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# Low Order Decentralized Stabilizing Controller Design for a Mobile Inverted Pendulum Robot

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***Abstract*— We propose a low order two-channel decentralized controller design for stabilizing the multi-input multi-output linearized model of a self balancing autonomous robot. The robot model is based on a form of an inverted pendulum and the robot was constructed into a mechanical system in order to implement the stabilizing controller design.**

## I. Introduction

Motivated by the inverted pendulum, we designed and constructed a mobile robot, obtained a mathematical model, and designed a stabilizing controller for the unstable plant model obtained by linearizing around the equilibrium. There are many variations of the inverted pendulum (e.g., [1], [2]). The robot, with two opposed wheels, is modeled from a schematic using Newton's laws of motion (e.g., [3]). The system has six states describing its motion with three degrees of freedom (DOF). The robot has linear motion characterized by position $x$ and velocity $\nu$. It can rotate about the $z$-axis characterized by pitch angle $\theta$ and angular velocity $\omega$, and about the $y$-axis characterized by the yaw angle $\delta$ and yaw velocity $\psi$. Any motion about the $x$-axis is considered negligible. The equations describing the nonlinear system are linearized about an operating point and the multi-input multi-output (MIMO) plant obtained from the linearized model has a $2 \times 2$ transfer matrix from the motor input voltages ($v_L$ and $v_R$) to the states ($x$ and $\theta$).

We designed the stabilizing controller so that it balances and regulates the position and orientation. With $x$ and $\theta$ as measured states used in feedback, the control system is a two-channel linear time-invariant (LTI) decentralized configuration, where each channel's controller actuates a control signal to one motor. We designed a second order integral-action controller for the first channel and a first order stable controller for the second channel. The decentralized PID control synthesis proposed here is a novel design for unstable MIMO systems and we prove that it achieves closed-loop stability.

*Notation*: Let $\mathbb{C}$, $\mathbb{R}_+$ denote complex and positive real numbers; $\mathbb{C}_{+e} = \{s \in \mathbb{C} \,|\, \mathcal{R}e(s) \geq 0\} \cup \{\infty\}$ is the extended closed right-half complex plane; $\mathbf{R_p}$ is real proper rational functions of $s$; $\mathbf{S}$ is the stable subset with no $\mathbb{C}_{+e}$ poles. A square stable matrix is called unimodular if $M^{-1}$ is also stable. The $H_\infty$-norm of $H \in \mathbf{S}$ is $\|H\| := \sup_\omega |H(j\omega)|$. We drop $(s)$ in transfer functions such as $G(s)$ where this causes no confusion. We use coprime factorizations over $\mathbf{S}$.

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## II. System dynamics and model

The dynamics of the three DOF system are described as nonlinear second order differential equations representing the motion of the robot. The nonlinear equations are linearized about an operating point (corresponding to the robot standing straight up at rest). See [5] for details. With $[\dot{x} \;\; \ddot{x} \;\; \dot{\theta} \;\; \ddot{\theta}]^T = [\dot{x} \;\; \dot{\nu} \;\; \dot{\theta} \;\; \dot{\omega}]^T$, $\chi^T := [x \;\; \nu \;\; \theta \;\; \omega]^T$, $\mu^T = [v_L \;\; v_R]^T$, the state-space representation is $\dot{\chi} = A\chi + B\mu$ $= \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & -9.4932 & 0.1894 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & 33.3538 & 9.2355 & 0 \end{bmatrix} \chi + \begin{bmatrix} \uparrow & \uparrow \\ b & b \\ \downarrow & \downarrow \end{bmatrix} \mu$, where $b = \begin{bmatrix} 0 & 0.4893 & 0 & -0.0367 \end{bmatrix}^T$. This model provides a linear representation of our plant close to the operating point and only describes the dynamics of two DOF. More detailed state-space representations including the yaw and yaw rate states can be found in [5]. Using the position and angle states $x$ and $\theta$ as output, we obtain $y = \begin{bmatrix} x \\ \theta \end{bmatrix} = C\chi + D\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \chi + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mu$. The plant is $P = C(sI - A)^{-1}B + D = \begin{bmatrix} P_1 & P_1 \\ P_2 & P_2 \end{bmatrix} = \frac{1}{d_u d_s} \begin{bmatrix} s^{-1}n_1 & s^{-1}n_1 \\ n_2 & n_2 \end{bmatrix} \in \mathbf{R_p}^{2\times 2}$, where $n_1 = (0.4893s^2 - 4.526)$, $n_2 = (-0.0367s + 15.97)$, $p = 3.1203$, $z_1 = 3.0414$, $z_2 = 435.1499$, $d_u = (s - p)$, $d_s = (s + 9.4136)(s + 3.1999)$, $n_2 = -0.0367s + 15.975 = -0.0367(s - z_2)$, $n_1 = 0.4893s^2 - 4.526 = 0.4893(s - z_1)(s + z_1)$. The eigenvalues are $\{0, 3.1203, -9.4136, -3.1999\}$. Due to the eigenvalues at 0 and 3.1203, the plant $P$ is unstable.

