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H_∞ Robust Control of Active Suspensions: A Practical Point of View

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***Abstract*—Control of vehicle suspension systems has been the focus of extensive work in the past two decades. Many control strategies have been developed to improve the overall vehicle performance, including both ride quality and stability conditions. However, the concerns regarding the wheel motions affecting significantly on the handling and steering are ignored by researchers in the control formulation. A H_∞ control methodology is employed to design an active suspension system so that ride quality and wheel motions are improved simultaneously. In addition, a three-dimensional kinematic model of a specific suspension system, namely Macpherson suspension, is developed to study the alteration of those of kinematic suspension parameters which represent the wheel motions. The results show that the proposed robust design provides superior kinematic and dynamic performances compared to those of the passive system.**

I. INTRODUCTION

The major roles of a suspension system are to adequately support the vehicle weight, to provide effective ride quality, to maintain the wheels in the appropriate position and to keep tire contact with the ground. In order to examine the extent to which these demands are met, three main categories of suspension systems, namely passive, active and semi-active systems are extensively studied in the literature. However, it is known that the active suspensions offer good road roughness isolation, stability, and handling performance in a wide range of frequencies compared to other type of systems.

For active control of a vehicle suspension system various approaches have been proposed in the literature, including, linear optimal control [1], fuzzy logic control [2], adaptive control [3], gain scheduling control [4], H_∞ robust control [5], and nonlinear control [6]. Nevertheless, robust control has received the most attention among researchers in the context of disturbance attenuation and robustness of the vehicle [1]. The main attempt in the robust control is on minimization of the energy exerted from the road to the vehicle. The main approaches have been defined based on either L_2 (H_2) or L_∞ (H_∞) norms of the transfer function between road disturbances and suspension responses. While in the former method, the transfer function is minimized in the whole range of frequencies, in the latter it is optimized in the worst case of disturbance.

Typically, the acceleration of the sprung mass, suspension deflection, i.e. the relative displacement between the car body and wheel assembly, and tire deflection are considered as measures of the ride quality, rattle space constraint and road-holding ability [1]. As is common in the formulation of H_∞ robust control for a vehicle suspension, all the requirements are weighted and formulated in a single objective function which is minimized to find an optimal control gain [7], [8]. However, in another formulation, the attempt is on minimization of the transfer characteristics from road irregularities to the chassis acceleration while holding the other requirements within their reasonable bounds [9].

Although the abovementioned control strategies can improve the overall performance of active suspensions in terms of ride quality and stability, one of the disadvantages in them is that the trend of the wheel motions has not been considered. The appropriate motions of the wheel are significantly important in the stability, handling, steering, tire life, and even fuel consumption. In the context of the suspension design, based on the kinematics of the suspension, some parameters such as toe angle, camber angle and track width have been defined to describe the different motions of a wheel. The variations of these parameters are really important in the overall performance of a vehicle especially in tire life and stability. For instance, while camber angle variation reduces tire life and increases tire temperature, it also may deteriorate stability of the vehicle by generating lateral forces acting on the tire. Lower toe angle alterations are important for reduced tire wear, less rolling resistance, and better directional stability. Track width alterations produce lateral forces affecting tire life, steering and even stability of the vehicle [10].

In the current study, the vertical displacement of the vehicle body is considered in the control formulation in order to improve the ride quality and wheel motions simultaneously. In addition, a three-dimensional kinematic model of a specific suspension system, namely Macpherson suspension system, is developed to investigate the alterations of the aforementioned parameters subject to active force variation. Further, it is shown how the integration of the vertical vehicle body displacement influences the overall suspension performance, the point which was ignored in the previous studies.

The rest of the paper is organized as follows. After introducing the dynamics of a quarter-car model in Section II, the H_∞ controller is formulated based on the dynamic model. A three-dimensional kinematic model of the Macpherson strut wheel suspension system is proposed for wheel motion investigation in Section III. While Section IV includes the simulation results and related discussion,

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Section V summarizes the results given in this paper.

