)$ and a scalar $\gamma > 0$ so that following condition holds

$$\begin{bmatrix} \left\{ \begin{array}{c} Q - K^T(z)\frac{\partial \mathcal{M}(z)^T}{\partial z} P_1 N(z) \\ -N^T(z)P_1 \frac{\partial \mathcal{M}(z)}{\partial z} K(z) + K^T(z)RK(z) \end{array} \right\} & \star \\ \widetilde{\phi}(z,x_d)\frac{\partial \mathcal{M}(z)^T}{\partial z} P_1 N(z) - P_2 \Gamma \widetilde{\phi}(z,x_d) & -\gamma^2 I \end{bmatrix} < 0 \tag{11}$$

where $\mathcal{M}(z)$ is a pre-specified monomial vector of $z$ and $N(z)$ is a matrix function of $z$ so that $\mathcal{M}(z) = N(z)z$, then problem (10) is solvable by an adaptive control law in the form of (3) with controller gain $K(z)$ and an adaptation law (6).

This result can be easily shown and its proof is omitted here. Nevertheless, the solvability condition (11) is bilinear about matrix functions $V(z,\widetilde{\theta})$ and $K(z)$. In general, there does not exist a computationally efficient algorithm to solve the condition (11). For this reason, we will resort to an iterative computational scheme for its effective solution.

The iterative algorithm will start from the simplest case. We start with a quadratic Lyapunov function, constant control and adaptation laws as following

$$V(z,\widetilde{\theta}) = z^T P_1 z + \widetilde{\theta}^T \Gamma^{-1} \widetilde{\theta},\ P_1 > 0, \Gamma^{-1} > 0 \tag{12}$$

$$u_s = -Kz,\ K > 0 \tag{13}$$

$$\dot{\hat{\theta}} = \Gamma \widetilde{\phi}(z,x_d)z. \tag{14}$$

Then the condition in optimization problem (10) becomes

$$\begin{aligned} &\dot{V}(z,\widetilde{\theta}) + (z^T Q z + u_s^T R u_s) - \gamma^2 \widetilde{\theta}^T \widetilde{\theta} \\ &= \dot{z}^T P_1 z + z^T P_1 \dot{z} + \dot{\widetilde{\theta}}^T \Gamma^{-1} \widetilde{\theta} + \widetilde{\theta}^T \Gamma^{-1} \dot{\widetilde{\theta}} \\ &\quad + (z^T Q z + u_s^T R u_s) - \gamma^2 \widetilde{\theta}^T \widetilde{\theta} \\ &= [\widetilde{\phi}^T(z,x_d)\widetilde{\theta} - Kz]^T P_1 z + z^T P_1 [\widetilde{\phi}^T(z,x_d)\widetilde{\theta} - Kz] \\ &\quad - z^T \widetilde{\phi}^T(z,x_d)\widetilde{\theta} - \widetilde{\theta}^T \widetilde{\phi}(z,x_d) z + (z^T Q z + u_s^T R u_s) \\ &\quad - \gamma^2 \widetilde{\theta}^T \widetilde{\theta} \\ &= z^T (Q + K^T R K - K^T P_1 - P_1 K) z \\ &\quad + z^T [P_1 \widetilde{\phi}^T(z,x_d) - \widetilde{\phi}^T(z,x_d)]\widetilde{\theta} \\ &\quad + \widetilde{\theta}^T [\widetilde{\phi}(z,x_d) P_1 - \widetilde{\phi}(z,x_d)] z - \gamma^2 \widetilde{\theta}^T \widetilde{\theta} \\ &< 0. \end{aligned}$$

By Schur complement, it is sufficient to solve the following condition

$$\mathbf{S_{itr0}} := \begin{bmatrix} -M - M^T & \star & \star & \star \\ \widetilde{\phi}(z,x_d) - \widetilde{\phi}^T(z,x_d)P_1^{-1} & -\gamma_i^2 & \star & \star \\ Q^{\frac{1}{2}} P_1^{-1} & 0 & -I & \star \\ R^{\frac{1}{2}} M & 0 & 0 & -I \end{bmatrix} < 0 \tag{15}$$

