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H_∞ Model Reduction of Linear Continuous-time Systems over Finite Frequency Interval-LMI based Approach

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Abstract— **This paper studies the model reduction problem for linear continuous-time systems over finite frequency interval. Different form the existing methods in the literature, we resort the problem to the aid of recently developed Generalized Kalman-Yakubovich-Popov (GKYP) lemma. A finite frequency H_∞ model reduction design method is presented in terms of solutions to a set of linear matrix inequalities (LMIs). Numerical examples are included for illustration.**

I. Introduction

The problem of model reduction is one of the fundamental problems of system theory and has been extensively investigated in the last three decades [1] [2]. In general, it can be stated as follows: Given a full order model $G(s)$, find a lower order model $G_r(s)$ such that G and G_r are close in some sense (such as H_∞, L_∞, H_2). Many approaches have been proposed in the literature [3] [4].

As have been pointed by many researchers (see [5]-[25] et al), many model reduction problems are inherently frequency dependent, i.e., the requirement on the approximation accuracy at some frequency ranges are more important than others. The behavior of the reduced-order model near resonances or at *a priori* known operating frequency interval should often be as close as possible to that of the high-order model, even at the expense of larger errors at other frequencies. To cope with such problems, the balanced truncation method and optimal Hankel-norm approximation method have been extensively extended. In this paper, the extensions of balanced truncation are mainly concerned since the balanced truncation method deals with the model reduction problem in H_∞ norm while the optimal Hankel-norm approximation method considers the model reduction problem in the sense of L_∞ norm. Generally speaking, there are two classes of extension for the balanced truncation. One is frequency weighted balanced truncation (see [5] [7] [8] [10] [11] [12] [18] [19] [20] [22] [24] et al), the basic idea is to introduce proper weighting functions for emphasizing the pre-specified frequency interval, however, the process of selecting appropriate weighting functions can be tedious and time consuming. The other is frequency Grammian-based balanced truncation, by extending the definitions of controllability/observability Grammian form entire frequency range to a given frequency interval, [12] [13] [14] [15] [16] [17] [25] suggested an alternative way to deal with the finite-frequency model reduction problem, in which the given information about the considered frequency interval can be combined directly. In some cases, the frequency Grammian-based balanced truncation method may produce a better reduced-order model than balanced truncation. Whereas, worse reduced-order model may be resulted in other cases. More important, those two extensions all provide no intrinsic connection between the reduced-order model design and the desired performance index (i.e., the maximum singular value of the approximation error transfer function over the given frequency interval), and we will show those observations by examples in the forthcoming section II.

This paper follows a different approach to revisit the finite-frequency H_∞ model reduction problem. In the recent papers [27] [29] [28], Iwasaki and Hara has generalized the fundamental machinery of dynamical systems analysis, the Kalman-Yakubovic-Popov lemma. By generalized KYP lemma, many finite-frequency characterized design performance specification can be treated directly. With the aid of the GKYP lemma, an LMI-based finite frequency reduced-order model design approach is proposed in this paper.

The main contributions of the paper include: I) GKYP lemma is introduced to deal with the finite-frequency model reduction problems for the first time. II) A proper structure of the key slack variable matrix is firstly proposed, which converts the finite-frequency model reduction problem into a convex programming problem expressed in terms of LMIs.

II. Preliminary/Review and Background

A. Problem formulation

In this paper, we consider the problem of continuous -time H_∞ model reduction over finite-frequency ranges. It can be generally stated as follows: Given a stable, $n^{th}-$ order continuous-time transfer function $G(s)$ with the following

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state-space realization:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \quad (1)$$

where $G(s) = C(sI - A)^{-1}B + D$, $s = j\omega$, and a user-specified frequency range $\omega \in \Theta$. Θ is defined in Table I, where LF, MF, and HF stand for low, middle, and high frequency ranges, respectively. The reduced-order stable r^{th}-order $(r < n)$ continuous-time transfer function G_r is supposed to be a linear time-invariant operator, which can be written in the following state-space realization

TABLE I

DIFFERENT FREQUENCY RANGES

	LF	MF	HF
Θ	$\lvert\omega\rvert \le \omega_l$	$\omega_1 \le \omega \le \omega_2$	$\lvert\omega\rvert \ge \omega_h$

$$\dot{x}_r(t) = A_r x_r(t) + B_r u(t)$$
$$y_r(t) = C_r x_r(t) + D_r u(t) \quad (2)$$

where $G_r(s) = C_r(sI - A_r)^{-1}B_r + D_r$.