II. QUARTER-CAR MODEL AND H_∞ CONTROL FORMULATION

A. Dynamic model

A quarter-car model shown in Figure 1 is composed of two lumped masses, connected via a spring and a damper, and the tire stiffness. This model considers the vertical motion of the sprung mass (vehicle body), z_s, and the vertical motion of the unsprung mass (wheel), z_u. Road disturbances are shown by z_r. It should be noted that all the coefficients are assumed to be linear. The equations of motion are given as:

$$\begin{aligned}\sum f_{m_s} &= -k_s(z_s - z_u) - c_p(\dot{z}_s - \dot{z}_u) + f_a = m_s \ddot{z}_s \\ \sum f_{m_u} &= k_s(z_s - z_u) + c_p(\dot{z}_s - \dot{z}_u) - k_t(z_u - z_r) - f_a = m_u \ddot{z}_u\end{aligned} \tag{1}$$

where f_a represents actuator force. k_s and k_t are suspension and tire stiffness coefficients, respectively. In addition, c_p stands for damping coefficient.

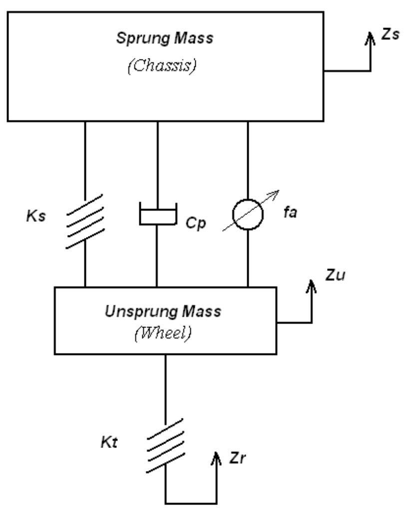

Figure 1. Two DOF active quarter-car suspension model

The state vector is defined in the state space as $x = [z_s, \dot{z}_s, z_u, \dot{z}_u]$. The state equations are expressed in matrix form as:

$$\dot{x} = Ax + B_1 f_a + B_2 z_r \tag{2}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_s}{m_s} & -\frac{c_p}{m_s} & \frac{k_s}{m_s} & \frac{c_p}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{c_p}{m_u} & -\frac{k_t + k_s}{m_u} & -\frac{c_p}{m_u} \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1/m_s \\ 0 \\ -1/m_u \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_t/m_u \end{bmatrix}$$

B. H_∞ control formulation

The major performance criteria of a suspension system are ride comfort, road-holding ability and suspension deflection. The RMS value of the body acceleration response is widely considered in order to quantify ride quality. Therefore, it is essential to keep the transfer characteristics from road irregularities to the body acceleration small. The measure of the road-holding ability of a vehicle is the dynamic contact force variation between the tire and ground which depends on the tire deflection (z_u-z_r). Accordingly, a vehicle has good stability if a strong contact force between the road and tire is maintained. The excessive suspension deflection may result in structure damage and deterioration of ride comfort. Hence, this limit should be considered in the control design as well. However, in previous suspension control designs while the focus was on the improvement of ride quality and road-holding ability, the concern pertaining to the wheel performance was not considered in the control design process. As mentioned before, the appropriate wheel performance results in a superior handling performance, better stability, and less steering. In the current study, it is recommended that in order to improve the wheel motion, the vertical displacement of the vehicle body should be integrated in the control formulation as well.

Based on the abovementioned situations, a H_∞ control problem is formulated in the following to manage different requirements of the suspension system. The following form of equations expresses dynamics of the system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 f_a + B_2 z_r \\ z_1(t) &= C_1 x(t) + D_1 f_a \\ z_2(t) &= C_2 x(t) + D_2 f_a\end{aligned} \tag{3}$$

subject to input constraint

$$|f_a(t)| \le f_{a,\max}$$

where $x(t)$, z_r, A, B_1 and B_2 are defined as (2) and $f_{a,\max}$ stands for actuator saturation. Moreover, z_1 and z_2 are the vectors of H_∞ performance controlled outputs and the constrained outputs, respectively. Despite previous controlled output formulations [7], [8] and [9], the vertical body displacement is integrated in the controlled vector as the following in the current control design

$$z_1(t) = \begin{bmatrix} \ddot{z}_s \\ z_s \end{bmatrix}, \quad C_1 = \begin{bmatrix} -\frac{k_s}{m_s} & -\frac{c_p}{m_s} & \frac{k_s}{m_s} & \frac{c_p}{m_s} \\ \lambda & 0 & 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} \frac{1}{m_s} \\ 0 \end{bmatrix} \tag{4}$$