where $M = KP_1^{-1}$. Note that the polynomial matrix inequality (15) depends on variables $z_d, x$ with decision variables $M, P_1^{-1}, \gamma_i^2$. Therefore, it can be formulated as an SOS optimization problem with compatible dimension free variable $v$

$$\begin{aligned} &min\ \gamma \\ \text{s.t.}\quad &\mathbf{S}_0 := -v^T \mathbf{S_{itr0}} v \in \Phi_{\text{SOS}}, \end{aligned} \tag{16}$$

and solved by SOS programming tool such as SOSTOOL [16], [17]. We will denote the controller gain $K$ as $K_0$ in sequel. Sum-of-squares (SOS) technique provides an efficient way to solve the polynomial matrix inequalities by recasting them as SOS optimization problems. The basic tool for testing if a given polynomial is a sum of squares is the Gramian matrix associated to the polynomial. Specifically, denote the monomial variable vector $\widetilde{x}$ and multi-index $p$ as

$$\begin{aligned} \widetilde{x} &= \begin{bmatrix} \widetilde{x}_1 & \dots & \widetilde{x}_{n_x+n_v} \end{bmatrix}^T \\ &= \begin{bmatrix} z_1 & \dots & z_{n_x} & v_1 & \dots & v_{n_v} \end{bmatrix}^T \\ p &= [p_1, \dots, p_n], \quad \bar{p} = \sum_{i=1}^{n} p_i \end{aligned}$$

so that any monomial can be written as $\widetilde{x}^p = \prod_{i=1}^{n_x+n_v} \widetilde{x}_i^{p_i}$. Then the representation of polynomial function $\mathbf{S}_0$ in (16) with degree $2k$ is given by $\mathbf{S} = \sum_{\bar{p} \le 2k} S_{[i]} \widetilde{x}_{[i]}^p$, where $k$ is a positive integer, $\tilde{x}_{[i]}^p$ is the $i$th monomial basis and $S_{[i]}$ is the corresponding coefficient. The maximal value of $i$ is determined by $k$.

Now for fixed $K_0$, one can generalize the Lyapunov function to a higher order form

$$V(z,\widetilde{\theta}) = \mathcal{M}^T(z)P_1\mathcal{M}(z) + \widetilde{\theta}^T P_2 \widetilde{\theta}, \tag{17}$$

where $P_1 > 0, P_2 > 0, \mathcal{M}(z)$ is a pre-specified monomial vector of $z$. Because the Lyapunov function is at least of 2nd order, any such $\mathcal{M}(z)$ could be represented as $\mathcal{M}(z) = N(z)z$, where $N(z)$ is a matrix function of $z$. Clearly, when $N(z) = I$, $\mathcal{M}(z)$ is the monomial vector of a quadratic Lyapunov function. Re-deriving the condition in optimization problem (10), we have

$$\begin{aligned} &\dot{V}(z,\widetilde{\theta}) + (z^T Q z + u_s^T R u_s) - \gamma^2 \widetilde{\theta}^T \widetilde{\theta} \\ &= -z^T [K_0^T \frac{\partial \mathcal{M}(z)^T}{\partial z} P_1 N(z) + N^T(z) P_1 \frac{\partial \mathcal{M}(z)}{\partial z} K_0 \\ &- Q - K_0^T R K_0] z + z^T [N^T(z) P_1 \frac{\partial \mathcal{M}(z)}{\partial z} \widetilde{\phi}^T(z,x_d) \\ &- \widetilde{\phi}^T(z,x_d) \Gamma^T P_2] \widetilde{\theta} + \widetilde{\theta}^T [\widetilde{\phi}(z,x_d) \frac{\partial \mathcal{M}(z)^T}{\partial z} P_1 N(z) \\ &- P_2 \Gamma \widetilde{\phi}(z,x_d)] z - \gamma^2 \widetilde{\theta}^T \widetilde{\theta} \\ &= \begin{bmatrix} z^T & \widetilde{\theta}^T \end{bmatrix} \begin{bmatrix} \left\{ \begin{array}{c} Q - K_0^T \frac{\partial \mathcal{M}(z)^T}{\partial z} P_1 N(z) \\ -N^T(z) P_1 \frac{\partial \mathcal{M}(z)}{\partial z} K_0 + K_0^T R K_0 \end{array} \right\} & \star \\ \widetilde{\phi}(z,x_d) \frac{\partial \mathcal{M}(z)^T}{\partial z} P_1 N(z) - P_2 \Gamma \widetilde{\phi}(z,x_d) & -\gamma^2 I \end{bmatrix} \begin{bmatrix} z \\ \widetilde{\theta} \end{bmatrix} < 0. \end{aligned} \tag{18}$$