## III. Decentralized controller design

We use a novel two-channel LTI decentralized control design with low order controllers in each channel. The feedback configuration we consider, called $Sys(P, C_D)$, is shown in Fig. 1. Write the plant $P \in \mathbf{R_p}^{2\times 2}$ as

$$P = D_p^{-1} N_p = \begin{bmatrix} D_1 & V \\ 0 & D_2 \end{bmatrix}^{-1} \begin{bmatrix} N_1 & N_1 \\ N_2 & N_2 \end{bmatrix}, \qquad (1)$$

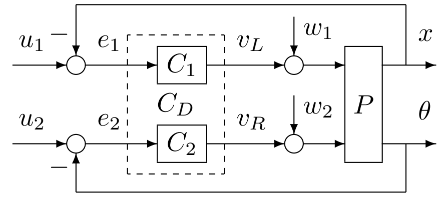


Fig. 1. The two-channel decentralized system $Sys(P, C_D)$

where, for any $a_1, a_2 \in \mathbb{R}_+$, $D_1 = \frac{s}{a_1 s+1}$, $D_2 = \frac{s-p}{a_2 s+1}$, $V = \frac{s-3.1353}{a_1 s+1}$, $N_1 = \frac{0.4526s+17.4973}{d_s(a_1 s+1)}$, $N_2 = \frac{n_2}{d_s(a_2 s+1)}$. Let the input and output vectors in the system $Sys(P, C_D)$ be $u := \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$, $\mu := \begin{bmatrix} v_L & v_R \end{bmatrix}^T$, $e := \begin{bmatrix} e_1 & e_2 \end{bmatrix}^T$, $w := \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$, $y := \begin{bmatrix} x & \theta \end{bmatrix}^T$. Then the closed-loop transfer function $H_{cl} \in \mathbf{R_p}^{4\times 4}$ from $(u, w)$ to $(\mu, y)$ is $H_{cl} = \begin{bmatrix} C_D(I+PC_D)^{-1} & -C_D(I+PC_D)^{-1}P \\ PC_D(I+PC_D)^{-1} & (I+PC_D)^{-1}P \end{bmatrix}$.

*Definition 1:* **a)** The system $Sys(P, C_D)$ is *stable* if the closed-loop transfer-function $H_{cl}$ from $(u, w)$ to $(\mu, y)$ is stable. **b)** The controller $C_D$ is a *stabilizing controller* for $P$ if $C_D$ is proper, and the system $Sys(P, C_D)$ is stable. **c)** The stable system $Sys(P, C_D)$ has *integral-action* if the closed-loop transfer-function $H_{eu}$ from $u$ to $e$ has blocking-zeros at $s = 0$, i.e., $H_{eu}(0) = 0$. □

Let $C_D$ be a two-channel LTI decentralized controller:

$$C_D = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = XY^{-1} = \begin{bmatrix} X_1 Y_1^{-1} & 0 \\ 0 & X_2 Y_2^{-1} \end{bmatrix}. \quad (2)$$