Here, $\lambda>0$ is a scalar weighting considered to normalize the controlled vector and to handle the trade-off between the control objectives. In addition, the suspension stroke limitation is considered in the following form

$$|z_s - z_u| \le SS, \qquad \forall t \ge 0 \tag{5}$$

and is considered in the vector of constrained output as

$$z_2(t) = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}, \quad C_2 = 0, \quad D_2 = 0 \tag{6}$$

Since road-holding is important during sharp turns and this study deals only with straight (forward) motion, this constraint is relaxed in the formulation.

Considering $f_a=Kx$, the gain matrix shall be designed so that the resulting closed loop system is asymptotically stable and the H_∞-norm from the road disturbance to the performance output, $z(t)$, is minimized. For the existence of such a control gain and for a given $\gamma>0$ (the upper bound of the transfer matrix H_∞ norm), the necessary and sufficient conditions are equivalent to the existence of matrices $Q^T=Q$ and Y satisfying the following LMI:

$$\begin{bmatrix} AQ+QA^T+B_1Y+Y^TB_1^T & B_2 & Q^TC_1^T+Y^TD_1^T \\ B_2^T & -\gamma I & 0 \\ C_1Q+D_1Y & 0 & -\gamma I \end{bmatrix} \le 0 \tag{7}$$

Accordingly, the feedback gain is equal to $K=YQ^{-1}$. In order to handle the force saturation and constrained output, the matrices Q and Y in Eq. (7) should satisfy the following LMIs as well,

$$\begin{bmatrix} \frac{1}{\alpha}X & Y \\ Y^T & Q \end{bmatrix} \ge 0, \qquad X \le f_{a,\max}^2 \tag{8}$$

$$\begin{bmatrix} \frac{1}{\alpha}Z & C_2Q+D_2Y \\ (C_2Q+D_2Y)^T & Q \end{bmatrix}, \qquad Z \le z_{\max}^2 \tag{9}$$

for some scalars X and Z. In this formulation, α/γ represent the upper bound of the energy exerted to the system. More details regarding the control scheme can be found in reference [9] and references therein. Therefore, the matrices Q and Y can be obtained from the solution of the following minimization problem

$$\min_{\gamma, Q=Q^T, Y} \gamma$$
$$\text{subject to LMIs 7, 8 and 9} \tag{10}$$

Subsequently, a state feedback control law with $K=YQ^{-1}$ can be obtained by solving the abovementioned convex optimization problem.

III. Kinematic Model of the Macpherson Suspension System

The Macpherson suspension is usually implemented in the vehicle body as a front suspension and categorized as an independent suspension. The main advantages of a Macpherson suspension are its simple structure, compact size, and low weight. However, it has some disadvantages such as less favorable kinematic performance, higher need of steering, higher tire wear and less isolation of the vehicle body from road roughness compared to other types of independent suspensions [10].

In order to investigate the wheel performance of this type of suspension system subject to the control force variation, a three-dimensional kinematic model, including two-degree-of-freedom (DOF) system, is developed in the following.

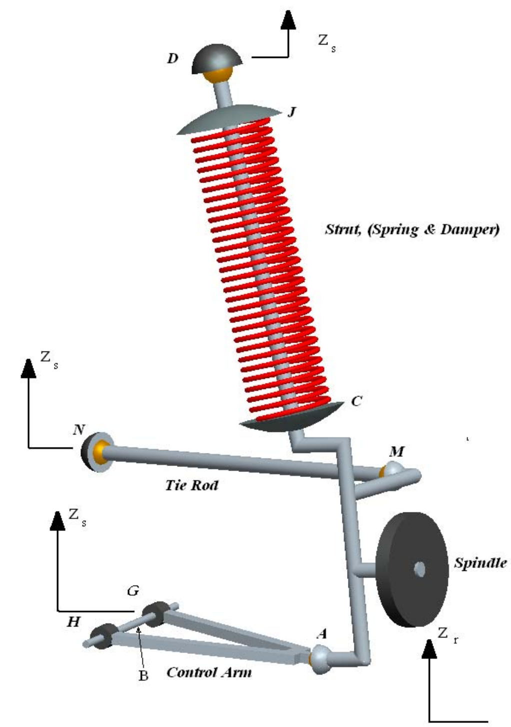

Figure 2. A three dimensional kinematic model of the Macpherson suspension system