Again, it is straightforward to solve this condition by transforming to its sufficient form

$$\mathbf{S_{itr1}} :=$$
$$\begin{bmatrix} \left\{ \begin{array}{c} Q - K_0^T \frac{\partial \mathcal{M}(z)^T}{\partial z} P_1 N(z) \\ -N^T(z) P_1 \frac{\partial \mathcal{M}(z)}{\partial z} K_0 + K_0^T R K_0 \end{array} \right\} & \star \\ \widetilde{\phi}(z, x_d) \frac{\partial \mathcal{M}(z)^T}{\partial z} P_1 N(z) - Y \widetilde{\phi}(z, x_d) & -\gamma^2 I \end{bmatrix} < 0, \tag{19}$$

where $Y = P_2 \Gamma$. By selecting an adaptation gain $\Gamma$, one can get the Lyapunov matrices $P - 1$ and $P_2 = Y \Gamma^{-1}$. Since $P_2$ and $\Gamma$ always show up in the multiplication form, it is possible to simplify the synthesis of $Y$ as the synthesis of $\Gamma$ by absorbing $P_2$ into $\Gamma$ so that the second part of the Lyapunov function keeps a simple form of $\widetilde{\theta}^T \widetilde{\theta}$.

For the polynomial matrix inequality (19) of $P_1, Y, \gamma^2$ involved, one can reformulate another SOS optimization problem

$$\begin{aligned} & min\ \gamma \\ \text{s.t.} \quad & \mathbf{S} := -v^T \mathbf{S_{itr1}} v \in \Phi_{\text{SOS}}. \end{aligned} \tag{20}$$

and solve it using SOSTOOL.

An interesting observation coming from the fundamental mechanism of the adaptive control is the form of Lyapunov function $V(z, \widetilde{\theta}) = \mathcal{M}^T(z) P_1 \mathcal{M}(z) + \widetilde{\theta}^T P_2 \widetilde{\theta},\ P_1 > 0, P_2 > 0$. The Lyapunov function involves not only the state $z$, but also the parameter estimation error $\widetilde{\theta}$. This is motivated by the fact that the dynamics of $\widetilde{\theta}$ is available and depends on $z$ as well. Therefore, augmenting the Lyapunov function to include complete information of the system dynamics will not obstruct determination of the Lyapunov function.

Keep in mind that whenever solving the synthesis condition (20), we will minimize the $\gamma$ value simultaneously. From this point of view, it is promising to obtain better $\gamma$ value through iteration because much more freedom is induced due to the extension on both of the Lyapunov function and the parameter adaptation forms.

Now we will state the iterative adaptive control algorithm for performance improvement using higher-order Lyapunov functions.

1) Let $i = 0$. Starting from a quadratic Lyapunov function (12) and adaptation law (14), solve the SOS optimization problem (16) to get the initial controller gain $K_0$ and performance level $\gamma_0$.
2) If problem (16) has a feasible solution, let $i = i+1$. For the given $\mathcal{M}(z)$ and fixed $K_{i-1}$, solve the following SOS optimization problem to get $Y_i, P_{1i} > 0$ and $\gamma_i$.

$$\begin{aligned} & min\ \gamma_i \\ \text{s.t.} \quad & -v^T \mathbf{S_{itri}} v \quad \in \Phi_{\text{SOS}}, \end{aligned} \tag{21}$$

where

$$\mathbf{S_{itri}} = \begin{bmatrix} \left\{ \begin{array}{c} -K_{i-1}^T(z) \frac{\partial \mathcal{M}(z)^T}{\partial z} P_{1i} N(z) \\ -N^T(z) P_{1i} \frac{\partial \mathcal{M}(z)}{\partial z} K_{i-1}(z) \\ +Q + K_{i-1}^T(z) R K_{i-1}(z) \end{array} \right\} & \star \\ \widetilde{\phi}(z, x_d) \frac{\partial \mathcal{M}(z)^T}{\partial z} P_{1i} N(z) - Y_i \widetilde{\phi}(z, x_d) & -\gamma_i^2 I \end{bmatrix}.$$

