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A correct characterization of strict positive realness for MIMO systems

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Abstract— **We present conditions which are necessary and sufficient for a transfer function (or transfer function matrix) to be strictly positive real. A counter example is given to illustrate that, in the MIMO (multi-input multi-output) case, the conditions presented here differ from those previously presented in the literature, and that these same conditions represent an incomplete characterization of strict positive realness (and passivity).**

I. INTRODUCTION

The concept of Strict Positive Realness (SPR) of a transfer function matrix appears frequently in various aspects of engineering. Application oriented areas such as optimal control, adaptive control, VLSI design [1], and in particular, stability theory, have all benefited greatly from the concept of SPR [2], [3], [4], [5], [6], [7]. It is therefore vitally important to characterize this property via conditions which can readily be computed or verified experimentally. While such conditions have been readily available in the literature [8], [9], [10], [11], [12] for some time, our main purpose here in this paper is demonstrate by means of an elementary counter-example that these conditions are in fact incomplete. We present an alternative characterisation of a strictly positive real transfer function matrix that takes care of the problems highlighted by the counter example.

Definition 1 (SPR) *A transfer function* $G:\mathbb{C}\to\mathbb{C}^{m\times m}$ *is* **strictly positive real (SPR)** *if there exists a real scalar* $\epsilon>0$ *such that* G *is analytic in a region containing* $\{s\in\mathbb{C}:\Re(s)\geq-\epsilon\}$ *and*

$$G(\jmath\omega-\epsilon)+G(\jmath\omega-\epsilon)^*\geq 0 \qquad \text{for all} \qquad \omega\in\mathbb{R}\,. \tag{1}$$

We say that G is **regular** if $\det[G(\jmath\omega)+G(\jmath\omega)^*]$ is not identically zero for all $\omega\in\mathbb{R}$.

The first appearance of the above definition seems to be [6], [3] in the scalar case. The definition was motivated by a desire to obtain conditions on a transfer function which satisfied the requirements of the Kalman-Yacubovic-Popov Lemma. Reference [6] also provides an electrical network interpretation of SPR. Assuming G is stable, rational and proper, it was known that the dissipativity condition, $G(\jmath\omega)+G(\jmath\omega)^*>0$ for all finite ω, was necessary but not sufficient for SPR. Requiring in addition that $G(\infty)+G(\infty)^*>0$ yields sufficiency but not necessity. Thus, starting with [6], a search was on for a side condition which in addition to the dissipativity condition yielded an equivalent characterization of SPR; some of this research is summarized in Section III. Such a side condition eliminates the need for ϵ in the characterization of SPR and this is important for two main reasons: The dissipativity condition is something that can be verified experimentally by looking at the frequency response of a system and the ϵ-free conditions are more computationally tractable.

We present here a new side condition which along with stability and the dissipativity condition yields ϵ-free conditions which are necessary and sufficient for a transfer function (or transfer function matrix) to be strictly positive real. This new side condition can be simply stated as

$$\boxed{\lim_{|\omega|\to\infty}\omega^{2\rho}\det[G(\jmath\omega)+G(\jmath\omega)^*]\neq 0} \tag{2}$$

where ρ is the nullity of the matrix $G(\infty)+G(\infty)^*$.

As we shall illustrate by means of an elementary counter-example, when $0<\rho<m$, the new side condition presented here is not equivalent to those previously presented in the literature. In those cases, the example presented highlights that the conditions given previously are only necessary but not sufficient to characterize an SPR transfer function matrix. Notwithstanding this fact, there are important situations where our condition does indeed coincide with previous conditions, for example, scalar transfer functions ($m=1$) and transfer functions for which ρ is either equal zero or m. In these important situations our condition is consistent with previous conditions given in the literature. However, for other cases, existing conditions do not give a correct characterization of positive realness whereas our conditions do.

II. MAIN RESULTS

We assume throughout G is analytic at infinity and we let

$$D=G(\infty):=\lim_{|\omega|\to\infty}G(\jmath\omega)\,. \tag{3}$$

We do not assume that G is rational. Our main result contains a new condition which is embodied in (2) above. It involves ρ, the nullity of $D+D^*$, that is, the dimension of the null space of $D+D^*$. This is the same as $m-m_1$ where m_1 is the rank of $D+D^*$. It is also the same as the number of zero eigenvalues of $D+D^*$.

Lemma 1: A transfer function G *is SPR and regular if and only if the following conditions hold.*

(a) *[Stability] There exists* $\beta>0$ *such that* G *is analytic in the region* $\{s\in\mathbb{C}:\Re(s)>-\beta\}$.