A typical model of the Macpherson suspension system is shown in Figure 2. Generally, a Macpherson suspension connects the chassis to the wheel through three links, namely the control arm, the tie rod and the strut. While the tie rod and control arm are rigid links, the length of the strut varies due to the relative motions between two ends of the damper and spring. The tie rod connects steering gear to the front wheel whilst the control arm and strut connect the wheel to the chassis. The assumptions in developing the kinematic model are; 1) all bodies are rigid; 2) control arm is modeled by a rod 3) the chassis has only the vertical displacement and the motion of the body in other directions is zero. The inputs of the system are the road disturbance, z_r, and the vertical displacement of the sprung mass, z_s. It should be pointed out that what connects the dynamic model represented in the previous section to the kinematic model is the vertical vehicle body displacement, obtained by solving Eq. (3).

The kinematic equations and constraints, governing on the system, are summarized in the following system of equations

$$\begin{cases}
1: x_M = a_{11}x_{M_1} + a_{12}y_{M_1} + a_{13}z_{M_1} + a_{14} \\
2: x_C = a_{11}x_{C_1} + a_{12}y_{C_1} + a_{13}z_{C_1} + a_{14} \\
3: x_J = a_{11}x_{J_1} + a_{12}y_{J_1} + a_{13}z_{J_1} + a_{14} \\
4: y_M = a_{21}x_{M_1} + a_{22}y_{M_1} + a_{23}z_{M_1} + a_{24} \\
5: y_C = a_{21}x_{C_1} + a_{22}y_{C_1} + a_{23}z_{C_1} + a_{24} \\
6: y_J = a_{21}x_{J_1} + a_{22}y_{J_1} + a_{23}z_{J_1} + a_{24} \\
7: z_M = a_{31}x_{M_1} + a_{32}y_{M_1} + a_{33}z_{M_1} + a_{34} \\
8: z_C = a_{31}x_{C_1} + a_{32}y_{C_1} + a_{33}z_{C_1} + a_{34} \\
9: z_J = a_{31}x_{J_1} + a_{32}y_{J_1} + a_{33}z_{J_1} + a_{34} \\
10: u_z(x_C - x_D) - u_x(z_C - z_D) = 0 \\
11: u_z(y_C - y_D) - u_y(z_C - z_D) = 0 \\
12: u_z(x_J - x_D) - u_x(z_J - z_D) = 0 \\
13: u_z(y_J - y_D) - u_y(z_J - z_D) = 0 \\
14: u_x^2 + u_y^2 + u_z^2 = 1 \\
15: (x_M - x_N)^2 + (y_M - y_N)^2 + (z_M - z_N)^2 = \\
(x_{M_1} - x_{N_1})^2 + (y_{M_1} - y_{N_1})^2 + (z_{M_1} - z_{N_1})^2 \\
16: (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 = \\
(x_{A_1} - x_{B_1})^2 + (y_{A_1} - y_{B_1})^2 + (z_{A_1} - z_{B_1})^2 \\
17: u_{x_0}(x_A - x_B) + u_{y_0}(y_A - y_B) + u_{z_0}(z_A - z_B) = 0
\end{cases}$$

where a_{ij}'s are the components of a rotation matrix around fixed coordinates RYP (Roll-Yaw-Pitch) and can be found in the robotic or dynamic text books such as in reference [11]. The items 10-14 show the kinematic constraint arising from prismatic motion of the strut where items (u_x, u_y, u_z) are the cosine directions of the strut (Line *CD*). In addition, items 15 and 16 show the constant length of the tie rod and control arm, respectively. The item 17 represents rotational motion of the control arm around the fixed axis GH.