The dynamics of (1) and (2) can be rewritten as the following augmented error system:

$$\dot{\xi}(t) = \bar{A}\xi(t) + \bar{B}u(t)$$
$$e(t) = \bar{C}\xi(t) + \bar{D}u(t) \quad (3)$$

where $e(t) = y_r(t) - y(t)$ is the approximation error, $\xi(t) = [\ x_r(t)^T \quad x(t)^T\ ]^T$, and

$$\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] = \left[\begin{array}{cc|c} A_r & 0 & B_r \\ 0 & A & B \\ \hline -C_r & C & -D_r + D \end{array}\right]$$

The H_∞ optimal model reduction problem is to find a reduced-order system (2) to minimize the worst case approximation error $e(t)$ over energy bounded input $u(t)$, this is

$$\min \sup_{\|u(t)\|_2 \neq 0} \frac{\|e(t)\|_2}{\|u(t)\|_2} \quad (4)$$

This is equivalent to minimizing the H_∞ norm of the transfer function G_{eu} between the input and the approximation error, which is to find a reduce-order system (2) such that

$$\max_{\omega} \|G_{eu}(j\omega)\|_\infty = \max_{\omega} \|G(j\omega) - G_r(j\omega)\|_\infty < \gamma \quad (5)$$

However, in this paper, we mainly consider the H_∞ model reduction over limited frequency interval, so condition (5) is converted to

$$\max_{\omega} \|G_{eu}(j\omega)\|_\infty =< \gamma, \quad \omega \in \Theta \quad (6)$$

B. Review of Related Work and Motivation

B.1 **Frequency Weighted Balanced Truncation (FWBT)**

The technique details are omitted here for the reasons of space, please see [1] [5] [7] [8] [10] [11] [12] [18] [19] [20] [22] [24] and references therein for details. Let us see the following example.

Example 1: [6] [9] [23]
Consider the 4^{th} order transfer function given by:

$$G(s) = \frac{(s^2 + 0.2s + 1.01)(s^2 + 3s + 9.01)}{(s^2 + 0.8s + 4.01)(s^2 + s + 16.02)} \quad (7)$$

The objective is to compute a 2^{nd} order reduced model which approximate this transfer function on the particular frequency range (0.1,2) rad/s. To emphasize the specified frequency interval, Scroletti et al [23] suggest the use of following weighting function

$$W_o(s) = \frac{4(s + 26.5)(s + 0.03774)}{(s + 2)^2(s^2 + 0.02653s + 1)} \qquad W_i(s) = 1 \quad (8)$$

see [23] for details. In the Table II, the reduced-order transfer functions and the maximal singular value of the error system $\overline{\sigma}(E(j\omega)) = \overline{\sigma}(G(j\omega) - G_r(j\omega))$ over the specified frequency range are obtained via BT and FWBT respectively. It can be seen that FWBT does not improve the desired performance index significantly (only 2.96% improvement achieved in this example). In other words, the selected weighting functions $W_i(s), W_o(s)$ (8) are not good enough. However, choosing an appropriate weighting function can be a quite difficult task.

