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LQR-based Optimal Linear Consensus Algorithms

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Abstract— **Laplacian matrices play an important role in linear consensus algorithms. This paper studies linear-quadratic regulator (LQR) based optimal linear consensus algorithms for multi-vehicle systems with single-integrator kinematics in a continuous-time setting. We propose two global cost functions, namely, interaction-free and interaction-related cost functions. With the interaction-free cost function, we derive the optimal (nonsymmetric) Laplacian matrix. It is shown that the optimal (nonsymmetric) Laplacian matrix corresponds to a complete directed graph. In addition, we show that any symmetric Laplacian matrix is inverse optimal with respect to a properly chosen cost function. With the interaction-related cost function, we derive the optimal scaling factor for a pre-specified symmetric Laplacian matrix associated with the interaction graph. Illustrative examples are given as a proof of concept.**

I. Introduction

Recently, distributed cooperative control for multiple autonomous vehicles, including unmanned aerial vehicles (UAVs), unmanned ground vehicles (UGVs), and unmanned underwater vehicles (UUVs), has become a very active research topic. Great benefits, including high adaptability, easy maintenance, and low complexity, can be achieved by having a group of vehicles work cooperatively with local interaction.

An important approach in distributed multi-vehicle cooperative control is consensus which means the agreement of all vehicles on some common features by negotiating with their local neighbors. Examples of the features include positions, phases, velocities, and attitudes. Consensus has been studied extensively in the recent literature. For vehicles with single-integrator kinematics, [1]–[5], to name a few, studied linear consensus algorithms in different settings. Consensus for vehicles with double-integrator dynamics was studied in [6]–[12], to name a few. For detailed discussions about linear consensus algorithms, refer to [13] and [14].

Optimality issues in consensus algorithms have also been studied in the literature. In [15], the authors proposed a (locally) optimal nonlinear consensus strategy by imposing individual objectives. In [16], the authors proposed the optimal interaction graph, a de Bruijn's graph, in the average consensus problem. In [17], the authors designed a semi-decentralized optimal control strategy by minimizing the individual cost. In addition, cooperative game theory was employed to ensure cooperation with a team cost function. In [18], the authors proposed an iterative algorithm to maximize the second smallest eigenvalue of a Laplacian matrix to optimize the control system performance. In [3], the authors studied the fastest converging linear iteration by using semi-definite programming.

Among various studies of linear consensus algorithms, a noticeable phenomenon is that the algorithms with different parameters (i.e., different Laplacian matrices) can be applied to the same system to ensure consensus. It is natural to ask these questions: Is there an optimal linear consensus algorithm with the associated optimal Laplacian matrix (under a given cost function)? How to find the optimal linear consensus algorithm? In contrast to [3], [15]–[18], the purpose of the current paper is to study the optimal linear consensus algorithms for vehicles with single-integrator kinematics in a continuous-time setting from an LQR perspective. Instead of studying locally optimal algorithms, this paper focuses on globally optimal algorithms. We first propose two global cost functions, namely, interaction-free and interaction-related cost functions. With the interaction-free cost function, we derive the optimal (nonsymmetric) Laplacian matrix. It is shown that the optimal (nonsymmetric) Laplacian matrix corresponds to a complete directed graph. In addition, we show that any symmetric Laplacian matrix is inverse optimal with respect to a properly chosen cost function. With the interaction-related cost function, we derive the optimal scaling factor for a pre-specified symmetric Laplacian matrix associated with the interaction graph.

The remainder of this paper is organized as follows. In Section II, the graph theory notions and definitions used in this paper are introduced. Then the interaction-free and interaction-related cost functions are presented in Section III. Section IV is the main part of this paper focusing on the LQR-based optimal linear consensus algorithms in a continuous-time setting. A short conclusion is given in Section V.

II. Background and Definitions

A. Graph Theory Notions

The interaction graph among a team of n vehicles can be modeled by a weighted directed or undirected graph $\mathcal{G}$ consisting of a node set $\mathcal{V} = \{1, \ldots, n\}$, an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$. In this paper, we assume that $\mathcal{G}$ is fixed. Each edge $(i, j) \in \mathcal{E}$ in a directed graph denotes that vehicle j can obtain information from vehicle i, but not necessarily vice versa. In contrast, in an undirected graph, $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$. The weighted adjacency matrix $\mathcal{A}$ is defined as $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. In particular, we assume $a_{ii} = 0$, $i = 1, \cdots, n$. For undirected graphs, it is assumed that $a_{ij} = a_{ji}$.