Three parameters, including toe angle, camber angle, and track width, defined in the context of suspension design, describes different motions of a wheel. Toe and camber angles show the roll and yaw rotational motions of the wheel around fixed coordinate axes while the track width alteration indicates lateral motion of the wheel. The variations of these parameters significantly affect handling, steering and stability of a vehicle [10]. Using the three-dimensional kinematic model of the Macpherson suspension system developed in the current study, alterations of these parameters are investigated subject to controlled force variations. More details regarding the mathematical definitions of the abovementioned parameters and validation of the model can be found in reference [12].

IV. SIMULATION RESULTS

For the simulation purposes, the following data are considered

$$\begin{array}{ll}
m_s = 320 \ (Kg) & m_u = 40 \ (Kg) \\
K_s = 180 \ (KN/m) & K_t = 200 \ (KN/m) \\
C_p = 1000 \ (N.\text{sec}/m) & f_{a,\ \max} = 2000 \ (N) \\
SS = 0.08 \ \ (m) & \alpha = 0.03
\end{array}$$

The positions of the key points on the Macpherson suspension are considered as below and taken from ADAMS software default (the origin of the coordinate system at the equilibrium position assumed to be at point *B* and all dimensions are in *mm*):

$$\begin{array}{ll}
A_1 = (206.5,\ 249.05,\ -60.77) & C_1 = (222,\ 152.6,\ 236.25) \\
J_1 = (229.2,\ 134.5,\ 374.75) & P_1 = (211.1,\ 292.15,\ 27.5) \\
D_1 = (240,\ 107.35,\ 582.5) & G_1 = (-217.3,\ 76.05,\ 27.5) \\
H_1 = (217.3,\ -76.05,\ -27.5) & M_1 = (\ 332.9,\ 212.35, -31.6) \\
N_1 = (317.3,\ -94.05,\ -0.5) &
\end{array}$$

Based on the data given above and by setting λ equal to 2500, the control gain matrix obtained from Eq. (10) is equivalent to

$$K = 10^4 \times [\ -1.5052 \quad -0.0798 \quad 0.1478 \quad -0.0013]$$

In addition, the road profile is considered as $z_r(t) = 25.4\sin(2\pi t) + d(t)$ (mm) where $d(t) = 5\sin(10.5\pi t) + \sin(21.5\pi t)$ (mm) representing the high frequency disturbances. In the simulations, two cases of the robust control are considered. While in the first one, so-called R1 in this paper, both the vertical sprung mass displacement and acceleration are considered in the formulation of the controlled output vector, in the second one the objective vector just includes the vertical acceleration (R2). The results are illustrated in Figures 3-8. The RMS values of both the dynamic and kinematic responses are calculated and summarized in Tables 1 and 2. Referring to those results, one can see that both controllers improve the dynamic responses of the suspension system compared to passive one and keep the suspension deflection within the assumed bounds. However, R1 has a slightly better performance in comparison with R2 in terms of ride quality. Regarding the kinematic performance, R1 improves the toe angle and track width variations significantly compared to R2 and passive cases at the expense of violation of camber angle alteration.