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(b) *[Dissipativity]*

$$G(\jmath\omega) + G(\jmath\omega)^* > 0 \qquad \textit{for all} \qquad \omega \in \mathbb{R} \qquad (4)$$

(c) *[Asymptotic side condition]*

$$\lim_{|\omega|\to\infty} \omega^{2\rho} \det[G(\jmath\omega) + G(\jmath\omega)^*] \neq 0$$

where ρ is the nullity of $G(\infty) + G(\infty)^*$.
In either case, the above limit is positive.

A proof of this lemma is given in Section IV.

A. Another characterization of the side condition

Here we provide another characterization of side condition (2). This is sometimes useful for computational purposes. To this end, recall that G is analytic at infinity. Specifically we require that, for some $\epsilon > 0$, the function G has the following power series expansion

$$G(s) = D + \frac{1}{s}G_1 + \frac{1}{s^2}G_2 + \cdots \qquad (5)$$

for $|s|$ large and $\Re(s) \geq -\epsilon$. Here $D, G_1, G_2, \cdots$ are constant $m \times m$ matrices. As before, let ρ be the nullity of the matrix $D + D^*$, that is, ρ is the dimension of the null space of $D + D^*$. We distinguish between three cases:

$\rho = 0$: $\det(D + D^*) \neq 0$
$\rho = m$: $D + D^* = 0$
$0 < \rho < m$: $\det(D + D^*) = 0$ and $D + D^* \neq 0$

When $0 < \rho < m$, we let U and V be any matrices of sizes $m \times (m - \rho)$ and $m \times \rho$, respectively, where the columns of U form a basis for the range of $D + D^*$ and the columns of V form a basis for the null space of $D + D^*$. Note that these matrices can be reliably and efficiently obtained from a singular value decomposition of $D + D^*$. Now let

$$E := \begin{cases} D & \text{if} \quad \rho = 0 \\ -G_2 & \text{if} \quad \rho = m \\ \begin{pmatrix} U^*DU & U^*G_1V \\ -V^*G_1U & -V^*G_2V \end{pmatrix} & \text{if} \quad 0 < \rho < m \end{cases} \qquad (6)$$

Remark 1 We will see later in Lemma 4 that side condition (2) is equivalent to

$$\det(E + E^*) \neq 0\,. \qquad (7)$$

Also, under the strict dissipativity hypothesis (4), side condition (2) is also equivalent to $E + E^* > 0$. Note that the matrices U and V are not unique. However, the above results hold for any U any V satisfying the above requirements on U and V.

B. Proper rational transfer functions

Consider a proper rational transfer function given by

$$G(s) = C(sI - A)^{-1}B + D \qquad (8)$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{m\times m}$. If all the eigenvalues of A have negative real part then there is a $\beta > 0$ such that G is analytic in $\{s \in \mathbb{C} : \Re(s) > -\beta\}$. Also, for large s,

$$G(s) = D + \frac{1}{s}CB + \frac{1}{s^2}CAB + \cdots$$

Hence $G_1 = CB$, $G_2 = CAB$ and

$$E = \begin{cases} D & \text{if} \quad \rho = 0 \\ -CAB & \text{if} \quad \rho = m \\ \begin{pmatrix} U^*DU & U^*CBV \\ -V^*CBU & -V^*CABV \end{pmatrix} & \text{if} \quad 0 < \rho < m \end{cases} \qquad (9)$$

III. Previous conditions in the literature

For rational transfer functions corresponding to real A, B, C, D matrices, references [8], [9], [10], [11], [12] present conditions which are necessary and sufficient for SPR. Reference [8] considers scalar transfer functions whereas [9] considers the special cases of $D = 0$ or $D > 0$ in the matrix case. For these special cases, their conditions are basically the same as those here.

References [10], [11] consider the general matrix case. Their conditions are the same as here except in the case when $0 < \rho < m$ where ρ is the nullity of $D + D^*$. In this case, instead of side condition (2), they have the following condition:

$$\lim_{|\omega|\to\infty} \omega^2[G(\jmath\omega) + G(\jmath\omega)^*] > 0 \qquad \text{and} \qquad D + D^* \geq 0$$

However, for the above limit to exist one must have $D + D^* = 0$. So, the limit does not exist when $\rho < m$.

Reference [12] also considers the general matrix case. In Lemma 6.1 it is claimed that a regular G is SPR if and only if hypotheses (a) and (b) of Lemma 1 hold and the following side condition holds.

Either $G(\infty) + G(\infty)^T$ is positive definite or it is positive semi-definite and

$$\lim_{|\omega|\to\infty} \omega^2 V^T[G(\jmath\omega) + G(\jmath\omega)^*]V > 0$$

where V is any $m \times \rho$ full rank matrix for which $V^T[G(\infty) + G(\infty)^T]V = 0$ and ρ is the nullity of $G(\infty) + G(\infty)^T$.

