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Delay-dependent Robust Model Predictive Control for Time-delay Systems with Input Constraints

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Abstract—**In this paper, we present a delay-dependent robust model predictive control (MPC) algorithm for a class of discrete-time linear state-delayed systems subjected to polytopic-type uncertainties and input constraints. The state-feedback MPC law is calculated by minimizing an upper bound of the worst-case quadratic cost function over an infinite time horizon at each sampling instant. In contrast to existing robust MPC techniques, the main advantage of the proposed approach is that the algorithm is derived by using a descriptor model transformation of the time-delay system and by applying a result on bounding of cross products of vectors. This has significantly reduced the conservativeness. It has been shown that robust stability of the closed-loop system is guaranteed by the feasible MPC from the optimization problem. The effectiveness of the algorithm is demonstrated by a simulation.**

I. Introduction

Time-delay often occurs in many dynamical systems, such as chemical processes, transportation systems, and communication networks etc. Time-delay often leads to serious deterioration of system stability and performance. Moreover, since it is difficult to obtain the exact model of system dynamics, thus uncertainties are unavoidable. Therefore, robust control of uncertain time-delay systems has received much attention, and many research results have been reported in control literature (see [1-4]). There are mainly two types of stabilization results: one is delay-independent and the other is delay-dependent. It has been shown that delay-dependent results taking into account the size of delays are generally less conservative than delay-independent ones, which do not include any information on the size of delays.

Most of the literatures on time-delay systems have neglected input constraints, which normally represent physical limits (such as valve saturation and power limitations, etc) and widely exist in many processes. A well method with ability to handle constraints on input /output is the MPC[5].

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At present, only a few results of MPC for delayed systems have been reported, such as [6-9]. A simple control method based on the receding horizon concept for delayed systems has been established in [6]. In [7], the authors have improved the technique of [6] by applying a cost function that includes two terminal weighting terms, which are crucial to guarantee closed-loop stability. In [6,7] both MPC algorithms have considered the continuous-time systems with state-delays. However, neither model uncertainty nor input constraints are considered. Constrained MPC for discrete-time system with state-delays has been addressed in [8,9]. In [8] the robust constrained MPC scheme for delay-free systems has been extended to a delayed system by simply employing equivalent augmented systems without delay. However, this is not an effective alternative for general time-delay systems, especially for systems with unknown delays or systems with time-varying delays because it could lead to a high degree of complexity in the control design. In [9], an MPC algorithm for uncertain systems with input constraints and unknown state-delay has been presented. Due to unknown delay indices, the authors relaxed the optimization problem that minimizes a cost function to two other optimization problems and checked the closed-loop stability under an assumption that the weighting matrix is fixed to a constant matrix at all time. The assumption is very restricted and may lead to conservatism. On all accounts, all these design methods with regard to MPC for the delayed systems are delay-independent. A delay -dependent robust constrained MPC for the uncertain delayed systems is developed to reduce conservatism compared with delay-independent ones, which motivates our research.

This idea of this paper has been inspired by the method in [4]. The descriptor system approach and a bounding technique, which have been presented recently in [2] and [3] respectively, are used to develop a delay-dependent robust constrained MPC algorithm for a class of discrete time-varying systems with state-delay and polytopic uncertainties. Furthermore, we will prove the closed-loop stability of the proposed algorithm.

II. Problem Statement

Consider the following discrete-time uncertain time-varying systems with state-delay:

$$x(k+1) = A(k)x(k) + \overline{A}(k)x(k-d) + B(k)u(k) \qquad (1)$$

subject to input constraints

$$-\overline{u} \le u(k) \le \overline{u} \quad ,\overline{u} \ge 0, \quad \text{for all } k \in [0,\infty) \qquad (2)$$

where $x(k) \in R^n$ is the system state, $u(k) \in R^m$ is the control

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input, d is a positive integer for time delay and $x(k)=\phi(k)$, $-d\le k\le 0$ is the initial condition. Moreover, we assume that

$$\begin{aligned}&[A(k)\ \overline{A}(k)\ B(k)]\in\Omega\\&=Co\{[A_1\ \overline{A}_1\ B_1],[A_2\ \overline{A}_2\ B_2],\cdots,[A_p\ \overline{A}_p\ B_p]\}\end{aligned}\tag{3}$$

This means that for every k there exist p nonnegative coefficients $\lambda_i(k), i=1,2,\cdots,p$ such that

$$[A(k)\ \overline{A}(k)\ B(k)]=\sum_{i=1}^{p}\lambda_i(k)[A_i\ \overline{A}_i\ B_i],\ \sum_{i=1}^{p}\lambda_i(k)=1 \tag{4}$$

It is assumed that system (1) is stabilizable and that state $x(k)$ is available at each time k.