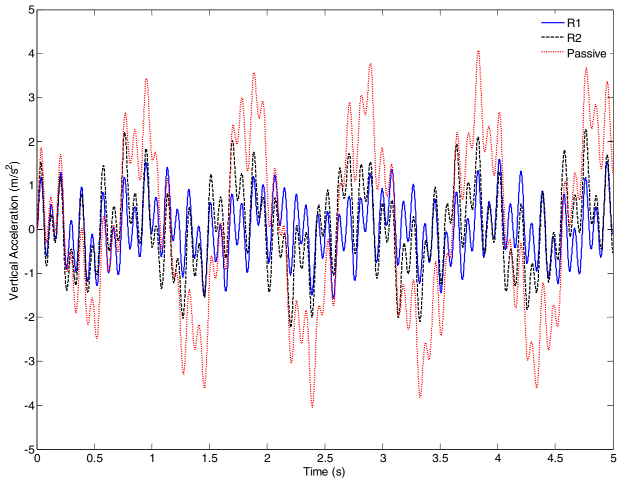

Figure 3. Vertical chassis Acceleration

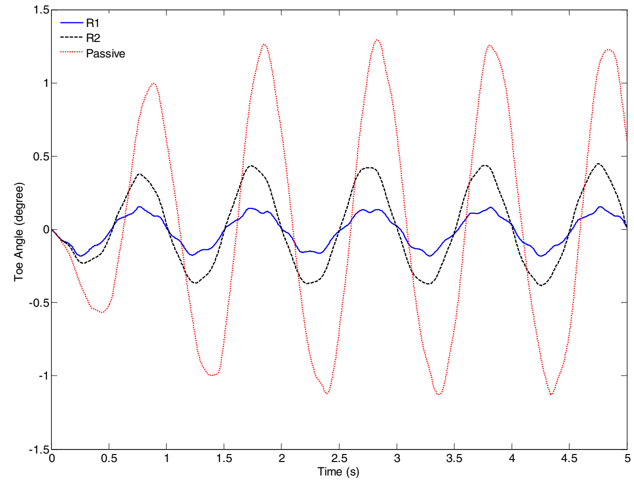

Figure 6. Toe Angle Alteration

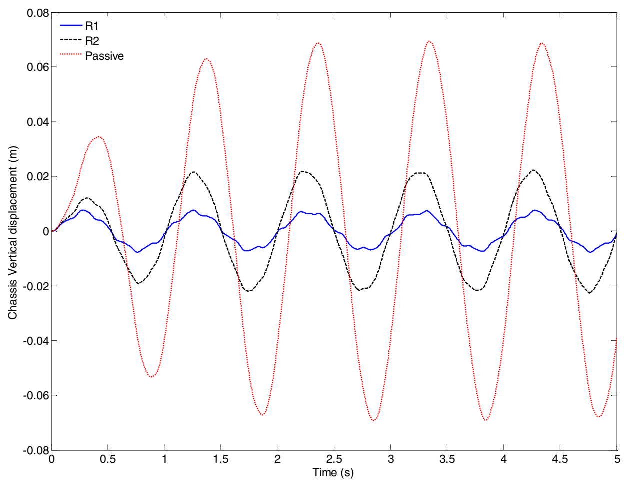

Figure 4. Vertical Chassis Displacement

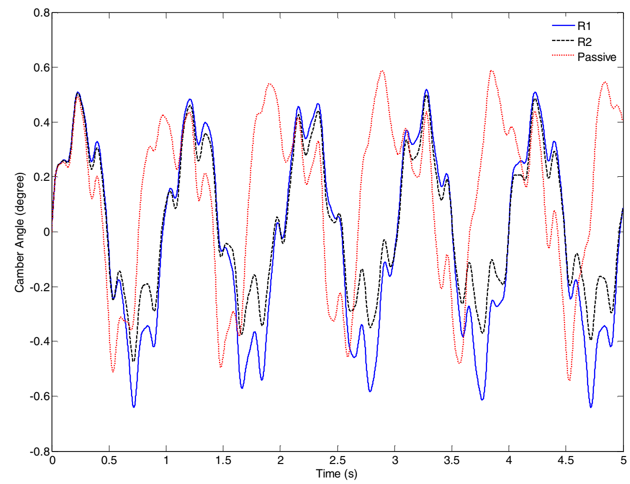

Figure 7. Camber Angle Alteration

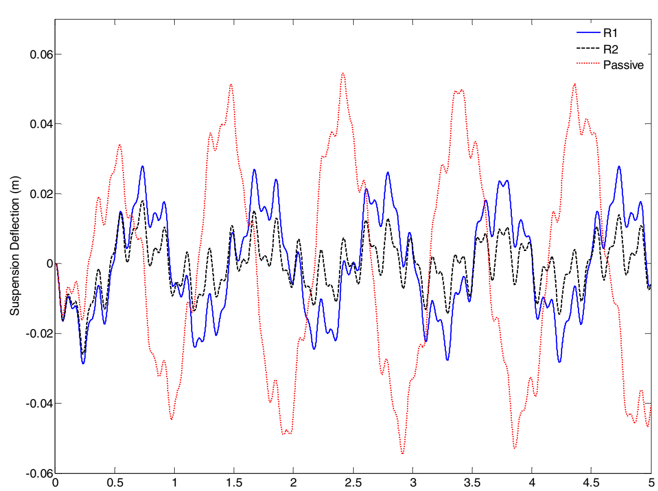

Figure 5. Suspension Deflection

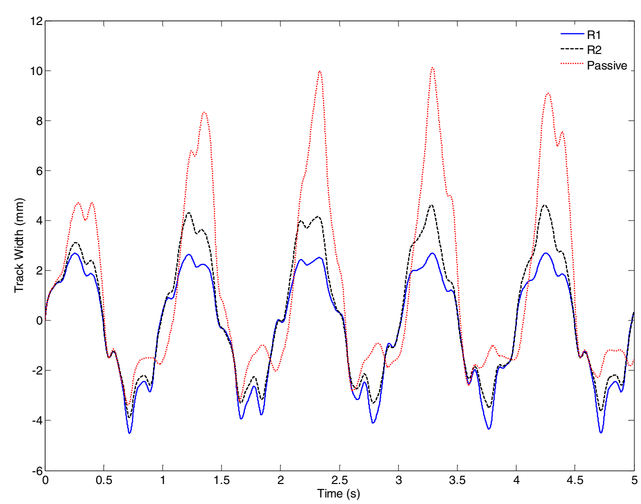

Figure 8. Track Width Alteration

Table 1. RMS values of the dynamic responses

	Acceleration (m/s2)	Displacement (mm)	Suspension Deflection (mm)
R1	0.8255	3.7216	15.6019
R2	0.9249	14.5137	7.4695
Passive	1.9393	44.9255	30.5309

Table 2. RMS values of the kinematic responses

	Track width (mm)	Camber angle (degree)	Toe angle (degree)
R1	2.1596	0.3477	0.0880
R2	2.4877	0.2547	0.2711
Passive	3.9167	0.3350	0.7775

Since both track width and toe angle changes play a major role in directional stability, it is obvious that R1 has a better effect on the directional stability of the vehicle compared to R2. These results indicate how different control strategies can enhance stability of the vehicle differently while their contribution in improving the ride quality and road-holding ability are close to each other.