TABLE II

EXAMPLE 1: BT VS FWBT

Methods	$G_r(s)$	$\|E(jw)\|_\infty$ $\omega \in [0.1, 2]$
BT	$\frac{1.051s^2+2.771s+2.086}{s^2+1.091s+14.78}$	0.7542
FWBT	$\frac{s^2+0.2813s+0.9568}{s^2+0.6191s+8.72}$	0.7319

B.2 **Frequency Grammian-based Balanced Truncation (FGBT)**

The technique details are omitted here for the reasons of space, please see [12] [13] [14] [15] [16] [17] [25] and references therein for details. Let us study the following example.

Example 2.
Consider the following 4-th order original system:

$$A = \begin{bmatrix} -2.218 & -1.974 & 0.306 & -1.287 \\ -1.529 & -2.809 & 0.909 & -1.141 \\ -1.049 & -0.050 & -2.144 & -1.210 \\ -1.299 & 1.745 & 0.345 & -2.141 \end{bmatrix}$$

$$B = \begin{bmatrix} -1.712 & -0.089 & -0.671 & -1.800 \end{bmatrix}^T$$

$$C = \begin{bmatrix} -0.877 & -1.252 & 0.774 & -0.111 \end{bmatrix}$$

Here, the specified frequency range is $|\omega| \leq 2$. Based on BT and FGBT, the following 2^{rd} reduced-order model can be attained (See Tabel III)

TABLE III

EXAMPLE 2: BT VS FGBT

Methods	(A_r, B_r, C_r)	$\|E(jw)\|_\infty$ $|\omega| \leq 2$
BT	$A_{r1} = \begin{bmatrix} -0.4956 & -1.2894 \\ -1.2894 & -5.9626 \end{bmatrix}$ $B_{r1} = \begin{bmatrix} 0.6120 & 0.9584 \end{bmatrix}^T$ $C_{r1} = \begin{bmatrix} 0.6012 & 0.9584 \end{bmatrix}$	0.1533
FGBT	$A_{r2} = \begin{bmatrix} -0.3714 & -0.2700 \\ 0.2700 & -0.1707 \end{bmatrix}$ $B_{r2} = \begin{bmatrix} 0.5448 & -0.0629 \end{bmatrix}^T$ $C_{r2} = \begin{bmatrix} 0.5448 & 0.0629 \end{bmatrix}$	0.2597

It can be seen that FGBT even gives a worse result than the standard BT method for this example. In fact, provides no intrinsic connection between the design method and the desired maximum singular value of the error systems over given frequency interval. In other words, FGBT cannot be considered as a reliable solution for finite frequency model reduction problem although it works well for many numerical examples.

C. Preliminaries

The following preliminaries are essential to the later developments.

Lemma 1: (*GKYP Lemma, Iwasaki and Hara [27] [29]*) Consider a transfer function matrix $G(j\omega) = C(j\omega I - A)^{-1}B + D$, and let a symmetric matrix Π be given, the following statements are equivalent:

i) The finite frequency inequality

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^T \Pi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \forall \omega \in \Theta \tag{9}$$

ii) There exist symmetric matrices P and Q of appropriate dimensions, satisfying $Q > 0$, and

$$\begin{bmatrix} A & I \\ C & 0 \end{bmatrix} \Xi \begin{bmatrix} A & I \\ C & 0 \end{bmatrix}^T + \begin{bmatrix} B & 0 \\ D & I \end{bmatrix} \Pi \begin{bmatrix} B & 0 \\ D & I \end{bmatrix}^T < 0 \tag{10}$$

where

$$\Xi = \begin{bmatrix} -Q & P + j\omega_c Q \\ P + j\omega_c Q & -\omega_1\omega_2 Q \end{bmatrix}$$

Lemma 2 *(Projection Lemma, Gahinet and Apkarian [30])* Given a real symmetric matrix Ψ and two real matrices P, Q, the following LMI problem

$$\Psi + PXQ^T + QX^TP^T < 0$$

is solvable with respect to decision variable X if and only if

$$\mathcal{N}_P \Psi \mathcal{N}_P^T < 0, \mathcal{N}_Q \Psi \mathcal{N}_Q^T < 0,$$

