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A geometrical characterization of a class of 0-flat affine dynamical systems

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***Abstract*— This paper gives a description of a class of 0-flat dynamical systems. This class is characterized by the involutivity of a distribution associated naturally to multi-output affine dynamical systems and the Lie bracket of some control vector fields fulfilling some conditions. We will also show that these conditions are a generalization of the well-known result on 0-flatness of codimension 1 affine systems.**

I. INTRODUCTION

One important problem in control theory is to invert dynamical systems in order to compute the inputs required to perform a given task, for example the trajectory planning problem.

One classical solution to this problem is feedback linearization. Necessary and sufficient conditions for static state feedback linearizability were given in ([20], [22],[24]). For dynamic state feedback linearization, several results can be found in (see [2], [5], [6], [18], [19],[33], [28], [31], [34], [37], [38]).

Another approach to solve the trajectory planning problem, is the concept of differential flatness. This concept was first addressed by Fliess, Lévine, Martin, and Rouchon ([11], [13]), using the differential algebra theory.

A second approach to deal with flatness is exterior differential systems where a control dynamical system is regarded as a Pfaffian system on an appropriate jet space ([1], [7], [8], [25], [28], [39]), and flatness is related to absolute equivalence introduced by E. Cartan [4]. Another geometrical approach by means of Lie-Bäcklund equivalence was addressed in ([12], [14], [15], [23]).

Flat systems are a generalization of Linear dynamical systems in the sense that all linear controllable dynamical systems are flat and static feedback linearizable (in Brunovsky's form). In contrast to the feedback linearization, the flatness does not need to convert nonlinear systems into linear ones to design different kinds of feedback laws[1] [41], [42], [43]. Therefore, when a system is flat, we can use its structure to design control for motion, trajectory generation and stabilization.

A problem in the flatness theory is to give a general criterion for checking flatness and an algorithm to build the so-called flat outputs in a constructive way. In addition to the fact that feedback linearizable dynamical systems are flat, some results in this direction exist; [5] controllable codimension 1 affine dynamical systems or codimension 2 non holonome dynamical systems are flat.

In [25], the authors gave a characterization of the so-called k-flatness with the Cartan-Kähler approach.

In this paper, we will characterize a 0-flatness of particular classes of affine nonlinear dynamical systems for which we can build the flat outputs in a constructive way. As we will show with some examples, for this particular class, our method presents a new direction to solve the flatness problem.

This paper is outlined as follows. In the next section, we address notations, definition and the problem statement. In section 3, we give a class of 0-flat of nonlinear dynamical systems. This class can be seen as a normal form which is structurally 0-flat. In section 4, we give the necessary and sufficient geometrical conditions for affine dynamical systems to belong to the described class in section 3.

II. DEFINITIONS, NOTATIONS AND PROBLEM STATEMENT

Let us consider the following class of nonlinear dynamical systems:

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$ and f is a smooth function on $\mathcal{X} \times \mathcal{U}$.

Definition 1: Dynamical system (1) is flat if there exist m functions $y = (y_1, ..., y_m)$ called the flat outputs such that:

1) $y(x, u, \dot{u}, ..., u^{(r_1)})$ is a function of state x, input u, and the derivatives $u^{(i)}$,
2) $x = \varphi(y, \dot{y}, ..., y^{(r_2)})$ is a function of the flat outputs and their derivatives,
3) $u = \gamma(y, \dot{y}, ..., y^{(r_2+1)})$ is a function of the flat outputs and their derivatives.

In this paper, we will deal with multi-input affine dynamical systems in the following form:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \tag{2}$$

Without loss of generality, we will assume within this work that:

ASSUMPTION 1: $G = [g_1, ..., g_m]$ is of rank m.

We will characterize a class of dynamical systems for which the flat outputs are only functions of states x. Thus,

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in point 1) of definition 1 we have $y(x)$. This class of dynamical systems is called 0-flat [25].

Among the flat dynamic systems we can quoted the class of controllable linear dynamical systems. Therefore, another class of 0-flat dynamical systems is given by dynamical systems which are linearizable by means of a diffeomorphism and a static feedback. This class was characterized in ([20],[22],[24]).
Finally, recall that a dynamical affine system with n states and $n-1$ inputs is 0-flat as soon as it is controllable ([23],[30]).
Hereafter, we give another class of dynamical systems which are 0-flat locally.