3) Let $i = i + 1$. For the given $\mathcal{M}(z)$ and fixed $P_{1(i-1)}$, solve an equivalent SOS optimization problem to get $K_i, Y_i$ and $\gamma_i$.

$$\begin{aligned} & min\ \gamma_i \\ \text{s.t.} \quad & -v^T \tilde{\mathbf{S}}_{\mathbf{itri}} v \in \Phi_{\text{SOS}}, \end{aligned} \tag{22}$$

where

$$\tilde{\mathbf{S}}_{\mathbf{itri}} = \begin{bmatrix} \left\{ \begin{array}{c} -K_i^T(z) \frac{\partial \mathcal{M}(z)^T}{\partial z} P_{1(i-1)} N(z) \\ -N^T(z) P_{1(i-1)} \frac{\partial \mathcal{M}(z)}{\partial z} K_i(z) + Q \end{array} \right\} & \star & \star \\ \widetilde{\phi}(z, x_d) \frac{\partial \mathcal{M}(z)^T}{\partial z} P_{1(i-1)} N(z) - Y_i \widetilde{\phi}(z, x_d) & -\gamma_i^2 I & \star \\ R^{\frac{1}{2}} K_i(z) & 0 & -I \end{bmatrix}$$

Note that $K_i$ could be a $z$ dependent polynomial function with pre-specified polynomial form. If $|\gamma_i - \gamma_{i-1}| < \epsilon$ or the iteration number $i$ is sufficiently large, let $\gamma = \gamma_i$ and STOP. Otherwise, go back to step 2.

Although we start from a quadratic Lyapunov function, the final Lyapunov function could be a complicated polynomial form. With a polynomial representation of Lyapunov function, it is possible to achieve a better closed-loop performance $\gamma_i < \gamma_{i-1}$. In step 2, it is clear that (21) is always feasible (at least $\mathcal{M}(z) = z, \frac{\partial \mathcal{M}(z)}{\partial z} = I, N(z) = I, Y_i = I$ should work). When iterating on step 3, condition (22) to be solved is also feasible (at least keep $K_{i-1} = K_i, Y_{i-1} = Y_i, P_{1(i-1)} = P_{1i}$ should work).

Finally, from derivation (18), it is clear that the iterative algorithm guarantees

$$\begin{aligned} & \dot{V}(z, \widetilde{\theta}) + (z^T Q z + u_s^T R u_s) - \gamma^2 \widetilde{\theta}^T \widetilde{\theta} \\ & = \begin{bmatrix} z^T & \widetilde{\theta}^T \end{bmatrix} \mathbf{S_{itri}} \begin{bmatrix} z \\ \widetilde{\theta} \end{bmatrix} < 0. \end{aligned}$$

The feasibility of above condition indicates that the function of $z, \widetilde{\theta}$ is nonnegative and the squares of $z(t), \widetilde{\theta}(t)$ are integrable with respect to time, i.e. $z(t), \widetilde{\theta}(t) \in \mathcal{L}_2$. Then by Barbalat's lemma [18], equation (8) and (6) imply that $\dot{z}(t), \dot{\widetilde{\theta}}(t) \in \mathcal{L}_\infty$ for any initial condition, which in turn implies that $z(t), \widetilde{\theta}(t) \to 0$ as $t \to 0$.

## IV. Example

Consider a second-order uncertain nonlinear system with one unknown parameter $\theta$ described by

$$\dot{x} = A(x) x + \phi^T(x) \theta + u,$$

where

$$A(x) = \begin{bmatrix} x_1^2 & x_1x_2 \\ 2x_2 & x_2^2 \end{bmatrix} \quad \phi^T(x) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