With $P$ as in (1) and $C_D$ as in (2), the system $Sys(P, C_D)$ is stable if and only if $M := \begin{bmatrix} D_1 Y_1 + N_1 X_1 & V Y_2 + N_1 X_2 \\ N_2 X_1 & D_2 Y_2 + N_2 X_2 \end{bmatrix}$ is unimodular [4]. We first design $C_2 = X_2 Y_2^{-1}$ such that $M_2 := D_2 Y_2 + N_2 X_2$ is unimodular, equivalently, $C_2$ is a stabilizing controller for $P_2$. We design $C_2$ as a stable first order controller

$$C_2 = (\alpha + p) K_2 \frac{(f_2 s + 1)}{(\rho_2 s + 1)}. \quad (3)$$

In (3), let $K_2 = N_2(0)^{-1} = 1.8862$, and define $\Phi_2 := \frac{1}{s}\left[\frac{(f_2 s+1)}{(\rho_2 s+1)} d_u(s) P_2(s) K_2 - 1\right]$, where $f_2, \rho_2 \in \mathbb{R}_+$ are chosen so that $p < \|\Phi_2\|^{-1}$. Choosing $\rho_2 = 0.01$, $f = 0.5$ we have $\|\Phi_2\|^{-1} = 10.1174 > p = 3.1203$. Choose any $\alpha \in \mathbb{R}_+$ satisfying $\alpha < (\|\Phi_2\|^{-1} - p)$. If $\alpha = 6.8 < 6.9971$, the controller $C_2$ in (3) becomes $C_2 = 18.7117 \frac{(0.5s+1)}{(0.01s+1)}$. The order of $C_2$ is one, which is lower than the order of $P_2$. To prove $C_2 = Y_2^{-1} X_2 = I^{-1} C_2$ in (3) stabilizes $P_2 = D_2^{-1} N_2$, write $M_2 = D_2 + N_2 C_2 = \frac{(s-p)}{(a_2 s+1)} + \frac{n_2}{d_s(a_2 s+1)}(\alpha + p) K_2 \frac{(f_2 s+1)}{(\rho_2 s+1)} = \frac{(s+\alpha)}{a_2 s+1)}(\frac{s-p}{s+\alpha} + \frac{(\alpha+p)}{(s+\alpha)} d_u P_2 K_2 \frac{(f_2 s+1)}{(\rho_2 s+1)}) = \frac{(s+\alpha)}{(a_2 s+1)}(1 + \frac{(\alpha+p)}{(s+\alpha)}[d_u P_2 K_2 \frac{(f_2 s+1)}{(\rho_2 s+1)} - 1]) = \frac{(s+\alpha)}{(a_2 s+1)}(1 + \frac{(\alpha+p)\, s}{s+\alpha}\Phi_2)$. The norm $\|\frac{(\alpha+p)\, s}{s+\alpha}\Phi_2\| \le (\alpha + p)\|\Phi_2\| < 1$ by choice of $\alpha$; hence, $M_2$ is unimodular, i.e., $C_2$ stabilizes $P_2$.

With $M_2$ unimodular due to the design of $C_2$ in (3), the closed-loop $Sys(P, C_D)$ is stable if and only if $M$ is unimodular, equivalently, $M_1 := D_1 Y_1 + [N_1 - (V Y_2 + N_1 X_2) M_2^{-1} N_2] X_1 = D_1 Y_1 + D_1 P_1 (I + C_2 P_2)^{-1} X_1$ is unimodular, equivalently, $C_1 = X_1 Y_1^{-1}$ stabilizes the system $W := P_1(I + C_2 P_2)^{-1}$. The controller $C_1$ should be designed to stabilize $W = P_1(I + C_2 P_2)^{-1} = \frac{n_1}{s d_u d_s}\frac{(\rho_2 s+1) d_u d_s}{d_m} = \frac{(0.01s+1)\,(0.4893s^2-4.5260)}{s\, d_m}$, where $d_m = d_s d_u(\rho_2 s+1) + (\alpha+p) K_2 (f_2 s+1) n_2 = (0.01s^4 + 1.0949s^3 + 9.0575s^2 + 138.5506s + 204.8335)$. We design $C_1$ as a second order integral-action controller

$$C_1 = K_1 \frac{(f_1 s + 1)(2\beta s + \beta^2)}{s\,(\rho_1 s + 1)}. \quad (4)$$