III. A CLASS OF 0-FLAT DYNAMICAL SYSTEM

Let us in this section give a class of affine controllable dynamical systems in the (2) form which are 0-flat.
Let $\nu_1 \geq \nu_2 \geq ... \geq \nu_m$ be m integers such that:

$$\nu_1 + ... + \nu_m = n,$$

and let r an integer such that:

$$\begin{array}{rcl} \nu_i & \geq 2 & \text{if } \; i \leq r, \\ \nu_i & = & 1 \;\; \text{if } \; i > r \end{array}$$

Let us set

$$z = \{z_{i,j}, 1 \leq j \leq m \;\text{ and }\; 1 \leq i \leq \nu_j\},$$

and consider the following dynamical system:
For $1 \leq j \leq m$ we set

$$\dot{z}_{i,j} = z_{i+1,j} + \sum_{l=k}^{m} \alpha_{i,j}^{l}(z)u_l \;\text{ if }\; 1 \leq i \leq \nu_j - 1 \quad (3)$$

$$\dot{z}_{\nu_j,j} = a_j(z) + \sum_{l=k}^{m} \alpha_{\nu_j,j}^{l}(z)u_l \text{ if }\; i = \nu_j \quad (4)$$

where $k = min\{l; \;\; \nu_l \leq i\}$, and
functions a_j and $\alpha_{i,j}^l$ satisfy to the following conditions:

ASSUMPTION 2:

1) For $1 \leq k \leq m$, functions a_k depend only on the following variables:
   - $z_{i,j}$ such that $\nu_j > \nu_k$ and $1 \leq i \leq \nu_k + 1$
   - $z_{i,j}$ such that $\nu_j \leq \nu_k$ and $1 \leq i \leq \nu_j$
2) Functions $\alpha_{i,j}^k$ are as follows:
   - if $\nu_k > i$, then $\alpha_{i,j}^k = 0$,
   - if $\nu_k \leq i$, then $\alpha_{i,j}^k$ depend only on the following variables
     - $z_{s,l}$ for $1 \leq s \leq \nu_l$ if $\nu_l \leq i \leq \nu_j$,
     - $z_{s,l}$ for $1 \leq s \leq i+1$ if $i < \nu_l \leq \nu_j$.
   - $\alpha_{\nu_j,j}^j \neq 0$ on $\mathcal{X}$

Remark 1: a) For a fixed i and for all $1 \leq j \leq m$ dynamics $(\dot{z}_{i,j})_{\nu_j \geq i}$ depend on the variables:

1) u_k for $\nu_k \leq i$
2) $z_{s,l}$ for $1 \leq s \leq \nu_l$ if $\nu_l \leq i \leq \nu_j$,
3) $z_{s,l}$ for $1 \leq s \leq i+1$ if $i < \nu_l \leq \nu_j$.

Thus, for $1 \leq j \leq m$ dynamics $\dot{z}_{1,j}$ depend only on $(u_k)_{\nu_k=1}$, $(z_{1,l})_{1 \leq l \leq m}$ and $(z_{2,l})_{\nu_l \geq 2}$.
Dynamics $(\dot{z}_{2,j})_{\nu_j \geq 2}$ depend only on: $(u_k)_{\nu_k=1}$, $(u_k)_{\nu_k=2}$, $(z_{1,l})_{1 \leq l \leq m}$, $(z_{2,l})_{1 \leq l \leq m}$ and $(z_{3,l})_{\nu_l \geq 3}$.
And so on.
b) We can use the fact that $\alpha_{\nu_j,j}^j \neq 0$ on $\mathcal{X}$ to have $a_{\nu_j} = 0$ and $u_j = 1$. Indeed, we consider the following static feedback:

$$u_j = \frac{1}{\alpha_{\nu_j,j}^j}(v_j - a_{\nu_j}).$$

To give a geometrical interpretation of the above conditions, let us give some notations. We set dynamical system (3)-(4) in the following compact form:

$$\dot{z} = \overline{f} + \sum_{k=1}^{m} \overline{g}_k u_k, \quad (5)$$

with

$$\overline{f} = \begin{pmatrix} \overline{f}_1 \\ \overline{f}_2 \\ .. \\ .. \\ \overline{f}_m \end{pmatrix}$$

where for $1 \leq j \leq m$ we have:

$$f_j = \begin{pmatrix} z_{2,j} \\ z_{3,j} \\ .. \\ z_{\nu_j,j} \\ a_j \end{pmatrix}.$$