The uncertain system can be transformed to its corresponding error dynamics form with

$$\widetilde{A}(z, x_d) = \begin{bmatrix} \begin{Bmatrix} (z_1 + x_{d1})^2 \\ +z_1x_{d1} + 2x_{d1}^2 \end{Bmatrix} & \begin{Bmatrix} (z_1 + x_{d1})(z_2 + x_{d2}) \\ +z_1x_{d2} + x_{d1}x_{d2} \end{Bmatrix} \\ 2(z_2 + x_{d2}) & \begin{Bmatrix} (z_2 + x_{d2})^2 \\ +z_2x_{d2} + 2x_{d2}^2 \end{Bmatrix} \end{bmatrix}$$

$$\widetilde{\phi}^T(z, x_d) = \begin{bmatrix} z_1 + x_{d1} \\ 0 \end{bmatrix}$$

$$f(z, x_d, \dot{x}_d) = \begin{bmatrix} x_{d1}^3 + z_1x_{d2}^2 + x_{d1}x_{d2}^2 \\ 2z_2x_{d1} + 2x_{d1}x_{d2} + x_{d2}^3 \end{bmatrix}.$$

Given $\theta = 0.5, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$. We will apply the proposed iterative algorithm to solve the adaptive control problem.

Starting from a simple Lyapunov function (12) and an adaptation law (14), solve condition (16) to get an initial performance $\gamma_0 = 1.1853$ and its corresponding control gain

$$K_0 = \begin{bmatrix} 0.5350 & 0 \\ 0 & 0.5350 \end{bmatrix}.$$

Then we will examine the performance of controller $K_0$. During simulation, we set references trajectories as following

$$x_{1d}(t) = \begin{cases} 2 + 0.4t, & 0 \le t \le 7s, \\ 4.8, & 7s < t \le 10s. \end{cases}$$

$$x_{2d}(t) = \begin{cases} -1 - 0.5t, & 0 \le t \le 3s, \\ -2.5, & 3s < t \le 7s, \\ -2.5 + 0.5(t - 7), & 7s < t \le 10s. \end{cases}$$

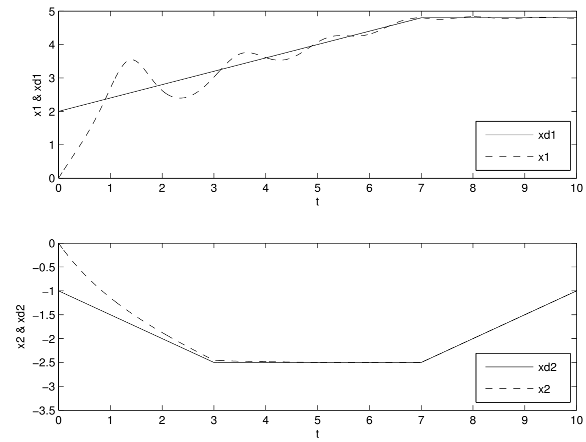
The starting point is chosen as $[x_{10}; x_{20}; \theta_0] = [0; 0; 0]$. The simulated performance of this initial adaptive controller is shown in Fig. 1.

To improve adaptive control performance, we assume that the Lyapunov function is 4th order function of $z$. Specifically, $V = \mathcal{M}^T(z)P_1\mathcal{M}(z) + \widetilde{\theta}^T P_2 \widetilde{\theta}$ with $\mathcal{M}(z) = [z_1^2; z_1z_2; z_2^2; z_1; z_2]$. For fixed $K_0$, we solve condition (21) using SOSTOOL and obtain

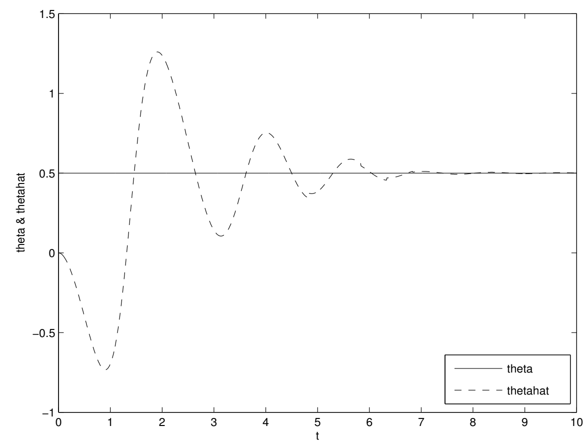
$$\gamma_1 = 1.1095$$

$$Y_1 = 26.166$$

$$P_1 = \begin{bmatrix} 0.0810 & 0 & -0.0526 & -0.0933 & 0 \\ 0 & 0.2272 & 0 & 0 & -0.0039 \\ -0.0526 & 0 & 6.7129 & -0.0061 & 0 \\ -0.0933 & 0 & -0.0061 & 36.1680 & 0 \\ 0 & -0.0039 & 0 & 0 & 30.4730 \end{bmatrix}$$



(a) Tracking performance of states



(b) Unknown parameter estimation

Fig. 1. Trajectories of states and adaptive parameter w.r.t their reference values with an initial adaptive controller.