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A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), \ldots$, where $i_j \in \mathcal{V}$. An undirected path is defined analogously. A directed graph has a directed spanning tree if there exists at least one node having a directed path to all other nodes. A directed graph is strongly connected if there is a directed path from every node to every other node. An undirected graph is connected if there is an undirected path between every pair of distinct nodes. A complete directed graph is a directed graph in which each pair of distinct nodes is connected by an edge and the edge is bidirectional.

Let the (nonsymmetric) Laplacian matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{A}$ be defined such that $\ell_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$ and $\ell_{ij} = -a_{ij},\ i \neq j$. For an undirected graph, $\mathcal{L}$ is symmetric positive semi-definite (PSD). For a directed graph, $\mathcal{L}$ is not necessarily symmetric. In the remainder of the paper, we use (nonsymmetric) Laplacian to emphasize the fact that a certain Laplacian matrix is not necessarily symmetric and use symmetric Laplacian to explicitly emphasize the fact that a certain Laplacian matrix is symmetric. From the definition of $\mathcal{L}$, it can be noted that 0 is an eigenvalue of $\mathcal{L}$ with an associated eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n$ is an all-one $n \times 1$ column vector. For undirected graphs, $\mathcal{L}$ has a simple zero eigenvalue if and only if undirected graph $\mathcal{G}$ is connected [19]. For directed graphs, $\mathcal{L}$ has a simple zero eigenvalue if and only if directed graph $\mathcal{G}$ has a directed spanning tree [20], [5].

B. Definitions

Definition 2.1: We define $Z^{n\times n} := \{B = [b_{ij}] \in \mathbb{R}^{n\times n} | b_{ij} \leq 0, i \neq j\}$, $\mathbf{0}_{m\times n} \in \mathbb{R}^{m\times n}$ as an all-zero matrix, and $I_n \in \mathbb{R}^{n\times n}$ as an identity matrix.

Definition 2.2: A matrix $E \in \mathbb{R}^{m\times n}$ is said positive (nonnegative), i.e., $E > (\geq) 0$, if each entry of E is positive (nonnegative). A square nonnegative matrix is (row) stochastic if all of its row sums are 1.

Definition 2.3: [21] A real matrix $B = [b_{ij}] \in \mathbb{R}^{n\times n}$ is called an M-matrix if it can be written as

$$B = sI_n - C, \quad s > 0, C \geq 0$$

where $C \in \mathbb{R}^{n\times n}$ satisfies $\rho(C) \leq s$, where $\rho(\cdot)$ is the spectral radius of a matrix. The matrix B is called a nonsingular M-matrix if $\rho(C) < s$.

Definition 2.4: [21] A matrix $D \in \mathbb{R}^{n\times n}$ is called semi-convergent if $\lim_{i\to\infty} D^i$ exits.

III. Global Cost Functions

Consider vehicles with single-integrator kinematics given by

$$\dot{x}_i(t) = u_i(t) \tag{1}$$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are, respectively, the state and control input of the ith vehicle. A common linear consensus algorithm is studied in [1], [2], [4], [5] as

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}[x_i(t) - x_j(t)], \tag{2}$$

where a_{ij} is the (i,j)th entry of the weighted adjacency matrix $\mathcal{A}$ associated with $\mathcal{G}$. The objective of (2) is to guarantee consensus, i.e., for any $x_i(0)$, $x_i(t) \to x_j(t)$ as $t \to \infty$. Substituting (2) into (1) and writing the closed-loop system in matrix form gives

$$\dot{X}(t) = -\mathcal{L}X(t), \tag{3}$$

where $X(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T$ and $\mathcal{L}$ is the (nonsymmetric) Laplacian matrix associated with $\mathcal{A}$. It can be noted that (3) is a linear differential equation. Consensus is reached for (3) if and only if $\mathcal{L}$ has a simple zero eigenvalue or equivalently the directed graph associated with $\mathcal{L}$ has a directed spanning tree [5].