The trend of force efforts generated by R1 and R2 are plotted in Figure 9. In both cases, the force has been limited between the assumed bounds well.

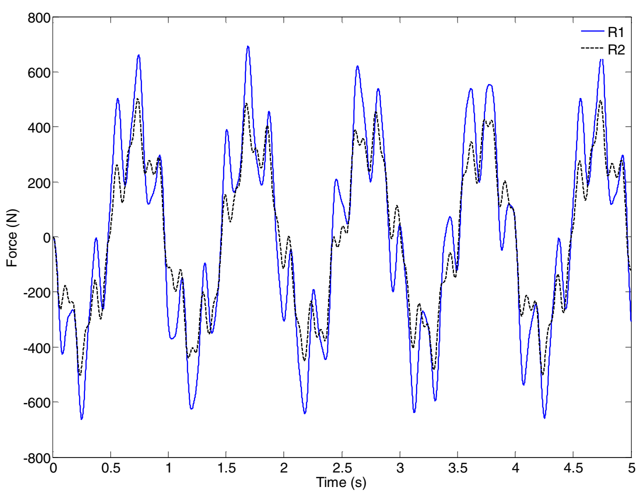

Figure 9. Active force variation

V. Conclusion

A three-dimensional kinematic model of a specific suspension system, namely Macpherson suspension, is developed in order to investigate the influence of the control force variation on the wheel motions. In order to improve the suspension system performance, H_∞ robust control strategy is employed. It is shown that the formulation of the control strategy can significantly affect the performance of a wheel, the point that was ignored in the previous studies. In order to study wheel performance, toe angle, camber angle and track width are explained and investigated. Using these parameters, more details regarding suspension performance as well as stability and handling of a vehicle are studied.

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