Our goal is to design a stabilizing state-feedback controller $u(k)=K(k)x(k)$ for system (1) by the MPC strategy and achieves the following robust performance index at each time k:

$$\min_{u(k+j|k),j\ge0}\ \max_{[A(k+j)\ \overline{A}(k+j)\ B(k+j)]\in\Omega} J(k) \tag{5}$$

subject to

$$J(k)=\sum_{j=0}^{\infty}\{\|x(k+j\,|\,k)\|_Q^2+\|u(k+j\,|\,k)\|_R^2\} \tag{6}$$

$$\begin{aligned}x(k+j+1\,|\,k)&=A(k+j)x(k+j\,|\,k)+\\&\overline{A}(k+j)x(k+j-d\,|\,k)+B(k+j)u(k+j\,|\,k)\end{aligned}\tag{7}$$

$$-\overline{u}\le u(k+j\,|\,k)=K(k)x(k+j\,|\,k)\le\overline{u},\ j\in[0,\infty) \tag{8}$$

where $Q>0$ and $R>0$ are given symmetric weighting matrices, $x(k+j\,|\,k)$ and $u(k+j\,|\,k)$ denote the predicted state of the plant at time $k+j$ and the future control move at time $k+j$, respectively, with $x(k-j\,|\,k)=x(k-j)$ for $j\ge 0$.

Eqs.(5)-(8) is a constrained min-max optimization problem corresponding to a worst-case infinite-horizon MPC with a quadratic objective. According to the principle of MPC, only the first computed input $u(k\,|\,k)=K(k)x(k\,|\,k)$ is implemented until the next sampling time. Updated by the actual state, the above optimization problem is repeated.

Now set $x(k+j+1|k)=x(k+j|k)+y(k+j|k)$. Then we have

$$x(k+j-d\,|\,k)=x(k+j\,|\,k)-\sum_{i=1}^{d}y(k+j-i\,|\,k) \tag{9}$$

As in [4], Eq. (7) can be transformed into an equivalent descriptor form as follows

$$\begin{aligned}&[A(k+j)+B(k+j)K(k)+\overline{A}(k+j)-I]x(k+j\,|\,k)\\&-y(k+j\,|\,k)-\overline{A}(k+j)\sum_{i=1}^{d}y(k+j-i\,|\,k)=0\end{aligned}\tag{10}$$

Before ending this section, we present the following lemma.

Lemma 1[3]: Assume that $\alpha\in R^{n_a}$, $\beta\in R^{n_b}$ and $N\in R^{n_a\times n_b}$. Then for any matrices $X\in R^{n_a\times n_a}$, $Y\in R^{n_a\times n_b}$ and $Z\in R^{n_b\times n_b}$, the following inequality holds:

$$-2\alpha^{\mathrm{T}}N\beta\le\begin{bmatrix}\alpha\\\beta\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}X & Y-N\\ Y^{\mathrm{T}}-N^{\mathrm{T}} & Z\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}$$

where $\begin{bmatrix}X & Y\\ Y^{\mathrm{T}} & Z\end{bmatrix}\ge 0$

III. Delay-dependent MPC Algorithm

The exact solution to this min-max problem (5)-(8) is not in general tractable. To obtain a practical optimization problem, the following quadratic function is defined:

$$\begin{aligned}V(x(k+j\,|\,k))&=x^{\mathrm{T}}(k+j\,|\,k)P_1x(k+j\,|\,k)+\\&\sum_{\theta=1}^{d}\sum_{l=1}^{\theta}y^{\mathrm{T}}(k+j-l\,|\,k)S_1y(k+j-l\,|\,k)+\\&\sum_{l=1}^{d}x^{\mathrm{T}}(k+j-l\,|\,k)S_2x(k+j-l\,|\,k)\end{aligned}\tag{11}$$

An upper bound on the worst value of the cost function $J(k)$ is obtained whenever the following inequality is satisfied for any $[A(k+j)\ \overline{A}(k+j)\ B(k+j)]\in\Omega$, $j\ge 0$:

$$\begin{aligned}&V(x(k+j+1\,|\,k))-V(x(k+j\,|\,k))\\&\le-[\|x(k+j\,|\,k)\|_Q^2+\|u(k+j\,|\,k)\|_R^2]\end{aligned}\tag{12}$$

For $J(k)$ to be finite, we must have $x(\infty\,|\,k)=0$. By the definition of $y(k+j\,|\,k)$, we can obtain $y(\infty\,|\,k)=0$. Hence $V(x(\infty\,|\,k))=0$. Summing both sides of the inequality (12) form $j=0$ to $j=\infty$ yields $-V(x(k\,|\,k))\le -J(k)$. Thus

$$\max_{[A(k+j)\ \overline{A}(k+j)\ B(k+j)]\in\Omega,\,j\ge0} J(k)\le V(x(k\,|\,k)) \tag{13}$$

where

$$\begin{aligned}V(x(k|k))=&x^{\mathrm{T}}(k|k)P_1x(k|k)+\sum_{\theta=1}^{d}\sum_{l=1}^{\theta}y^{\mathrm{T}}(k-l|k)S_1y(k-l|k)\\&+\sum_{l=1}^{d}x^{\mathrm{T}}(k-l|k)S_2x(k-l|k)\end{aligned}\tag{14}$$