Similar to the cost function used in optimal control problems for systems with linear differential equations, we propose the following two consensus cost functions for system (1) as

$$J_f = \int_0^\infty \left\{ \sum_{i=1}^{n}\sum_{j=1}^{i} c_{ij}[x_i(t) - x_j(t)]^2 + \sum_{i=1}^{n} r_i u_i^2(t) \right\} dt, \tag{4}$$

$$J_r = \int_0^\infty \left\{ \sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij}[x_i(t) - x_j(t)]^2 + \sum_{i=1}^{n} u_i^2(t) \right\} dt, \tag{5}$$

where $c_{ij} \geq 0$, $r_i > 0$, and a_{ij} is defined in (2). In (4), both c_{ij} and r_i can be chosen freely. Therefore J_f is called the *interaction-free cost function*. Because (5) depends on weighted adjacency matrix $\mathcal{A}$, J_r is called the *interaction-related cost function*. The motivation of (4) and (5) is to weigh both the consensus errors and the control effort. The optimization problems can be written as

$$\min_{a_{ij}}\ J_f,\ \text{subject to (1) and (2)} \tag{6}$$

$$\min_{\beta}\ J_r,\ \text{subject to (1) and}$$

$$u_i(t) = -\sum_{j=1}^{n} \beta a_{ij}[x_i(t) - x_j(t)]. \tag{7}$$

IV. LQR-based Optimal Linear Consensus Algorithms in a Continuous-time Setting

In this section, we derive the optimal linear consensus algorithms in a continuous-time setting from an LQR perspective. We first derive the optimal (nonsymmetric) Laplacian matrix using continuous-time interaction-free cost function (4). We will then find the optimal scaling factor for a pre-specified symmetric Laplacian matrix using continuous-time interaction-related cost function (5). Finally, illustrative examples will be provided.

A. Optimal Laplacian Matrix Using Interaction-free Cost Function

With interaction-free cost function (4), optimal control problem (6) can be written as

$$\min_{\mathcal{L}} J_f = \int_0^\infty [X^T(t)QX(t) + U^T(t)RU(t)]dt \tag{8}$$

subject to: $\dot{X}(t) = U(t),\ U(t) = -\mathcal{L}X(t)$, where $X(t)$ is defined in (3), $U(t) = [u_1(t), \cdots, u_n(t)]^T$, $Q \in \mathbb{R}^{n\times n}$ is symmetric with the (i, j)th entry given as $-c_{ij}$ for $i \neq j$ and the (i, i)th entry given as $\sum_{j=1, j\neq i}^{n} c_{ij}$, $R \in \mathbb{R}^{n\times n}$ is a positive definite (PD) diagonal matrix with r_i being the ith diagonal entry, and $\mathcal{L}$ is the (nonsymmetric) Laplacian matrix defined in (3). It can be noted that Q is a symmetric PSD Laplacian matrix.

We need the following lemmas to derive our main theorem.

Lemma 4.1: [21] An M-matrix $B \in \mathbb{R}^{n\times n}$ has exactly one M-matrix as its square root if the characteristic polynomial of B has at most a simple zero root.

If the characteristic polynomial of M-matrix B has at most a simple zero root, we use $\sqrt{B}$ hereafter to represent the M-matrix that is the square root of B.

Lemma 4.2: An M-matrix that has a zero eigenvalue with a corresponding eigenvector $\mathbf{1}_n$ is a (nonsymmetric) Laplacian matrix.

Proof: Follow Definition 2.3 and definition of a (nonsymmetric) Laplacian matrix. ■

Lemma 4.3: Let Q and R be defined in (8). Suppose that Q has a simple zero eigenvalue. There exists exactly one (nonsymmetric) Laplacian matrix W satisfying $W = \sqrt{R^{-1}Q}$ and W has a simple zero eigenvalue.