In (4), choose any $f_1, \rho_1 \in \mathbb{R}_+$; one choice is $\rho_1 = 0.009$, $f = 0.1$. Let $K_1 = (sW)(0)^{-1} = -45.2571$. Define $\Psi_1 := \frac{1}{s}\left[\frac{(f_1 s+1)}{(\rho_1 s+1)}\, s\, W\, K_1 - 1\right]$. Choose any $\beta \in \mathbb{R}_+$ satisfying $\beta < 0.5\|\Psi_1\|^{-1}$, where $\|\Psi_1\|^{-1} = 1.7379$ for the chosen $\rho_1, f_1$. If $\beta = 0.867 < 0.8689$, the controller $C_1$ in (4) becomes $C_1 = -45.2571 \frac{(0.1s+1)(1.7340s+0.7517)}{s\,(0.009s+1)}$. The order of $C_1$ is two, which is lower than the order of the fifth order $W$. To prove that $C_1$ in (4) stabilizes $W = P_1(I + C_2 P_2)^{-1}$, write $C_1 = X_1 Y_1^{-1}$ with $Y_1 = \frac{s}{s+\beta}$ and $X_1 = C_1 \frac{s}{s+\beta}$; then $M_1 = D_1 Y_1 + D_1 W X_1 = \frac{(s+\beta)}{(a_1 s+1)}(\frac{s^2}{(s+\beta)^2} + \frac{(2\beta s+\beta^2)}{(s+\beta)^2}\frac{(f_1 s+1)}{(\rho_1 s+1)}\, s\, W\, K_1) = \frac{(s+\beta)}{(a_1 s+1)}(1 + \frac{(2\beta s+\beta^2)}{(s+\beta)^2}[\frac{(f_1 s+1)}{(\rho_1 s+1)}\, s\, W\, K_1 - 1]) = \frac{(s+\beta)}{(a_1 s+1)}(1 + \frac{(2\beta s+\beta^2)\, s}{(s+\beta)^2}\Psi_1)$. The norm $\|\frac{s(2\beta s+\beta^2)}{(s+\beta)^2}\Psi_1\| \le \beta\|\frac{s(2s+\beta)}{(s+\beta)^2}\|\|\Psi_1\| \le \beta\|\frac{s}{s+\beta}\|\|\frac{(2s+\beta)}{(s+\beta)}\|\|\Psi_1\| = 2\beta\|\Psi_1\| < 1$ by choice of $\beta$ and hence, $M_1$ is unimodular, i.e., $C_1$ stabilizes $P_1$. Therefore, the decentralized controller $C_D = \mathrm{diag}[\,C_1, C_2\,]$ as in (4) and (3) stabilizes the system $Sys(P, C_D)$. The closed-loop poles are $\{-115.19, -101.36, -0.56, -0.82 \pm j9.11, -0.92 \pm j1.51\}$. The different choices of the PD and PID-controller parameters in $C_2$ and $C_1$ obviously effect the closed-loop input-output transfer-function $H_{yu}$ from $u$ to $y = \begin{bmatrix} x & \theta \end{bmatrix}^T$. Due to the integral-action in $C_1$, the steady-state error in the first channel for step input references at both $u_1$ and $u_2$ goes to zero asymptotically. Although the first channel output $x$ asymptotically tracks constant input references with no steady-state error, the steady-state error in the second channel is small but not zero. We tested the robot with different initial conditions and reference inputs. Simulation results can be found in [5].

## IV. Conclusions

We presented a low order two-channel decentralized controller synthesis to stabilize the linearized MIMO model of an autonomous mobile robot that was constructed based on the inverted pendulum. The controller in the first channel is second order and has integral-action, which provides asymptotic tracking of constant reference inputs with zero steady-state error. The second channel has a first order controller achieving very small steady-state error. The decentralized PD/PID controller is programmed into a 8-bit microcontroller to validate the accuracy of the physical implementation of the modeling.

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