Thus, from (13) the original min-max optimization problem (5)-(8) is turned into the following optimization problem that minimizes this upper bound $V(x(k\,|\,k))$

$$\min_{K(k),P_1,S_1,S_2} V(x(k\,|\,k)) \tag{15}$$

subject to (8), (10) and (12)

Theorem 1: Consider the time-delay system (1) with input constraints(2), and the system matrix $[A(k)\ \overline{A}(k)\ B(k)]$ belongs to a polytope Ω. For some prescribed scalar ε, if there exist matrices with appropriate dimension $X>0, Y, Z, \overline{K}$, $U_1>0, U_2>0, \overline{W}_1, \overline{W}_2, \overline{W}_3, E$ and scalar $\gamma>0$, such that the following semi-definite programming problem is solvable:

$$\min_{\gamma,X,Y,Z,\overline{K},U_1,U_2,\overline{W}_1,\overline{W}_2,\overline{W}_3,E}\ \gamma \tag{16}$$

subject to

$$\begin{bmatrix}1 & x^{\mathrm{T}}(k\,|\,k) & x^{\mathrm{T}}(k-1\,|\,k) & \cdots & x^{\mathrm{T}}(k-d\,|\,k)\\ * & X & 0 & \cdots & 0\\ * & * & U_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ * & * & * & \cdots & U_2\\ * & * & * & \cdots & *\\ * & * & * & \cdots & *\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ * & * & * & \cdots & *\end{bmatrix}$$

$$
\begin{bmatrix}
dy^{\mathrm{T}}(k-1|k) & (d-1)y^{\mathrm{T}}(k-2|k) & \cdots & y^{\mathrm{T}}(k-d|k) \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
dU_1 & 0 & \cdots & 0 \\
* & (d-1)U_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & U_1
\end{bmatrix} \geq 0 \quad (17)
$$

$$
\begin{bmatrix}
\Theta_1 & \Theta_2 & 0 & Z^{\mathrm{T}} & dZ^{\mathrm{T}} & X & XQ^{1/2} & \bar{K}^{\mathrm{T}}R^{1/2} \\
* & \Theta_3 & (1-\varepsilon)\bar{A}_\sigma U_2 & Y^{\mathrm{T}} & dY^{\mathrm{T}} & 0 & 0 & 0 \\
* & * & -U_2 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -X & 0 & 0 & 0 & 0 \\
* & * & * & * & -dU_1 & 0 & 0 & 0 \\
* & * & * & * & * & -U_2 & 0 & 0 \\
* & * & * & * & * & * & -\gamma I & 0 \\
* & * & * & * & * & * & * & -\gamma I
\end{bmatrix} \leq 0 \quad (18)
$$

$$
\begin{bmatrix}
\bar{W}_1 & \bar{W}_2 & 0 \\
\bar{W}_2^{\mathrm{T}} & \bar{W}_3 & \varepsilon\bar{A}_\sigma U_1 \\
0 & \varepsilon U_1 \bar{A}_\sigma^{\mathrm{T}} & U_1
\end{bmatrix} \geq 0 \quad (19)
$$

$$
\begin{bmatrix} E & \bar{K} \\ \bar{K}^{\mathrm{T}} & X \end{bmatrix} \geq 0 \ , \quad E_{ii} \leq \bar{u}_i^2 \ , \quad i = 1, \cdots, m \quad (20)
$$

where $\Theta_1 = Z + Z^{\mathrm{T}} + d\bar{W}_1$, $\Theta_2 = X(A_\sigma^{\mathrm{T}} + \varepsilon\bar{A}_\sigma^{\mathrm{T}} - I) + Y + \bar{K}^{\mathrm{T}}B_\sigma^{\mathrm{T}} - Z^{\mathrm{T}} + d\bar{W}_2$, $\Theta_3 = -Y - Y^{\mathrm{T}} + d\bar{W}_3$, $\sigma = 1, \cdots, p$, E_{ii} is the i th diagonal entry of E and $\bar{u}_i$ is the i th element of $\bar{u}$ .Then the MPC law $u(k+j|k) = \bar{K}X^{-1}x(k+j|k)$, $j \geq 0$ minimizes the upper bound $V(x(k|k))$ of the robust performance index.

Proof: Let $X = \gamma P_1^{-1}, U_1 = \gamma S_1^{-1}$ and $U_2 = \gamma S_2^{-1}$. Then minimization of $V(x(k|k))$ is equivalent to

$$
\min_{\gamma, P_1, S_1, S_2} \gamma
$$

subject to

$$
V(x(k|k)) = x^{\mathrm{T}}(k|k)P_1 x(k|k) + \sum_{\theta=1}^{d}\sum_{l=1}^{\theta} y^{\mathrm{T}}(k-l|k)S_1 y(k-l|k) + \sum_{l=1}^{d} x^{\mathrm{T}}(k-l|k)S_2 x(k-l|k) \leq \gamma
$$

By the Schur complement we easily deduce (16) and (17).