III. MAIN RESULTS

Theorem 1.(*model reduction over low-frequency ranges*) Consider the approximation error system (3) with given original system (A, B, C, D) and a non-negative scalar ω_l be given. If there exist matrices $G_1, G_2, G_3, \tilde{A}_r, \tilde{B}_r, \tilde{C}_r, \tilde{D}_r$, and Hermitian matrices P, Q

$$P = \begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix} \quad Q = \begin{bmatrix} Q_1 & Q_2^T \\ Q_2 & Q_3 \end{bmatrix} > 0$$

satisfying the following LMI

$$\begin{bmatrix} -Q_1 & * & * & * & * & * \\ -Q_2 & -Q_3 & * & * & * & * \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & * & * & * \\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} & * & * \\ 0 & 0 & \Phi_{53} & \Phi_{54} & -\gamma^2 I & * \\ 0 & 0 & \Phi_{63} & \Phi_{64} & -D_r^T + D^T & -I \end{bmatrix} < 0 \tag{11}$$

where

$$\begin{aligned}
\Phi_{31} &= P_1 + G_1^T \\
\Phi_{41} &= P_2 + \begin{bmatrix} I & 0 \end{bmatrix}^T G_1^T \\
\Phi_{32} &= P_2^T + G_2^T \\
\Phi_{42} &= P_3 + G_3^T \\
\Phi_{33} &= \omega_l^2 Q_1 - \tilde{A}_r - \tilde{A}_r^T \\
\Phi_{43} &= \omega_l^2 Q_2 - AG_2 - \begin{bmatrix} I & 0 \end{bmatrix}^T \tilde{A}_r^T \\
\Phi_{53} &= \tilde{C}_r - CG_2 \\
\Phi_{63} &= \tilde{B}_r^T \\
\Phi_{44} &= \omega_l^2 Q_3 - AG_3 - G_3 A^T \\
\Phi_{54} &= \tilde{C}_r \begin{bmatrix} I & 0 \end{bmatrix} - CG_3 \\
\Phi_{64} &= B^T
\end{aligned}$$

then there exists a reduced-order system (2) which can be reconstructed as follows:

$$G_r(s) =: \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right] = \left[\begin{array}{c|c} \tilde{A}_r G_1^{-1} & \tilde{B}_r \\ \hline \tilde{C}_r G_1^{-1} & \tilde{D}_r \end{array}\right]$$

satisfying the specification

$$\sigma_{\max}(G(j\omega) - G_r(j\omega)) < \gamma \quad \forall |\omega| \leq \omega_l \tag{12}$$

Proof.

According matrix manipulations and Schur complement, inequality (11) can be rewritten as

$$\begin{bmatrix} -Q & * & * \\ P + G & \omega_l^2 Q - He(\bar{A}G) - G^T \bar{A}^T & * \\ 0 & -\bar{C}G & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} \bar{D} \\ \bar{B} \\ 0 \end{bmatrix} \begin{bmatrix} \bar{D} \\ \bar{B} \\ 0 \end{bmatrix}^T < 0 \tag{13}$$

where

$$G = \begin{bmatrix} G_1 & \begin{bmatrix} G_1 & 0 \end{bmatrix} \\ G_2 & G_3 \end{bmatrix} \tag{14}$$

which can be rewritten as follows:

$$\underbrace{\begin{bmatrix} -Q & P & 0 \\ P & \omega_l^2 Q + \bar{B}\bar{B}^T & \bar{B}\bar{D}^T \\ 0 & \bar{D}\bar{B}^T & \bar{D}\bar{D}^T - \gamma^2 I \end{bmatrix}}_{\Psi} - \underbrace{\begin{bmatrix} -I \\ \bar{A} \\ \bar{C} \end{bmatrix}}_{H} \underbrace{G}_{\mathcal{M}} \underbrace{\begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}^T}_{\mathcal{R}^T} - \underbrace{\begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}}_{R} \underbrace{G}_{\mathcal{M}} \underbrace{\begin{bmatrix} -I \\ \bar{A} \\ \bar{C} \end{bmatrix}^T}_{\mathcal{H}^T} < 0 \tag{15}$$