Proof: The proof is divided into the following three steps:

Step 1: $R^{-1}Q$ is a (nonsymmetric) Laplacian matrix with a simple zero eigenvalue. We first note that $R^{-1}Q$ is a (nonsymmetric) Laplacian matrix because Q is a symmetric Laplacian matrix and R is a PD diagonal matrix. Because Q is a symmetric Laplacian matrix with a simple zero eigenvalue, it follows that the undirected graph associated with Q is connected, which implies that the directed graph associated with $R^{-1}Q$ is strongly connected. It thus follows that (nonsymmetric) Laplacian matrix $R^{-1}Q$ also has a simple zero eigenvalue.

Step 2: W has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}_n$. Let the ith eigenvalue of W be λ_i with an associated eigenvector ν_i. Noting that $W^2 = R^{-1}Q$, it follows that the ith eigenvalue of $R^{-1}Q$ is λ_i^2 with an associated eigenvector ν_i. Because $R^{-1}Q$ has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}_n$, it follows that W has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}_n$.

Step 3: W is a (nonsymmetric) Laplacian matrix. Note that a (nonsymmetric) Laplacian matrix is a special case of an M-matrix according to Definition 2.3. It follows from Lemma 4.1 and Step 1 that $R^{-1}Q$ has exactly one square root W that is also an M-matrix. Because W has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}_n$ as shown in Step 2, it follows from Lemma 4.2 that W is a (nonsymmetric) Laplacian matrix. ■

We next show that the (nonsymmetric) Laplacian matrix W in Lemma 4.3 corresponds to a complete directed graph.

Lemma 4.4: Let Q and R be defined in (8). Suppose that Q has a simple zero eigenvalue. Then (nonsymmetric) Laplacian matrix $\sqrt{R^{-1}Q}$ corresponds to a complete directed graph.

Proof: We show that each entry of $\sqrt{R^{-1}Q}$ is nonzero, which implies that $\sqrt{R^{-1}Q}$ corresponds to a complete directed graph. Before moving on, we let q_{ij} denote the (i, j)th entry of Q. We also let $W = \sqrt{R^{-1}Q}$ and denote w_{ij}, $w_{i,:}$, and $w_{:,i}$, respectively, as the (i, j)th entry, the ith row, and the ith column of W.

First, we will show that $w_{ij} \neq 0$ if $q_{ij} \neq 0$. We show this statement by contradiction. Assume that $w_{ij} = 0$. Because $R^{-1}Q = W^2$, it follows that $\frac{q_{ij}}{r_i} = w_{i,:}w_{:,j}$. When $i = j$, it follows from $w_{ii} = 0$ that $w_{i,:} = \mathbf{0}_{n\times 1}$ because W is a (nonsymmetric) Laplacian matrix, which then implies that $\frac{q_{ij}}{r_i} = w_{i,:}w_{:,j} = 0$. This contradicts the assumption that $q_{ij} \neq 0$. When $i \neq j$, because we assume that $w_{ij} = 0$, it follows that $\frac{q_{ij}}{r_i} = w_{i,:}w_{:,j} = \sum_{k=1,k\neq i,k\neq j}^{n} w_{ik}w_{kj} \geq 0$ due to the fact $w_{i,k} \leq 0, \forall i \neq k$, because W is a (nonsymmetric) Laplacian matrix. Because Q is a symmetric Laplacian matrix, it follows that $q_{ij} \leq 0, \forall i \neq j$. Therefore, $\frac{q_{ij}}{r_i} \geq 0, \forall i \neq j$, implies $q_{ij} = 0$, which also contradicts the assumption that $q_{ij} \neq 0$.