Using $V(x(k+j|k))$ defined in (11), we have

$$
\begin{aligned}
&V(x(k+j+1|k)) - V(x(k+j|k)) \\
&= 2x^{\mathrm{T}}(k+j|k)P_1 y(k+j|k) + x^{\mathrm{T}}(k+j|k)S_2 x(k+j|k) \\
&\quad + y^{\mathrm{T}}(k+j|k)(P_1 + dS_1)y(k+j|k) \\
&\quad - x^{\mathrm{T}}(k+j-d|k)S_2 x(k+j-d|k) \\
&\quad - \sum_{l=1}^{d} y^{\mathrm{T}}(k+j-l|k)S_1 y(k+j-l|k)
\end{aligned} \quad (21)
$$

With descriptor system (10) and Lemma 1, we obtain

$$
\begin{aligned}
&2x^{\mathrm{T}}(k+j|k)P_1 y(k+j|k) \\
&= 2\eta^{\mathrm{T}}(k+j|k)P^{\mathrm{T}}\begin{bmatrix} y(k+j|k) \\ 0 \end{bmatrix} \\
&= 2\eta^{\mathrm{T}}(k+j|k)P^{\mathrm{T}}\left\{\begin{bmatrix} y(k+j|k) \\ \Theta_4 \end{bmatrix} - \sum_{i=1}^{d}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix} y(k+j-i|k)\right\} \\
&\leq 2\eta^{\mathrm{T}}(k+j|k)P^{\mathrm{T}}\begin{bmatrix} y(k+j|k) \\ \Theta_4 \end{bmatrix} \\
&\quad + \sum_{i=1}^{d}\begin{bmatrix} \eta(k+j|k) \\ y(k+j-i|k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} W & M - P^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix} \\ * & S_1 \end{bmatrix} \begin{bmatrix} \eta(k+j|k) \\ y(k+j-i|k) \end{bmatrix} \\
&= 2\eta^{\mathrm{T}}(k+j|k)P^{\mathrm{T}}\begin{bmatrix} 0 & I \\ A(k+j) + B(k+j)K - I & -I \end{bmatrix}\eta(k+j|k) \\
&\quad + 2\eta^{\mathrm{T}}(k+j|k)P^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix} x(k+j|k) + d\eta^{\mathrm{T}}(k+j|k)W\eta(k+j|k) \\
&\quad + 2\eta^{\mathrm{T}}(k+j|k)\left(M - P^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix}\right)(x(k+j|k) - x(k+j-d|k)) \\
&\quad + \sum_{i=1}^{d} y^{\mathrm{T}}(k+j-i|k)S_1 y(k+j-i|k)
\end{aligned} \quad (22)
$$

where $\Theta_4 = (A(k+j) + B(k+j)K(k) + \bar{A}(k+j) - I)x(k+j|k) - y(k+j|k)$

$\eta(k+j|k) = \begin{bmatrix} x(k+j|k) \\ y(k+j|k) \end{bmatrix}$, $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$ and W, M are matrices with appropriate dimensions satisfying

$$
\begin{bmatrix} W & M \\ M^{\mathrm{T}} & S_1 \end{bmatrix} \geq 0 \quad (23)
$$

After substituting Eq. (22) into inequality (21) the following inequality is obtained

$$
\begin{aligned}
&V(x(k+j+1|k)) - V(x(k+j|k)) \\
&\leq \eta^{\mathrm{T}}(k+j|k)\Phi\eta(k+j|k) \\
&\quad + 2\eta^{\mathrm{T}}(k+j|k)\left(P^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix} - M\right)x(k+j-d|k) \\
&\quad - x^{\mathrm{T}}(k+j-d|k)S_2 x(k+j-d|k) \\
&= \xi^{\mathrm{T}}(k+j|k)\begin{bmatrix} \Phi & P^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix} - M \\ * & -S_2 \end{bmatrix}\xi(k+j|k)
\end{aligned} \quad (24)
$$

where $\xi^{\mathrm{T}}(k+j|k) = \begin{bmatrix} x^{\mathrm{T}}(k+j|k) & y^{\mathrm{T}}(k+j|k) & x^{\mathrm{T}}(k+j-d|k) \end{bmatrix}$

$$
\begin{aligned}
\Phi = &P^{\mathrm{T}}\begin{bmatrix} 0 & I \\ A(k+j) + B(k+j)K - I & -I \end{bmatrix} \\
&+ \begin{bmatrix} 0 & I \\ A(k+j) + B(k+j)K - I & -I \end{bmatrix}^{\mathrm{T}} P \\
&+ dW + \begin{bmatrix} M & 0 \end{bmatrix} + \begin{bmatrix} M^{\mathrm{T}} \\ 0 \end{bmatrix} + \begin{bmatrix} S_2 & 0 \\ 0 & P_1 + dS_1 \end{bmatrix}
\end{aligned}
$$