And $1 \leq k \leq m$ we set :

$$\overline{g}_k = \begin{pmatrix} \overline{g}_1^k \\ \overline{g}_2^k \\ .. \\ \overline{g}_m^k \end{pmatrix}$$

where for $1 \leq j \leq m$

$$\overline{g}_j^k = \begin{pmatrix} \alpha_{1,j}^k \\ \alpha_{2,j}^k \\ .. \\ \alpha_{\nu_j,j}^k \end{pmatrix}.$$

Thanks to condition (2) in assumption 2, we have for $1 \leq k \leq m$:

$$\overline{g}_j^k = 0 \;\text{ if }\; \nu_j < \nu_k,$$

$$\alpha_{i,j}^k = 0 \;\text{ if }\; \nu_k < i \leq \nu_j.$$

Remark 2:

- Let us consider the following distribution:
$$\overline{\Delta} = span\{ad_f^k \overline{g}_i, \text{ for all } \nu_i \geq 2 \text{ and } 0 \leq k \leq \nu_i - 2\},$$
which is involutive. In fact its dual codistribution is given by:
$$\overline{\Delta}^T = \text{span}\{dz_{1,j}\}_{1\leq j\leq m}.$$
This is compatible with the form of dynamic (3)-(4) and the fact that functions a_j satisfy point (1) of assumption 2.
- Conditions 2) are equivalent to the following fact: for $1 \leq k \leq m$ with $\nu_k \geq 2$ and for indices l such that: $\nu_l < \nu_k$ we have for $0 \leq s \leq \nu_k - \nu_l - 1$:
$$[\overline{g}_l, ad_f^{\nu_k-\nu_l-1-s}\overline{g}_k] \in$$
$$span\{ad_f^j \overline{g}_i \text{ for } j = 0 : \nu_i - \nu_l - s \text{ and } j \geq 0\}.$$

Remark 3: Using a linear change of coordinates we can assume that: $a_k = O^2(z)$ and for $\alpha_{i,j}^k$ such that $i \neq \nu_j$ we have $\alpha_{i,j}^k = O^1(z)$.

Now, we have the following preliminary result.

Proposition 1: Under assumptions (1-2), dynamical system (3)-(4) is 0-flat and the flat outputs are $(z_{1,j})_{1\leq j\leq m}$locally.

Proof: By assumption 2, for a fixed $1 \leq s \leq \max_{j=1}^m(\nu_j)$ the following set of dynamics:
$$\{\dot{z}_{s,j}, \quad 1 \leq j \leq m \text{ with } s \leq \nu_j\}$$
depend only on the $\mathfrak{S}_s$ set of the following variables :

1) u_k for $\nu_k \leq i$
2) $z_{s,l}$ for $1 \leq s \leq \nu_l$ if $\nu_l \leq i \leq \nu_j$,
3) $z_{s,l}$ for $1 \leq s \leq i+1$ if $i < \nu_l \leq \nu_j$.

We will show that $y_j = z_{1,j}$ for $1 \leq j \leq m$ are the flat outputs. For this, we start by writing all variables in the $\mathfrak{S}_1$ set by means of $y_j = z_{1,j}$ and $\dot{y}_j = z_{1,j}$. However, in $\mathfrak{S}_1$ we already know the variables $(y_j = z_{1,j})_{1\leq j\leq m}$. We then have to determine all the state variables $(z_{2,j})_{\nu_j\geq 2}$ and outputs $(u_k)_{\nu_k=1}$.