Second, we will show that $w_{ij} \neq 0$ if $q_{ij} = 0$. We also show this statement by contradiction. Assume that $w_{ij} = 0$. To ensure that $q_{ij} = 0$, it follows from $\frac{q_{ij}}{r_i} = w_{i,:}w_{:,j} = \sum_{k=1,k\neq i,k\neq j}^{n} w_{ik}w_{kj}$ that $w_{ik}w_{kj} = 0, \forall k \neq i, k \neq j, k = 1, \cdots, n$. Denote $\hat{k}_1$ as the node set such that $w_{im} \neq 0$ for each $m \in \hat{k}_1$. Then we have $w_{mj} = 0$ for each $m \in \hat{k}_1$ because $w_{ik}w_{kj} = 0$. Similarly, denote $\bar{k}_1$ as the node set such that $w_{mj} \neq 0$ for each $m \in \bar{k}_1$. Then we have $w_{im} = 0$ for each $m \in \bar{k}_1$ because $w_{ik}w_{kj} = 0$. From the discussion in the previous paragraph, when $w_{mj} = 0$, we have $q_{mj} = 0$, which implies that $w_{mp}w_{pj} = 0, \forall p \neq m, p \neq j, p = 1, \cdots, n$. By following a similar analysis, we can consequently define $\hat{k}_i$ and $\bar{k}_i, i = 2, \cdots, \kappa$, where $\hat{k}_i \cap \hat{k}_j = \emptyset, \bar{k}_i \cap \bar{k}_j = \emptyset, \forall j < i$. Noting that the undirected graph associated with Q is connected and the directed graph associated with W has a directed spanning tree because W has a simple zero eigenvalue, it follows that $\kappa \leq n$. Therefore, each entry of $w_{:,j}$ is equal to zero by following the previous analysis for at most n times. This implies that $q_{ij} = 0, \forall i \neq j$, because $\frac{q_{ij}}{r_i} = w_{i,:}w_{:,j}$. Considering the fact that Q is a symmetric Laplacian matrix, it follows that $q_{ii} = 0$, which also contradicts the fact that the undirected graph associated with Q is connected. ■

Theorem 4.1: For optimal control problem (8), where Q has a simple zero eigenvalue, the optimal consensus algorithm is $U(t) = -\sqrt{R^{-1}Q}X(t)$, that is, the optimal (nonsymmetric) Laplacian matrix is $\sqrt{R^{-1}Q}$.[1] In addition, $\sqrt{R^{-1}Q}$ corresponds to a complete directed graph.

Proof: Consider the following standard LQR problem

$$\min_{U(t)} J_f \quad \text{subject to:} \quad \dot{X}(t) = AX(t) + BU(t), \quad (9)$$

where J_f is defined by (8), $A = \mathbf{0}_{n\times n}$, and $B = I_n$. It can be noted that (A, B) is controllable, which implies that there exists a P satisfying the continuous-time algebraic Riccati

[1]Obviously, consensus is reached for (1) using $U(t) = -\sqrt{R^{-1}Q}X(t)$ since $\sqrt{R^{-1}Q}$ has a simple zero eigenvalue.

equation (ARE)

$$A^TP + PA - PBR^{-1}B^TP + Q = \mathbf{0}_{n\times n}. \tag{10}$$

It follows from (10) that $PR^{-1}P = Q$, which implies $R^{-1}PR^{-1}P = R^{-1}Q$. It then follows from Lemma 4.3 that $R^{-1}P = \sqrt{R^{-1}Q}$ is also a (nonsymmetric) Laplacian matrix when Q has a simple zero eigenvalue. Therefore, the optimal control is $U(t) = -R^{-1}B^TPX(t) = -\sqrt{R^{-1}Q}X(t)$, which implies that $\sqrt{R^{-1}Q}$ is the optimal (nonsymmetric) Laplacian matrix. It also follows from Lemma 4.4 that $\sqrt{R^{-1}Q}$ corresponds to a complete directed graph. ■

Remark 4.2: Note that $\sqrt{R^{-1}Q}$ is not necessarily symmetric in general. When R is a diagonal matrix with identical diagonal entries (i.e., $R = cI_n$, where $c > 0$), $\sqrt{R^{-1}Q}$ is symmetric.

Remark 4.3: Theorem 4.1 requires that Q be a symmetrical PSD Laplacian matrix with a simple zero eigenvalue. When Q has more than one zero eigenvalue, $X^T(t)QX(t)$ can be written as the sum of at least two parts as

$$X^T(t)QX(t) = X_1^T(t)Q_1X_1(t) + X_2^T(t)Q_2X_2(t) + \cdots,$$

where $X_i \bigcap X_j = \varnothing, \forall i \neq j$. Therefore, the requirement that Q has a simple zero eigenvalue is necessary to ensure consensus.