Replacing (12) with (24), inequality (12) can be written as:

$$\begin{bmatrix} \Phi + \begin{bmatrix} Q+K^{\mathrm{T}}RK & 0 \\ 0 & 0 \end{bmatrix} & P^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix} - M \\ * & -S_2 \end{bmatrix} \le 0 \qquad (25)$$

In order to obtain LMI, we define

$M = \varepsilon P^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}(k+j) \end{bmatrix}$, $\bar{K} = KX$, $\gamma P^{-1} = \begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix}$,

$\bar{W} = \gamma (P^{-1})^{\mathrm{T}} W (P^{-1}) = \begin{bmatrix} \bar{W}_1 & \bar{W}_2 \\ \bar{W}_2^{\mathrm{T}} & \bar{W}_3 \end{bmatrix}$

Pre- and post-multiplying (25) by $\mathrm{diag}\{\gamma^{1/2}(P^{-1})^{\mathrm{T}},\ \gamma^{-1/2}U_2\}$ and $\mathrm{diag}\{\gamma^{1/2}P^{-1},\ \gamma^{-1/2}U_2\}$ respectively. Pre- and post-multiplying (23) by $\mathrm{diag}\{\gamma^{1/2}(P^{-1})^{\mathrm{T}},\ \gamma^{-1/2}U_1\}$ and $\mathrm{diag}\{\gamma^{1/2}P^{-1},\ \gamma^{-1/2}U_1\}$ respectively. Using the Schur complement, (25) is equivalent to

$$\begin{bmatrix} \Theta_5 & \Theta_6 & 0 & Z^{\mathrm{T}} & dZ^{\mathrm{T}} & X & XQ^{1/2} & \bar{K}^{\mathrm{T}}R^{1/2} \\ * & \Theta_7 & \Theta_8 & Y^{\mathrm{T}} & dY^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & -U_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -X & 0 & 0 & 0 & 0 \\ * & * & * & * & -dU_1 & 0 & 0 & 0 \\ * & * & * & * & * & -U_2 & 0 & 0 \\ * & * & * & * & * & * & -\gamma I & 0 \\ * & * & * & * & * & * & * & -\gamma I \end{bmatrix} \le 0 \qquad (26)$$

where

$\Theta_5 = Z + Z^{\mathrm{T}} + d\bar{W}_1$

$\Theta_6 = X(A^{\mathrm{T}}(k+j) + \varepsilon \bar{A}^{\mathrm{T}}(k+j) - I) + Y + \bar{K}^{\mathrm{T}}B^{\mathrm{T}}(k+j) - Z^{\mathrm{T}} + d\bar{W}_2$

$\Theta_7 = -Y - Y^{\mathrm{T}} + d\bar{W}_3$, $\Theta_8 = (1-\varepsilon)\bar{A}(k+j)U_2$, and (23) is equivalent to

$$\begin{bmatrix} \bar{W}_1 & \bar{W}_2 & 0 \\ \bar{W}_2^{\mathrm{T}} & \bar{W}_3 & \varepsilon \bar{A}(k+j)U_1 \\ 0 & \varepsilon U_1 \bar{A}^{\mathrm{T}}(k+j) & U_1 \end{bmatrix} \ge 0 \qquad (27)$$

Since (26) and (27) are affine in terms of system matrices $[A(k)\ \ \bar{A}(k)\ \ B(k)]$, which are satisfied for all $[A(k)\ \ \bar{A}(k)\ \ B(k)] \in \Omega$, if and only if inequalities (18) and (19) hold respectively.

In order to transform the input constraint (8) into LMI, we should introduce an invariant ellipsoid as follows

$$\varpi = \{ z \in R^{n(2d+1)} \mid z^{\mathrm{T}}\Xi^{-1}z \le 1 \} \qquad (28)$$

where $\Xi = \mathrm{diag}\begin{bmatrix} X & U_2 & \cdots & U_2 & d^{-1}U_1 & (d-1)^{-1}U_1 & \cdots & U_1 \end{bmatrix}$

$z = [x^{\mathrm{T}}(k+j|k), \cdots, x^{\mathrm{T}}(k+j-d|k), y^{\mathrm{T}}(k+j-1|k), \cdots, y^{\mathrm{T}}(k+j-d|k)]^{\mathrm{T}}$