Then we fix $P_{11}$ and specify the control gain $K_2 = K(z) = K_{20} + K_{21}z_1 + K_{22}z_2$. Solving condition (22) to minimize $\gamma$, finally we have

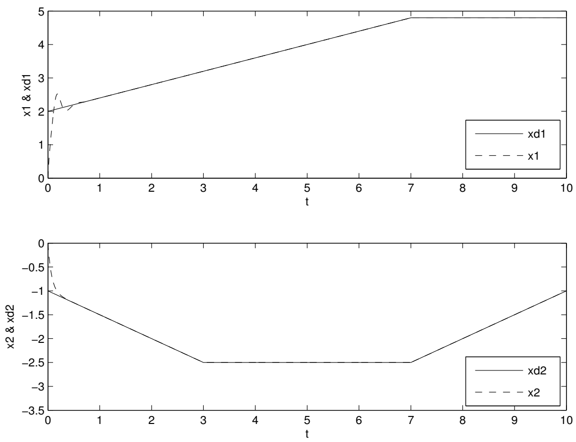
$$\gamma = \gamma_2 = 1.0410$$

$$Y_2 = 36.405$$

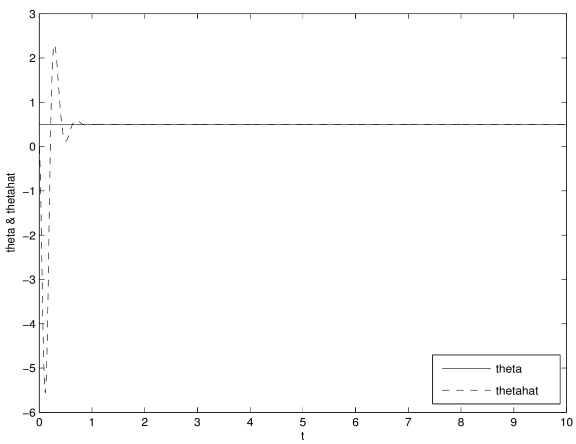
$$K(z) = \begin{bmatrix} 14.511 + 0.1107z_1 & 0.0117z_2 \\ -0.0014z_2 & 14.019 + 0.0021z_1 \end{bmatrix}$$

Select $\Gamma = Y_2$, then $P_2 = 1$. Through three iterations, we are able to minimize the $\gamma$ value by extending the Lyapunov function from 2nd order to 4th order and generalizing the control gain $K$ from scalar form to parameter dependent form.

For comparison purpose, we will use the same command trajectories and initial conditions to conduct simulation. Shown in Fig. 2(a) is the tracking performance of $x_1$ and $x_2$. As can be seen, both of the states track the command signals quickly compared with the initial adaptive controller. Moreover, $\hat{\theta}$ converges to its true value as shown in Fig. 2(b).




(a) Tracking performance of states




(b) Unknown parameter estimation

Fig. 2. Trajectories of states and adaptive parameter w.r.t their reference values using redesigned adaptive controller.

## V. CONCLUSION

In this paper, we developed an adaptive control redesign approach to attenuate the effect of the unknown parameters on the controlled output for a class of polynomial nonlinear systems. Motivated by classical adaptive control theory, the system is transformed into its error dynamics and the adaptive control law as well as the parameter adaptation law are synthesized through an iterative algorithm based on SOS programming. To improve adaptive controlled performance, the Lyapunov functions are relaxed from quadratic to higher order and the control gain is generalized from constant to parameter dependent. It has been shown that the system trajectories track their command profiles and the estimated parameters converge to their true value. All of the synthesis conditions are formulated in the frame work of polynomial/constant linear matrix inequalities and solvable using available SOS programming software package. Our future work will generalize the proposed nonlinear adaptive control approach to uncertain nonlinear systems with unmatched parameter uncertainties.

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