- For this, we use the implicit function theorem to compute variables $(z_{2,j})_{\nu_j\geq 2}$ and the inputs $(u_k)_{\nu_k=1}$, from the dynamics $\dot{z}_{1,l}$ for $1 \leq l \leq m$. In fact from the following equations:
$$\begin{aligned} \dot{z}_{1,j} &- z_{2,j} - \sum_{l=r+1}^{m} \alpha_{1,l}^j u_l = 0 \text{ if } \nu_j \geq 2 \\ \dot{z}_{1,j} &- a_j - \sum_{l=r+1}^{m} \alpha_{1,l}^j u_l = 0 \text{ if } \nu_j = 1, \end{aligned}$$
where $\alpha_{1,l}^j$ depend only on $(z_{1,l})_{1\leq l\leq m}$ and $(z_{2,l})_{\nu_l\geq 2}$. Therefore, we have to compute m variables $(z_{2,l})_{\nu_l\geq 2}$, $(u_j)_{\nu_j=1}$. The differential of the above equation by means of $\frac{\partial}{\partial(z_{2,j},u_k)}$ is equal to $I + O^1(z,u)$ which is locally invertible (here we used point (b) of remark 1 and remark 3).
Therefore, we have :
$$\begin{aligned} z_{2,l} &= \varphi_l(y_k, \dot{y}_k) \text{ for } \nu_l \leq 2 \quad (6) \\ u_k &= \gamma_k(y_k, \dot{y}_k) \text{ for } \nu_k = 1 \quad (7) \end{aligned}$$
where $y_k = z_{1,k}$ and $1 \leq k \leq m$. Thus, we know all the variables in $\mathfrak{S}_1$.
- Second step, we put expressions (6)-(7) in the dynamics $(\dot{z}_{2,j})_{\nu_j\geq 2}$. Then, we use the same argument to compute $(z_{3,l})_{\nu_l\geq 3}$ and u_k for indices k such that $\nu_k = 2$ (if $\nu_k > 2$ we only have to compute $(z_{3,l})_{\nu_l\geq 3}$). Thus, we know all the variables in $\mathfrak{S}_2$
- Then, by induction we assume that we had computed $\mathfrak{S}_s$, and from dynamics $(\dot{z}_{s+1,j})_{s+1\leq\nu_j}$ we will compute the variables in $\mathfrak{S}_{s+1}$ by using the same argument .

■

Let us give an example of dynamical systems which are in the 0-flat form (3)-(4), to show the procedure of computation of the state variables and the inputs by means of the flat outputs and their derivatives.

Example 1: Consider the following dynamical system :
$$\left\{ \begin{array}{c} \dot{z}_{1,1} = z_{2,1} \\ \dot{z}_{2,1} = z_{3,1} + \frac{z_{2,1}}{z_{2,2}-1}u_2 \\ \dot{z}_{3,1} = (z_{2,2}-1)\,u_1 + \frac{z_{3,1}}{z_{2,2}-1}u_2 \\ \dot{z}_{1,2} = z_{2,2} \\ \dot{z}_{2,2} = u_2 \end{array} \right.$$
We will give the procedure to compute all variable states and inputs from $y_1 = z_{1,1}$ and $y_2 = z_{1,2}$. For this, let us consider the following sub-dynamics:
$$\left\{ \begin{array}{l} \dot{z}_{1,1} - z_{2,1} = 0 \\ \dot{z}_{1,2} - z_{2,2} = 0 \end{array} \right.$$
we obtain $z_{2,1} = \ddot{y}_1$ and $z_{2,2} = \dot{y}_2$.
Now, from the following dynamics:
$$\left\{ \begin{array}{c} \dot{z}_{2,1} = z_{3,1} + \frac{z_{2,1}}{z_{2,2}-1}u_2 \\ \dot{z}_{2,2} = u_2 \end{array} \right.$$
we obtain: $u_2 = \dot{y}_2$ and $z_{3,1} = y_1^{(2)} - \frac{\dot{y}_1}{\dot{y}_2-1}\ddot{y}_2$.
Finally, from the third equation of the dynamical system we obtain:
$$\dot{z}_{3,1} - (z_{2,2}-1)\,u_1 - \frac{z_{3,1}}{z_{2,2}-1}u_2 = 0$$
we obtain
$$u_1 = \frac{1}{\dot{y}_2 - 1}\left(y_1^{(3)} + (\widehat{\frac{\dot{y}_1}{\dot{y}_2-1}}\ddot{y}_2) - \frac{y_1^{(2)} - \frac{\dot{y}_1}{\dot{y}_2-1}\ddot{y}_2}{\dot{y}_2 - 1} y_2^{(2)} \right).$$

IV. MAIN RESULT

In this section, we will give the geometrical necessary and sufficient conditions for the existence of a local diffeomorphism which transforms an affine dynamical system in (2) form into the (3)-(4) form.