Following the reasoning in [8], it is proved that

$$\max_{j\ge 0} |u_i(k+j|k)|^2 = \max_{j\ge 0} |(\bar{K}X^{-1}x(k+j|k))_i|^2$$
$$\le \max_{z\in\varpi} |(\bar{\bar{K}}\Xi^{-1}z)_i|^2 \le (\bar{\bar{K}}\Xi^{-1}\bar{\bar{K}}^{\mathrm{T}})_{ii}$$

where $\bar{\bar{K}} = \begin{bmatrix} \bar{K} & 0 & \cdots & 0 \end{bmatrix}$. Then there exists a symmetric matrix E such that

$$\begin{bmatrix} E & \bar{\bar{K}} \\ \bar{\bar{K}}^{\mathrm{T}} & \Xi \end{bmatrix} \ge 0,\ \ E_{ii} \le \bar{u}_i^2,\ i = 1, \cdots, m \qquad (29)$$

From definitions of $\bar{\bar{K}}$ and Ξ , it is easily shown that (29) is equivalent to (20). This establishes (20) and the proof is complete.

Theorem 1 has given a sufficient condition for the existence of the robust MPC controller at sampling time k . Feasibility and robust asymptotic stability of the closed-loop system are guaranteed in the following theorem.

Theorem 2: Once a feasible solution of the optimization problem (16)-(20) is found, the state feedback MPC law obtained from Theorem 1 robustly asymptotically stabilizes the closed-loop system.

Proof: Firstly, we check the feasibility of the optimization problem (16)-(20). Suppose that a feasible control sequence $u(k+j|k), j \ge 0$ exists in optimization problem (16)-(20) at time k . Then at the next time $k+1$, a feasible solution is guaranteed to exist, since we can choose the following control sequence calculated at time k , which is a feasible solution, i.e.

$$u(k+j+1|k+1) = u^*(k+1+j|k),\ \ j \ge 0 \qquad (30)$$

where symbol $*$ denotes a solution obtained from the optimization problem at time k .The control sequence satisfies the input constraint (8) at time $k+1$,which implies that the optimization problem (16)-(20) is feasible at time $k+1$. Applying the same process, we observe that the optimization problem is feasible at all future instants.

Secondly, we give the stability of the closed-loop system. Let $P_1^*(k), S_1^*(k), S_2^*(k)$ and $P_1^*(k+1), S_1^*(k+1), S_2^*(k+1)$ denote the optimal values of optimization problem (16)-(20) at time k and at time $k+1$ respectively. Consider a quadratic function

$$V^*(x(k|k)) = x^{\mathrm{T}}(k|k)P_1^*(k)x(k|k) + \sum_{\theta=1}^{d}\sum_{l=1}^{\theta} y^{\mathrm{T}}(k-l|k)S_1^*(k)y(k-l|k) + \sum_{l=1}^{d} x^{\mathrm{T}}(k-l|k)S_2^*(k)x(k-l|k) \qquad (31)$$

Since $P_1^*(k+1), S_1^*(k+1), S_2^*(k+1)$ are optimal, whilst $P_1^*(k), S_1^*(k), S_2^*(k)$ are only feasible at time $k+1$, we have

$$\begin{aligned} V^*(x(k+1|k+1)) &= x^{\mathrm{T}}(k+1|k+1)P_1^*(k+1)x(k+1|k+1) \\ &+ \sum_{\theta=1}^{d}\sum_{l=1}^{\theta} y^{\mathrm{T}}(k+1-l|k+1)S_1^*(k+1)y(k+1-l|k+1) \\ &+ \sum_{l=1}^{d} x^{\mathrm{T}}(k+1-l|k+1)S_2^*(k+1)x(k+1-l|k+1) \\ &\le x^{\mathrm{T}}(k+1|k+1)P_1^*(k)x(k+1|k+1) \\ &+ \sum_{\theta=1}^{d}\sum_{l=1}^{\theta} y^{\mathrm{T}}(k+1-l|k+1)S_1^*(k)y(k+1-l|k+1) \\ &+ \sum_{l=1}^{d} x^{\mathrm{T}}(k+1-l|k+1)S_2^*(k)x(k+1-l|k+1) \end{aligned} \qquad (32)$$

Moreover, it follows from (12) that

$V(x(k+1|k))-V(x(k|k))\le 0$. Then we have

$$x^{\mathrm T}(k+1|k)P_1^*(k)x(k+1|k)+\sum_{\theta=1}^{d}\sum_{l=1}^{\theta}y^{\mathrm T}(k+1-l|k)S_1^*(k)y(k+1-l|k)$$
$$+\sum_{l=1}^{d}x^{\mathrm T}(k+1-l|k)S_2^*(k)x(k+1-l|k)$$
$$\le x^{\mathrm T}(k|k)P_1^*(k)x(k|k)+\sum_{\theta=1}^{d}\sum_{l=1}^{\theta}y^{\mathrm T}(k-l|k)S_1^*(k)y(k-l|k)$$
$$+\sum_{l=1}^{d}x^{\mathrm T}(k-l|k)S_2^*(k)x(k-l|k)=V^*(x(k|k)) \tag{33}$$