Combining Lemma 2 and null space bases calculations, we have (15) holds means the following two inequalities hold

$$\begin{aligned}&\mathcal{N}(\mathcal{H})\Psi\mathcal{N}(\mathcal{H})^T\\&=\begin{bmatrix}\bar{A} & I\\ \bar{C} & 0\end{bmatrix}\begin{bmatrix}-Q & P\\ P & \omega^2 Q\end{bmatrix}\begin{bmatrix}\bar{A} & I\\ \bar{C} & 0\end{bmatrix}^T\\&\quad+\begin{bmatrix}\bar{B} & 0\\ \bar{D} & I\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & -\gamma^2 I\end{bmatrix}\begin{bmatrix}\bar{B} & 0\\ \bar{D} & I\end{bmatrix}^T<0\end{aligned}\tag{16}$$

$$\mathcal{N}(\mathcal{R})\Psi\mathcal{N}(\mathcal{R})^T=\begin{bmatrix}-Q & 0\\ 0 & \bar{D}\bar{D}^T-\gamma^2 I\end{bmatrix}<0\tag{17}$$

From Lemma 1 and (16) one can conclude that (11) gives a sufficient conditions such that

$$\begin{bmatrix}G_{eu}(j\omega)\\ I\end{bmatrix}^T\begin{bmatrix}I & 0\\ 0 & -\gamma^2 I\end{bmatrix}\begin{bmatrix}G_{eu}(j\omega)\\ I\end{bmatrix}<0,\forall|\omega|\le\omega_l\tag{18}$$

which means the performance specification (23) holds, thus, the proof is completed. □

Remark 1
The matrix $\mathcal{R}$ of (15) is called multiplier and the matrix G of (14) called key slack variable here, to derive the convex condition (11), G is restricted to a particular structure. Hence, some conservatism may be introduced here, a possible way to reduce the conservatism is letting G as follows

$$G=\begin{bmatrix}G_1 & \begin{bmatrix}\lambda G_1 & 0\end{bmatrix}\\ G_2 & G_3\end{bmatrix}\tag{19}$$

and using line searches to obtain the final result, where λ is a scalar. It deserves to emphasize that due to the structured slack variable G, the Lyapunov matrices (P,Q) are avoided to be structure restricted, therefore, the conservatism due to the structure restriction of G can be expected under an acceptable level.

The following two corollaries provide LMI conditions of the middle-frequency and high-frequency counterparts of Theorem 1.

Corollary 1.(*model reduction over middle-frequency ranges*)
Consider the approximation error system (3) with given original system (A,B,C,D) and non-negative scalars (ω_1,ω_2) be given. If there exist matrices $G_1,G_2,G_3,\tilde{A}_r,\tilde{B}_r,\tilde{C}_r,\tilde{D}_r$, and Hermitian matrices P,Q

$$P=\begin{bmatrix}P_1 & P_2^T\\ P_2 & P_3\end{bmatrix}\quad Q=\begin{bmatrix}Q_1 & Q_2^T\\ Q_2 & Q_3\end{bmatrix}>0$$

satisfying the following LMI

$$\begin{bmatrix}-Q_1 & * & * & * & * & *\\ -Q_2 & -Q_3 & * & * & * & *\\ \Phi_{31} & \Phi_{32} & \Phi_{33} & * & * & *\\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} & * & *\\ 0 & 0 & \Phi_{53} & \Phi_{54} & -\gamma^2 I & *\\ 0 & 0 & \Phi_{63} & \Phi_{64} & -D_r^T+D^T & -I\end{bmatrix}<0\tag{20}$$