When $\rho = 0$ or $\rho = m$ or $U^*(G_1 - G_1^*)V = 0$, the above condition is the same as $E + E^* > 0$ which is equivalent to side condition (2). When $0 < \rho < m$ and $U^*(G_1 - G_1^*)V \neq 0$, the condition is less restrictive than (2). So, according to Lemma 1, this condition is necessary for SPR. However, in general it is not sufficient as the example below illustrates.

Example 1 Consider

$$G(s) = \begin{pmatrix} 1 & \frac{1}{s+1} \\ -\frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}.$$

The matrix $G(\jmath\omega-\epsilon)+G(\jmath\omega-\epsilon)^*$ is given by

$$2\begin{pmatrix} 1 & \frac{-\jmath\omega}{(1-\epsilon)^2+\omega^2} \\ \frac{\jmath\omega}{(1-\epsilon)^2+\omega^2} & \frac{1-\epsilon}{(1-\epsilon)^2+\omega^2} \end{pmatrix}$$

which has determinant

$$\delta(\omega,\epsilon) = 4\frac{(1-\epsilon)^3-\epsilon\omega^2}{[(1-\epsilon)^2+\omega^2]^2}. \tag{10}$$

Consider any $\epsilon > 0$; it can be seen that $\delta(\omega,\epsilon) < 0$ for large ω and, hence the matrix $G(\jmath\omega-\epsilon)+G(\jmath\omega-\epsilon)^*$ is not positive semi-definite. Since this is true for any $\epsilon > 0$, we conclude that G **is not SPR.**

Hypotheses (a) and (b) of Lemma 1 hold. Since

$$G(\infty)+G(\infty)^* = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{11}$$

the nullity ρ of $G(\infty)+G(\infty)^*$ is one and we can let

$$U = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad V = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and obtain that

$$\lim_{|\omega|\to\infty} \omega^2 V^*\big[G(\jmath\omega)+G(\jmath\omega)^*\big]V = \lim_{|\omega|\to\infty} \frac{2\omega^2}{\omega^2+1} = 2.$$

So, the requirements of [12] hold but G is not SPR.

Note that

$$G_1 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

which results in

$$U^*(G_1-G_1^*)V = 2 \neq 0.$$

This is the situation in which the side condition of [12] is only necessary but not sufficient.

Since $\rho = 1$ and

$$\det[G(\jmath\omega)+G(\jmath\omega)^*] = \frac{4}{(1+\omega^2)^2}$$

we see that

$$\lim_{|\omega|\to\infty} \omega^{2\rho}\det[G(\jmath\omega)+G(\jmath\omega)^*] = \lim_{|\omega|\to\infty} \frac{4\omega^2}{(1+\omega^2)^2} = 0.$$

Thus, hypothesis (c) of Lemma 1 is not satisfied and this lemma correctly predicts that G is not SPR.

a) A note on the proof of Lemma 6.1 in [12]: In the proof of Lemma 6.1 in [12] it is claimed that when $G(\jmath\omega)+G(\jmath\omega)^* > 0$ for all ω and the side condition in [12] holds with $D+D^*$ singular, one has

$$\lim_{|\omega|\to\infty} \sigma_1(\omega) > 0 \text{ where } \sigma_1(\omega) := \sigma_{\max}[G(\jmath\omega)+G(\jmath\omega)^*]$$

and

$$\lim_{|\omega|\to\infty} \omega^2\sigma_2(\omega) > 0 \text{ where } \sigma_2(\omega) := \sigma_{\min}[G(\jmath\omega)+G(\jmath\omega)^*]$$

In the example above, the last inequality does not hold. To see this, we first note that

$$\sigma_1(\omega)\sigma_2(\omega) = \det[G(\jmath\omega)+G(\jmath\omega)^*] = \frac{4}{(1+\omega^2)^2}.$$

It follows from (11) that

$$\lim_{|\omega|\to\infty} \sigma_1(\omega) = \sigma_{\max}[G(\infty)+G(\infty)^*] = 2.$$

Hence

$$\begin{aligned} \lim_{|\omega|\to\infty} \omega^2\sigma_2(\omega) &= \lim_{|\omega|\to\infty} \frac{\omega^2\sigma_1(\omega)\sigma_2(\omega)}{\sigma_1(\omega)} \\ &= \lim_{|\omega|\to\infty} \frac{4\omega^2}{(1+\omega^2)^2\sigma_1(\omega)} = 0. \end{aligned}$$

IV. Proof of Main Result

A. Preliminary results

First we need the following results.