Since $x(k+1|k+1)=x(k+1|k)$ for any $[A(k)\ \bar{A}(k)\ B(k)]\in\Omega$ and $x(k+1-j|k+1)=x(k+1-j|k),\ j=1,\cdots,d$ considering definition of $y(k+j|k)$, we have

$$y(k+1-j|k+1)=x(k+2-j|k+1)-x(k+1-j|k+1)$$
$$=x(k+2-j|k)-x(k+1-j|k)=y(k+1-j|k),\ j=1,\cdots,d$$

We replace $x(k+1-j|k+1),\ j=0,\cdots,d$ and $y(k+1-j|k+1),\ j=1,\cdots,d$ in the last inequality of (32) by $x(k+1-j|k),\ j=0,\cdots d$ and $y(k+1-j|k),\ j=1,\cdots d$, respectively. Combining this with (33), we obtain

$$V^*(x(k+1|k+1))\le V^*(x(k|k)) \tag{34}$$

Therefore, $V^*(x(k|k))$ is a monotonically non-increasing and bounded Lyapunov function. Hence, $x(k)\to 0$ as $k\to\infty$, which completes the proof.

Remark 1: In the moving horizon implementation of Theorem 1, $x(k-l|k)=x(k-l)$ and $y(k-l|k)=x(k-l+1)-x(k-l)$ $l=1,\cdots,d$ are determined at the previous time instance and held fixed. Therefore, (16)-(20) is an LMI optimization problem (for fixed ε) that can be efficiently solved numerically.

Remark 2: The form of the invariant ellipsoid (28) given in Theorem 1 is different from the traditional MPC approach. Furthermore, our defined invariant ellipsoid is high dimensional for delayed systems, and similar methods can be found in [10].

Remark 3: Theorem 1 shows that the suggested MPC algorithm depends on the size of delay, unlike references [8] and [9] where the delay-independent MPC techniques are given. It was pointed out in [1] that delay-dependent conditions have better performances compared with delay-independent ones. This paper is to combine the bounding method [3] and the descriptor system approach [2], which is equivalent to the original system to developing a delay-dependent robust MPC algorithm for the uncertain time-delay systems. Compared with existing MPC method provided in [8,9], the advantage of the suggested MPC algorithms has reduced the conservatism.

IV. A Simulation Example

In this section, the effectiveness of the proposed delay-dependent MPC algorithms is illustrated through a backing up control problem of a computer simulated truck-trailer. The model of truck-trailer is given as follows[11]:

$$\begin{aligned}
\dot{x}_1(t)&=-a\frac{v\bar{t}}{Lt_0}x_1(t)-(1-a)\frac{v\bar{t}}{Lt_0}x_1(t-\tau)+\frac{v\bar{t}}{lt_0}u(t)\\
\dot{x}_2(t)&=a\frac{v\bar{t}}{Lt_0}x_1(t)+(1-a)\frac{v\bar{t}}{Lt_0}x_1(t-\tau)\\
\dot{x}_3(t)&=\frac{v\bar{t}}{t_0}\sin[x_2(t)+a\frac{v\bar{t}}{2L}x_1(t)+(1-a)\frac{v\bar{t}}{2L}x_1(t-\tau)]
\end{aligned} \tag{35}$$

where the variables x_1, x_2, x_3 and u denote the angle difference between the truck and the trailer, the angle of the trailer, the y -coordinate of the rear end of the trailer and the steering angle, respectively. The model parameters are given as $l=2.8, L=5.5, v=-1, \bar{t}=2, t_0=0.5$ .The constant $a\in[0,1]$ is the retarded coefficient. Here we assume $a=0.7$ .Using Euler first-order approximation with sampling time $T=0.1$ sec., system (35) is transformed to the uncertain discrete-time delayed system as follows:

$$\begin{bmatrix}x_1(k+1)\\x_2(k+1)\\x_3(k+1)\end{bmatrix}=\begin{bmatrix}1.0509&0&0\\-0.0509&1&0\\0.0509\alpha(k)&-0.4\alpha(k)&1\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\\x_3(k)\end{bmatrix}+$$
$$\begin{bmatrix}0.0218&0&0\\-0.0218&0&0\\0.0218\alpha(k)&0&0\end{bmatrix}\begin{bmatrix}x_1(k-d)\\x_2(k-d)\\x_3(k-d)\end{bmatrix}+\begin{bmatrix}-0.1429\\0\\0\end{bmatrix}u(k) \tag{36}$$