where

$$\begin{aligned}\Phi_{31}&=P_1+G_1^T-j\omega_c Q_1\\ \Phi_{41}&=P_2+\begin{bmatrix}I & 0\end{bmatrix}^T G_1^T-j\omega_c Q_2^T\\ \Phi_{32}&=P_2^T+G_2^T-j\omega_c Q_2\\ \Phi_{42}&=P_3+G_3^T-j\omega_c Q_3\\ \Phi_{33}&=-\omega_1\omega_2 Q_1-\tilde{A}_r-\tilde{A}_r^T\\ \Phi_{43}&=-\omega_1\omega_2 Q_2-AG_2-\begin{bmatrix}I & 0\end{bmatrix}^T\tilde{A}_r^T\\ \Phi_{53}&=\tilde{C}_r-CG_2\\ \Phi_{63}&=\tilde{B}_r^T\\ \Phi_{44}&=-\omega_1\omega_2 Q_3-AG_3-G_3A^T\\ \Phi_{54}&=\tilde{C}_r\begin{bmatrix}I & 0\end{bmatrix}-CG_3\\ \Phi_{64}&=B^T\end{aligned}$$

then there exists a reduced-order system (2) which can be reconstructed as follows:

$$G_r(s)=:\left[\begin{array}{c|c}A_r & B_r\\ \hline C_r & D_r\end{array}\right]=\left[\begin{array}{c|c}\tilde{A}_rG_1^{-1} & \tilde{B}_r\\ \hline \tilde{C}_rG_1^{-1} & \tilde{D}_r\end{array}\right]$$

satisfying the specification

$$\sigma_{\max}(G(j\omega)-G_r(j\omega))<\gamma\quad\forall\omega_1\le\omega\le\omega_2\tag{21}$$

Proof.
Choosing the multiplier R in (15) as

$$R=\begin{bmatrix}0 & I & 0\end{bmatrix}$$

and following the same lines for that of Theorem 1, it is immediate.

Corollary 2.(*model reduction over high-frequency ranges*)
Consider the approximation error system (3) with given original system (A,B,C,D) and a non-negative scalar ω_h be given. If there exist matrices $G_1,G_2,G_3,\tilde{A}_r,\tilde{B}_r,\tilde{C}_r,\tilde{D}_r$, and Hermitian matrices P,Q

$$P=\begin{bmatrix}P_1 & P_2^T\\ P_2 & P_3\end{bmatrix}\quad Q=\begin{bmatrix}Q_1 & Q_2^T\\ Q_2 & Q_3\end{bmatrix}>0$$

satisfying the following LMI

$$\begin{bmatrix}\Phi_{11} & * & * & * & * & *\\ \Phi_{21} & \Phi_{22} & * & * & * & *\\ \Phi_{31} & \Phi_{32} & \Phi_{33} & * & * & *\\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} & * & *\\ \Phi_{51} & \Phi_{52} & \Phi_{53} & \Phi_{54} & -\gamma^2 I & *\\ 0 & 0 & \Phi_{63} & \Phi_{64} & -D_r^T+D^T & -I\end{bmatrix}<0\tag{22}$$

where

$$\begin{aligned}\Phi_{11}&=Q_1+G_1+G_1^T\\ \Phi_{21}&=Q_2+G_2+\begin{bmatrix}I & 0\end{bmatrix}^T G_1^T\\ \Phi_{31}&=P_1-G_1^T-A_r\\ \Phi_{41}&=P_2-\begin{bmatrix}I & 0\end{bmatrix}^T G_1^T-AG_2\\ \Phi_{51}&=\tilde{C}_r-CG_2\\ \Phi_{22}&=Q_3+G_3+G_3^T\\ \Phi_{32}&=P_2^T-G_2^T-A_r\begin{bmatrix}I & 0\end{bmatrix}\\ \Phi_{42}&=P_3-G_3^T-AG_3\\ \Phi_{52}&=\tilde{C}_r\begin{bmatrix}I & 0\end{bmatrix}-CG_3\\ \Phi_{33}&=-\omega_h^2Q_1+\tilde{A}_r+\tilde{A}_r^T\\ \Phi_{43}&=-\omega_h^2Q_2+AG_2+\begin{bmatrix}I & 0\end{bmatrix}^T\tilde{A}_r^T\\ \Phi_{53}&=-\tilde{C}_r+CG_2\\ \Phi_{63}&=\tilde{B}_r^T\\ \Phi_{44}&=-\omega_h^2Q_3+AG_3+G_3A^T\\ \Phi_{54}&=-\tilde{C}_r\begin{bmatrix}I & 0\end{bmatrix}+CG_3\\ \Phi_{64}&=B^T\end{aligned}$$