For this, we assume that there exist $\nu_1 \geq \nu_2 \geq ... \geq \nu_m$ integers such that :

1) $\sum_{i=1}^{m} \nu_i = n$,
2) $\Delta_0 = \{ad_f^k g_i \quad \text{for} \quad i = 1 : m \quad \text{and} \quad 1 \leq k \leq \nu_k - 1\}$ is of rank n on $\mathcal{X}$.

Let us also consider the following distribution:

$$\Delta = span\{ad_f^k g_i, \quad \text{for all} \quad \nu_i \geq 2 \quad \text{and} \quad 0 \leq k \leq \nu_i - 2\}$$

Theorem 1: There exists a local diffeomorphism which transforms dynamical system (2) into the (3)-(4) form if and only if

1) Δ is involutive and
2) for $1 \leq k \leq m$ with $\nu_k \geq 2$ and for indices l such that: $\nu_l < \nu_k$ we have for $0 \leq s \leq \nu_k - \nu_l - 1$:

$$[g_l, ad_f^{\nu_k - \nu_l - 1 - s} g_k] \in$$
$$span\{ad_f^j g_i \quad \text{for} \quad j = 0 : \nu_i - \nu_l - s \quad \text{and} \quad \nu_i - \nu_l \geq s\}.$$

Before giving the proof of the theorem below, let us state the following result.

Corollary 1: If $\nu_j \leq 2$ for all $j = 1 : m$ then there exists a local diffeomorphism which transforms dynamical system (2) into the (3)-(4) form if and only if the distribution

$$\Delta = \{g_j \quad \text{for} \quad 1 \leq j \leq m \quad \text{such} \quad \nu_j = 2\}$$

is involutive. (Thus, we do not need condition (2) of theorem 1).

In particular, a codimension 1 dynamical system $m = n - 1$ is flat (well-known result [5].)

Remark 4: In the case of a single input $m = 1$, we only have condition (1) of theorem 1 and this condition is equivalent to the linearization by means of a diffeomorphism and a static feedback.

Now, we will prove theorem 1.

Proof: Conditions (1)-(2) of theorem 1 are necessary as we showed in remark 2.

Let us show that these conditions are sufficient. For this, we assume that $\nu_i \geq 2$ for $i = 1 : r$ and $\nu_i = 1$ of $r + 1 \leq i \leq m$. Thus $dim\Delta = \nu_1 + ... + \nu_r - r$ and it is of codimension m.

If Δ is involutive then, there exist m independent functions $h_1, ..., h_r, h_{r+1}, ..., h_m$ such that:

1) $dh_i(\Delta) = 0 \quad \text{for} \quad 1 \leq i \leq m$,
2) $dh_i(ad_f^{\nu_i - 1} g_i) \neq 0$ on $\mathcal{X}$ for $1 \leq i \leq m$.

Now, let us consider the following coordinates:

$$z_{i,j} = L_f^{i-1} h_j \quad \text{for} \quad j = 1 : m \quad \text{and} \quad 1 \leq i \leq \nu_i.$$

and set $z = (z_j)_{1 \leq j \leq m}$ where for $1 \leq j \leq m$

$$z_j = (z_{i,j})_{1 \leq i \leq \nu_j}.$$

We consider the diffeomorphism $z = \phi(x)$, and for $1 \leq s \leq m$, we denote by $\overline{g_s} = \phi_* g_s$, $\overline{g_s} = (\alpha_j^s)_{1 \leq j \leq m}$ where $\alpha_j^s = (\alpha_{i,j}^s)_{1 \leq i \leq \nu_s}$.

By definition of the new coordinates for $1 \leq j \leq m$ and $1 \leq i \leq \nu_j$, we have :

$dz_{i,j} \overline{g_s} = 0$ for $\nu_s - i > 0$. Thus, $\alpha_{i,j}^s = 0$ for $1 \leq j \leq m$ and $1 \leq i \leq \nu_j$ such that $\nu_s - i > 0$.