It is assumed that input constraints $|u(t)|\le\pi$ and that the uncertain parameter $\alpha(k)\in[1, 1.5915]$ is time-varying. Therefore we have

$$[A(k)\ \bar{A}(k)\ B(k)]\in\Omega=Co\{[A_1\ \bar{A}_1\ B_1],[A_2\ \bar{A}_2\ B_2]\}$$

$$A_1=\begin{bmatrix}1.0509&0&0\\-0.0509&1&0\\0.0509&-0.4&1\end{bmatrix},\bar{A}_1=\begin{bmatrix}0.0218&0&0\\-0.0218&0&0\\0.0218&0&0\end{bmatrix},B_1=\begin{bmatrix}-0.1429\\0\\0\end{bmatrix}$$

$$A_2=\begin{bmatrix}1.0509&0&0\\-0.0509&1&0\\0.0810&-0.6366&1\end{bmatrix},\bar{A}_2=\begin{bmatrix}0.0218&0&0\\-0.0218&0&0\\0.0347&0&0\end{bmatrix},B_2=\begin{bmatrix}-0.1429\\0\\0\end{bmatrix}$$

Simulation parameters are as follows: the time-delay $d=3$, the initial value $x(0)=\begin{bmatrix}0.5\pi & 0.75\pi & -5\end{bmatrix}^{\mathrm T}$, the weighting matrices are $Q=diag(10,10,10)$ and $R=1$, choosing $\varepsilon=1.5$.

In order to test the advantage of the MPC algorithm proposed in this paper, which is compared with the technique presented in [8]. In [8] the robust MPC scheme can be extended to handle uncertain time-delayed system by transforming the original system into an equivalent augmented delay-free system. Fig1-(a), (b) show the state trajectories of closed-loop systems achieved by the above two MPC algorithms. It is obvious that our MPC algorithm has better performance as the state variables achieved to the steady state faster than that of the MPC given in [8]. Furthermore, in order to demonstrate that the input constraints are active, the delay-dependent MPC method without taking into account input constraints of systems is applied, and the corresponding unconstrained closed-loop response is shown in Fig1-(c). Fig1-(d), (e) show the control inputs and the upper bounds of the cost function obtained by the three MPC methods respectively. From Fig1-(c), (d), it is clear that the unconstrained closed-loop responses is satisfied, but the corresponding control inputs are out of constraint range at some time. From Fig1-(d), it is apparent that control inputs

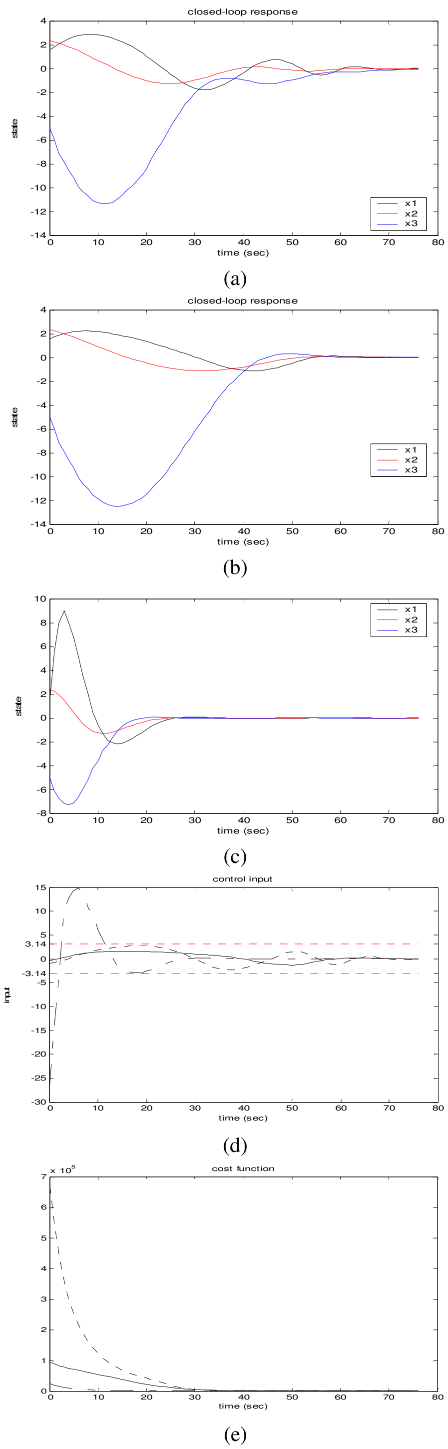

Fig. 1. Performance comparison of the MPC algorithm presented in the paper (solid line), the MPC algorithm given in [8] (dashed line) and unconstrained MPC algorithm (dash-dotted line)

calculated from the MPC proposed in this paper do not violate constraints. Moreover, from Fig1-(e), it is known that the upper bounds of the cost function obtained with our method is smaller than that obtained with the method in [8].

V. Conclusions

In this paper, we present a delay-dependent MPC algorithm for uncertain time-varying systems with state-delay and input constraints. We assume that the uncertainty of the model is of polytopic type. We transform the system to an equivalent descriptor system and combine the new bounding technique with LMI method, thus a less conservative stabilizing MPC algorithm is developed. Comparing with a pre-existing delay-independent MPC algorithm, the advantage of the proposed MPC algorithm is illuminated by means of a simulation.

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