then there exists a reduced-order system (2) which can be reconstructed as follows:

$$G_r(s) =: \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right] = \left[\begin{array}{c|c} \tilde{A}_r G_1^{-1} & \tilde{B}_r \\ \hline \tilde{C}_r G_1^{-1} & \tilde{D}_r \end{array}\right]$$

satisfying the specification

$$\sigma_{\max}(G(j\omega) - G_r(j\omega)) < \gamma \quad \forall |\omega| \geq \omega_h \qquad (23)$$

Proof.
Choosing the multiplier R in (15) as

$$R = \left[\begin{array}{ccc} I & -I & 0 \end{array}\right]$$

and following the same lines for that of Theorem 1, it is immediate.

Noticing that the stability of the reduced-order systems obtained from Theorem 1, Corollary 1 and Corollary 2 cannot be guaranteed. Thus, we need the following lemma to derive a stable reduced-order system.

Lemma 3.(*stability condition*)
Consider the approximation error system (3), $\bar{A}$ (implicitly A_r) is stable if there exist matrices $G_1, G_2, G_3, \tilde{A}_r$, and matrix P_s:

$$P_s = \left[\begin{array}{cc} P_{s1} & P_{s2}^T \\ P_{s2} & P_{s3} \end{array}\right] > 0$$

satisfying the following LMI

$$\left[\begin{array}{cccc} \Psi_{11} & * & * & * \\ \Psi_{21} & \Psi_{22} & * & * \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & * \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \end{array}\right] < 0 \qquad (24)$$

where

$$\begin{aligned}
\Psi_{11} &= -qG_1 - qG_1^T \\
\Psi_{21} &= -qG_2 - q\left[\begin{array}{cc} I & 0 \end{array}\right]^T G_1^T \\
\Psi_{31} &= P_{s1} + pG_1^T + qA_r \\
\Psi_{41} &= P_{s2} + p\left[\begin{array}{cc} I & 0 \end{array}\right]^T G_1^T + qAG_2 \\
\Psi_{22} &= -qG_3 - qG_3^T \\
\Psi_{32} &= P_{s2}^T + pG_2^T + qA_r\left[\begin{array}{cc} I & 0 \end{array}\right] \\
\Psi_{42} &= P_{s3} + pG_3^T + qAG_3 \\
\Psi_{33} &= -pA_r - pA_r^T \\
\Psi_{43} &= -pAG_2 - p\left[\begin{array}{cc} I & 0 \end{array}\right]^T A_r^T \\
\Psi_{44} &= -pAG_3 - pG_3^T A^T
\end{aligned}$$

and p, q are arbitrary given scalars satisfying $p^2 - q^2 < 0$.
Proof. It is omitted here for the reasons of space.

Algorithm: Combining Theorem 1, Corollaries 1 and 2 and Lemma 3, the desired reduced system $G_r : (A_r, B_r, C_r, D_r)$ over different frequency ranges can be obtained by solving the following optimization problem:

$$\begin{array}{ll} \min & \gamma \\ \text{s.t.} & (24)(11) \ \text{for low} - \text{frequency range} \\ & \text{or } (24)(20) \ \text{for middle} - \text{frequency range} \\ & \text{or } (24)(22) \ \text{for high} - \text{frequency range} \end{array} \qquad (25)$$

IV. Illustrative Examples

In this section , we will continue the study of examples presented in section II to give a comprehensive illustration.