Theorem 4.4: Any symmetric Laplacian matrix $\mathcal{L} \in \mathbb{R}^{n\times n}$ with a simple zero eigenvalue is the optimal symmetric Laplacian matrix for the cost function

$$J = \int_0^\infty [X^T(t)\mathcal{L}^2X^T + U^T(t)U(t)]dt.$$

Proof: By letting $Q = \mathcal{L}^2$ and $R = I_n$, it follows directly from the proof of Theorem 4.1 that $\mathcal{L}$ is the optimal symmetric Laplacian matrix. ■

B. Optimal Scaling Factor Using Interaction-related Cost Function

With interaction-related cost function (5), optimal control problem (7) can be written as

$$\min_\beta J_r = \int_0^\infty \left[X^T(t)\mathcal{L}X(t) + U^T(t)U(t)\right]dt \tag{11}$$
$$\text{subject to:}\quad \dot{X}(t) = U(t),\ U(t) = -\beta\mathcal{L}X(t),$$

where $\mathcal{L}$ is a pre-specified symmetric Laplacian matrix.

Theorem 4.5: For optimal control problem (11), where the symmetric Laplacian matrix $\mathcal{L}$ has a simple zero eigenvalue, the optimal β is $\sqrt{\frac{X^T(0)X(0)-X^T(0)m_1m_1^TX(0)}{X^T(0)\mathcal{L}X(0)}}$, where $m_1 = \frac{\mathbf{1}_n}{\sqrt{n}}$.

Proof: The cost function J_r can be written as

$$J_r = \int_0^\infty X^T(0)[e^{-\beta\mathcal{L}t}\mathcal{L}e^{-\beta\mathcal{L}t} + \beta^2e^{-\beta\mathcal{L}t}\mathcal{L}^2e^{-\beta\mathcal{L}t}]X(0)dt$$

Taking derivative of J_r with respect to β gives

$$\frac{dJ_r}{d\beta} = \int_0^\infty X^T(0)[-2\mathcal{L}te^{-\beta\mathcal{L}t}\mathcal{L}e^{-\beta\mathcal{L}t} + 2\beta e^{-\beta\mathcal{L}t}\mathcal{L}^2e^{-\beta\mathcal{L}t} - 2\beta^2\mathcal{L}te^{-\beta\mathcal{L}t}\mathcal{L}^2e^{-\beta\mathcal{L}t}]X(0)dt.$$

Setting $\frac{dJ_r}{d\beta} = 0$ gives

$$\begin{aligned}&\beta^2X^T(0)\left[\int_0^\infty \mathcal{L}te^{-\beta\mathcal{L}t}\mathcal{L}^2e^{-\beta\mathcal{L}t}dt\right]X(0)\\ &-\beta X^T(0)\left[\int_0^\infty e^{-\beta\mathcal{L}t}\mathcal{L}^2e^{-\beta\mathcal{L}t}dt\right]X(0)\\ &+X^T(0)\left[\int_0^\infty \mathcal{L}te^{-\beta\mathcal{L}t}\mathcal{L}e^{-\beta\mathcal{L}t}dt\right]X(0) = 0,\end{aligned} \tag{12}$$

where we have used the fact that $\mathcal{L}$ and $e^{-\beta\mathcal{L}t}$ are commutable. Because $\mathcal{L}$ is symmetric, $\mathcal{L}$ can be diagonalized as

$$\mathcal{L} = M\underbrace{\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \cdots & \cdots & & \\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}}_{\Lambda}M^T, \tag{13}$$

where M is an orthogonal matrix and λ_i is the ith eigenvalue of $\mathcal{L}$. It follows that

$$\begin{aligned}&\int_0^\infty \mathcal{L}te^{-\beta\mathcal{L}t}\mathcal{L}^2e^{-\beta\mathcal{L}t}dt\\ =&\int_0^\infty M\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & e^{-2\beta\lambda_2t}\lambda_2^3t & \cdots & 0\\ \cdots & \cdots & & \\ 0 & 0 & \cdots & e^{-2\beta\lambda_nt}\lambda_n^3t\end{bmatrix}M^Tdt\\ =&\frac{1}{4\beta^2}M\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \cdots & \cdots & & \\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}M^T = \frac{1}{4\beta^2}\mathcal{L}.\end{aligned} \tag{14}$$