It is clear that $\phi_* f$ is in (3)-(4) form.

Moreover, by the involutivity condition functions a_k fulfill points (1) of assumption 2.

Now, the (2) following conditions of theorem:

$$[g_l, ad_f^{\nu_k - \nu_l - 1 - s} g_k] \in$$
$$span\{ad_f^j g_i \quad \text{for} \quad j = 0 : \nu_i - \nu_l - s \quad \text{and} \quad \nu_i - \nu_l \geq s\},$$

implies that $\alpha_{p,q}^l$ with $p \leq \nu_q$ do not depend on variables $z_{\nu_l + s + 1, k}$ for $p \leq \nu_l + s$. Therefore, point (2) of assumption 2 is fulfilled.

■

Case 1: **Codimension 2 case**

Let us analyse the codimension 2 case, thus $m = n - 2$.

By reordering $(g_j)_{1 \leq j \leq m}$ we have two cases :

1) $\nu_1 = 2 \quad \text{and} \quad \nu_2 = 2$
2) $\nu_1 = 3$.

The first case is similar to corollary 1. Thus, we have to check the involutivity of distribution $\Delta = span\{g_1, g_2\}$.

For the second case, we have to check two conditions :

- distribution $\Delta = \{g_1, ad_f g_1\}$ is involutive, and
- for all $2 \leq k \leq m$ we must have $[g_k, g_1] \in span\{g_1, ad_f g_1\}$.

Consider the following academic example [25] modified for a regularity question.

Example 2: Consider the following dynamical system:

$$\begin{cases} \dot{x}_1 = x_2 + x_4 x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = x_5 \\ \dot{x}_4 = u_1 \\ \dot{x}_5 = u_2 \end{cases}$$

A simple calculation shows that distribution Δ_0 is spanned by the following vector fields:

$$g_1 = \frac{\partial}{\partial x_4},$$
$$ad_f g_1 = -\frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_1}$$
$$ad_f^2 g = (1 - x_5) \frac{\partial}{\partial x_1}$$
$$g_2 = \frac{\partial}{\partial x_5} \quad \text{and} \quad ad_f g_2 = -\frac{\partial}{\partial x_3}.$$

Thus, $dim\Delta_0 = 5$ on an open set of 0 such that $x_5 \neq 1$. Moreover, distribution

$$\Delta = span\{g_1, ad_f g_1, g_2\}$$

is involutive. Thus, condition (1) of theorem 1 is fulfilled. Condition (2) is obviously fulfilled, because g_2 commutes with g_1 and $ad_f g_1$ by means of Lie bracket, thus:

$$[g_2, g_1] = [g_2, ad_f g_1] = 0.$$

Now, we will give the diffeomorphism. For this, it is easy to see that codistribution Δ^T is spanned by dh_1 and dh_2 where:

$h_1 = x_3 x_2 - x_1$ and $h_2 = x_3$.
Therefore, the following diffeomorphism :

$$\begin{gathered} z_{1,1} = h_1 \\ z_{2,1} = L_f h_1 = (x_5 - 1)\, x_2 \\ z_{3,1} = L_f^2 h_1 = (x_5 - 1)\, x_4 \\ z_{1,2} = h_2 \\ z_{2,2} = L_f h_2 = x_5 \end{gathered}$$

transforms the dynamical system into the following 0-flat form studied in example 1 :

$$\left\{ \begin{gathered} \dot{z}_{1,1} = z_{2,1} \\ \dot{z}_{2,1} = z_{3,1} + \tfrac{z_{2,1}}{z_{2,2}-1} u_2 \\ \dot{z}_{3,1} = (z_{2,2} - 1)\, u_1 + \tfrac{z_{3,1}}{z_{2,2}-1} u_2 \\ \dot{z}_{1,2} = z_{2,2} \\ \dot{z}_{2,2} = u_2 \end{gathered} \right.$$

Remark 5: If, instead of the first dynamic $\dot{x}_1 = x_2 + x_4 x_3$ we take the same dynamic $\dot{x}_1 = x_4 x_3$ as in [25], then, Δ_0 is of rank 5 on an open dense of 0. In this case, the same flat outputs work well except that $dy_1(ad_f^3 g_1) \neq 0$ and $dy_2(ad_f^2 g_2) \neq 0$ on an open dense subset.
We think that we can generalize theorem 1 by assuming that distribution Δ_0 is of dimension n in a dense subset of $\mathcal{X}$ and Δ is regular on $\mathcal{X}$.