Example 1: (See details in section II)

Using the proposed LMI method of this paper, the reduced-order model and $\|E(jw)\|_\infty, \omega \in [0.1, 2]$ are provided in Table IV

TABLE IV
Example 1: Results obtain via LMI method

Methods	$G_{r3}(s)$	$\|E(jw)\|_\infty$ $\omega \in [0.1, 2]$
LMI Method of this paper $(p = 1, q = -1)$	$\frac{1.028s^2+1.906s-3.207}{s^2+1.317s+13}$	0.3874

To see the effectiveness and improvement intuitively, the singular value analysis of the error systems resulted by BT, FWBT, and LMI method of this paper are included here for comparison. See Fig 1.

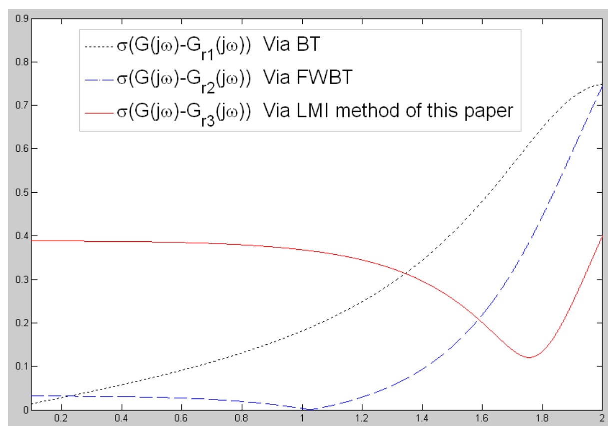

Fig. 1. Singular value analysis of the error systems in Example 1.

From Fig 1. it is obviously to see that the LMI method gives a better result.

Example 2.: (See details in section II)

The reduced-order model and $\|E(jw)\|_\infty, |\omega| \leq 2$ obtain via LMI method of this paper are presented in Table V

TABLE V
Example 2: Results obtain via LMI method

Methods	(A_r, B_r, C_r)	$\|E(jw)\|_\infty$ $\|\omega\| \leq 2$
LMI Method of this paper $(p = 1, q = -1)$	$A_{r2} = \left[\begin{array}{cc} -1.2598 & -2.6063 \\ -1.0435 & -3.2756 \end{array}\right]$ $B_{r2} = \left[\begin{array}{cc} -0.9093 & 0.1476 \end{array}\right]^T$ $C_{r2} = \left[\begin{array}{cc} -0.8036 & -1.4141 \end{array}\right]$	0.0998

Fig 2 gives a intuitive exhibition of singular values of the error systems resulted via different methods.

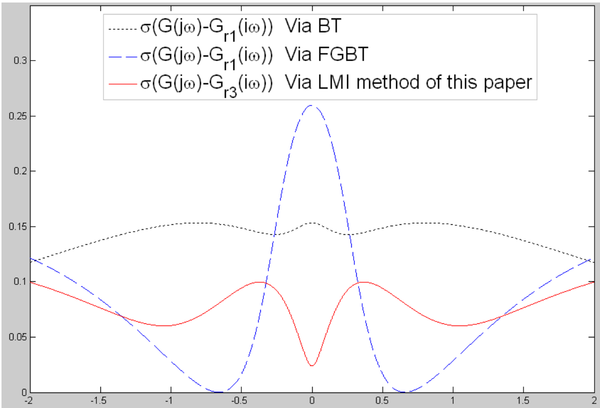

Fig. 2. Singular value analysis of the error systems in Example 2.

V. Conclusion

In this paper, we have investigated the problem of finite frequency H_∞ model reduction for linear continuous-time systems. The generalized Kalman-Yakubovich-Popov (GKYP) lemma is exploited and an appropriate block structure of the key slack variable is proposed to formulate it into a set of LMIs which can be computed by the LMI Toolbox. Two examples are analyzed to show the advantage of the LMI-based finite frequency H_∞ model reduction approach proposed in this paper.

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