Similarly, it follows that

$$\begin{aligned}&\int_0^\infty e^{-\beta\mathcal{L}t}\mathcal{L}^2e^{-\beta\mathcal{L}t}dt\\ =&\int_0^\infty M\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & e^{-2\beta\lambda_2t}\lambda_2^2 & \cdots & 0\\ \cdots & \cdots & & \\ 0 & 0 & \cdots & e^{-2\beta\lambda_nt}\lambda_n^2\end{bmatrix}M^Tdt\\ =&\frac{1}{2\beta}\mathcal{L}\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\int_0^\infty \mathcal{L}te^{-\beta\mathcal{L}t}\mathcal{L}e^{-\beta\mathcal{L}t}dt\\ =&\int_0^\infty M\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & e^{-2\beta\lambda_2t}\lambda_2^2t & \cdots & 0\\ \cdots & \cdots & & \\ 0 & 0 & \cdots & e^{-2\beta\lambda_nt}\lambda_n^2t\end{bmatrix}M^Tdt\\ =&\frac{I_n - m_1m_1^T}{4\beta^2},\end{aligned} \tag{16}$$

where $m_1 = \frac{\mathbf{1}_n}{\sqrt{n}}$. By substituting (14), (15) and (16) into (12), it follows that the optimal gain satisfies $\beta = \sqrt{\frac{X^T(0)X(0)-X^T(0)m_1m_1^TX(0)}{X^T(0)\mathcal{L}X(0)}}$. ■

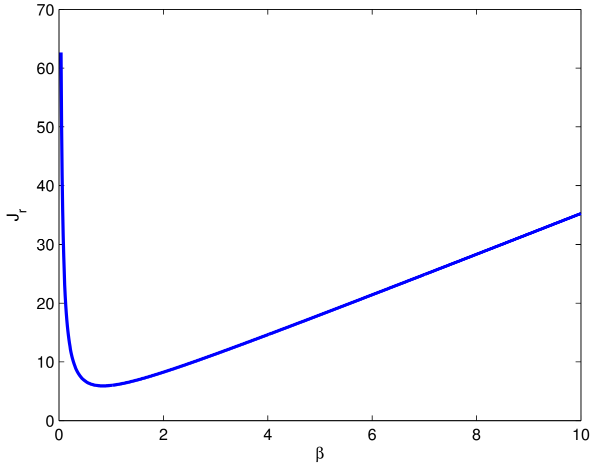

Fig. 1. Evolution of the cost function J_r as a function of β.

C. Illustrative Examples

In this subsection, we provide two illustrative examples about the optimal (nonsymmetric) Laplacian matrix and the optimal scaling factor derived in Subsection IV-A and Subsection IV-B, respectively.

In (8), let

$$Q = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

It follows from Theorem 4.1 that the optimal (nonsymmetric) Laplacian matrix is

$$\begin{bmatrix} 1.3134 & -0.5459 & -0.5964 & -0.1711 \\ -0.2730 & 0.8491 & -0.4206 & -0.1556 \\ -0.1988 & -0.2804 & 0.8218 & -0.3426 \\ -0.0428 & -0.0778 & -0.2570 & 0.3775 \end{bmatrix}.$$

Note that the optimal (nonsymmetric) Laplacian matrix corresponds to a complete directed graph.

In (11), let

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and the initial state $X(0) = [1, 2, 3, 4]^T$. Fig. 1 shows how the cost function J_r evolves as the scaling factor β increases. From Theorem 4.5, it can be computed that the optimal scaling factor is $\beta = 0.845$, which is consistent with the result shown in Fig. 1.

V. Conclusion

In this paper, we studied the LQR-based optimal linear consensus algorithms for multi-vehicle systems with single-integrator kinematics in a continuous-time setting. Two global cost functions, namely, interaction-free and interaction-related cost functions, were proposed. With an interaction-free cost function, the optimal (nonsymmetric) Laplacian matrix was derived. The interaction graph associated with the optimal (nonsymmetric) Laplacian matrix was shown to correspond to a complete directed graph. In addition, any symmetric Laplacian matrix was shown to be inverse optimal with respect to a properly chosen cost function. With an interaction-related cost function, the optimal scaling factor for a pre-specified symmetric Laplacian matrix was studied. Illustrative examples were given to validate the theoretical results.

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