Let us give another example to highlight the second conditions in theorem.

Example 3: Consider in $\mathbb{R}^6$ the following dynamical system:

$$\left\{ \begin{gathered} \dot{x}_1 = x_2 + \beta u_2 + ((1 + x_3)\beta + x_5)\, u_3 \\ \dot{x}_2 = x_3 + x_4 u_2 + x_3 u_3 \\ \dot{x}_3 = u_1 \\ \dot{x}_4 = x_5 \\ \dot{x}_5 = u_2 + x_3 u_3 \\ \dot{x}_6 = x_3 x_5 e^{x_4} + e^{x_4} u_1 + u_3 \end{gathered} \right.$$

where $\beta = x_6 - x_3 e^{x_4}$.

The generators of the distribution Δ_0 are:

$$\begin{aligned} g_1 &= \frac{\partial}{\partial x_3} + e^{x_4} \frac{\partial}{\partial x_6} \\ ad_f g_1 &= -\frac{\partial}{\partial x_2},\ ad_f^2 g_1 = \frac{\partial}{\partial x_1} \\ g_2 &= \frac{\partial}{\partial x_5} + \beta \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} \\ ad_f g_2 &= -\frac{\partial}{\partial x_4} - x_3 e^{x_4} \frac{\partial}{\partial x_6} + x_5 \frac{\partial}{\partial x_2} \\ g_3 &= \frac{\partial}{\partial x_6} + x_3 \frac{\partial}{\partial x_5} + x_3 \frac{\partial}{\partial x_2} + (x_5 + (1 + x_3)\beta) \frac{\partial}{\partial x_1}. \end{aligned}$$

It is easy to see that:

$$[g_2, g_1] = 0 \in span\{g_1, ad_f g_1\}$$

$$[g_3, g_1] = -\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_5} - \beta \frac{\partial}{\partial x_1} \in span\{g_1, ad_f g_1, g_2\}$$

Thus, conditions (2) of theorem 1 are fulfilled.
Condition (1) of theorem is also fulfilled. In fact, distribution

$$\Delta = span\{g_1, ad_f g_1, g_2\},$$

is involutive. Moreover,

$$\Delta^T = span\{dh_1, dh_2, dh_3\}.$$

where $h_1 = x_1 - x_5(x_6 - x_3 e^{x_4})$, $h_2 = x_4$ and $h_3 = x_6 - x_3 e^{x_4}$.
Let us set $z_{1,1} = h_1$, $z_{1,2} = h_2$ and $z_{1,3} = h_3$, we obtain the following diffeomorphism:

$$\begin{aligned} z_{1,1} &= x_1 - x_5\,(x_6 - x_3 e^{x_4}) \\ z_{2,1} &= L_f h_1 = x_2 \ \text{ and } \ z_{3,1} = L_f^2 h_1 = x_3 \\ z_{1,2} &= h_2 = x_4 \ \text{ and } \ z_{2,2} = L_f h_2 = x_5 \\ z_{1,3} &= h_3 = x_6 - x_3 e^{x_4} \end{aligned}$$

which transforms the dynamic into the following 0-flat form (3)-(4):

$$\left\{ \begin{gathered} \dot{z}_{1,1} = z_{2,1} + (z_{2.2} + z_{1,3}) u_3 \\ \dot{z}_{2,1} = z_{3,1} + z_{1,2} u_2 + z_{3,1} u_3 \\ \dot{z}_{3,1} = u_1 \\ \dot{z}_{1,2} = z_{2.2} \\ \dot{z}_{2,2} = u_2 + z_{3.1} u_3 \\ \dot{z}_{1,3} = u_3 \end{gathered} \right.$$

V. CONCLUSION

This paper deals with a characterization of a class of 0-flat dynamical systems. The conditions fulfilled by this class appear as a natural generalization of conditions of codimension 1 dynamical systems.
In our futur work, we try to characterize a class of k-flat dynamical systems by adapting the Charlet, Lévine and Marino method introduced for dynamic feedback linearization in [5].

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