Definition 2 (PR) *A transfer function G is* **positive real (PR)** *if G is analytic in the in a region containing $\{s \in \mathbb{C} : \Re(s) \geq 0\}$ and*

$$G(\jmath\omega)+G(\jmath\omega)^* \geq 0 \qquad \textit{for all} \qquad \omega \in \mathbb{R}. \tag{12}$$

Lemma 2: Suppose G is a positive real transfer function with a power series expansion of the form (5). If $u \in \mathbb{C}^m$ is any vector for which $u^*(D+D^*)u = 0$ then, u^*G_1u is real and non-negative, that is,

$$u^*G_1u = u^*G_1^*u \geq 0\,.$$

If in addition $u^*[G(\jmath\omega) + G(\jmath\omega)^*]u$ is not identically zero for all ω then, $u^*G_1u > 0$.

The Appendix contains a proof.

Corollary 1: Suppose G is a positive real transfer function with a power series expansion of the form (5). If V is any matrix for which $(D + D^*)V = 0$ then, V^*G_1V is hermitian and positive semi-definite, that is

$$V^*G_1^*V = V^*G_1V \geq 0\,. \tag{13}$$

If in addition V is full column rank and G is regular then, $V^*G_1V > 0$.

In the next result, ϵ is any real scalar and $L_\epsilon(s)$ is defined by

$$\begin{array}{ll} G(s-\epsilon) & \text{if} \quad \rho=0 \\ -s^2G(s-\epsilon) & \text{if} \quad \rho=m \\ \begin{pmatrix} U^*G(s-\epsilon)U & s\,U^*G(s-\epsilon)V \\ -sV^*G(s-\epsilon)U & -s^2V^*G(s-\epsilon)V \end{pmatrix} & \text{if} \quad 0<\rho<m \end{array} \tag{14}$$

where ρ is the nullity of $D + D^*$ and U and V are any matrices with the columns of U forming a basis for the range of $D + D^*$ and the columns of V forming a basis for the null space of $D + D^*$. Also, $L(s) = L_0(s)$. We define the matrix E_ϵ by

$$\begin{array}{ll} D & \text{if} \quad \rho=0 \\ -G_2-\epsilon G_1 & \text{if} \quad \rho=m \\ \begin{pmatrix} U^*DU & U^*G_1V \\ -V^*G_1U & -V^*G_2V-\epsilon V^*G_1V \end{pmatrix} & \text{if} \quad 0<\rho<m \end{array} \tag{15}$$

and we let

$$\tilde{G}_1 := \begin{cases} 0 & \text{if} \quad \rho=0 \\ G_1 & \text{if} \quad \rho=m \\ V^*G_1V & \text{if} \quad 0<\rho<m \end{cases} \tag{16}$$

Lemma 3: (i) *For all nonzero $\omega \in \mathbb{R}$,*

$$G(\jmath\omega-\epsilon) + G(\jmath\omega-\epsilon)^* \geq 0$$

if and only if

$$L_\epsilon(\jmath\omega) + L_\epsilon(\jmath\omega)^* \geq 0$$

This result also holds for strict inequalities.

(ii) *If $\tilde{G}_1$ is hermitian and $\{(\omega_k, \epsilon_k)\}_{k=1}^{\infty}$ is any sequence with $\lim_{k\to\infty} \epsilon_k = \epsilon$ and $\lim_{k\to\infty} |\omega_k| = \infty$ then,*

$$\lim_{k\to\infty} [L_{\epsilon_k}(\jmath\omega_k) + L_{\epsilon_k}(\jmath\omega_k)^*] = E_\epsilon + E_\epsilon^*\,. \tag{17}$$

(iii)

$$\det[L(s) + L(-\bar{s})^*] = c\,(-s^2)^\rho \det[G(s) + G(-\bar{s})^*] \tag{18}$$

where $c > 0$.

The Appendix contains a proof.

Note that $E = E_0$ where E is defined in (6). It follows from Lemma 3 above that, for all nonzero $\omega \in \mathbb{R}$,

$$G(\jmath\omega) + G(\jmath\omega)^* > 0 \Longleftrightarrow L(\jmath\omega) + L(\jmath\omega)^* > 0 \tag{19}$$

$$\det[L(\jmath\omega) + L(\jmath\omega)^*] = c\,\omega^{2\rho} \det[G(\jmath\omega) + G(\jmath\omega)^*] \tag{20}$$

for some constant $c > 0$, and

$$\lim_{|\omega|\to\infty} [L(\jmath\omega) + L(\jmath\omega)^*] = E + E^*\,. \tag{21}$$

The above observations lead us to the following result.

Lemma 4: The following conditions are equivalent.

$$\lim_{|\omega|\to\infty} \omega^{2\rho} \det[G(\jmath\omega) + G(\jmath\omega)^*] \neq 0 \tag{22}$$

$$\lim_{|\omega|\to\infty} \det[L(\jmath\omega) + L(\jmath\omega)^*] \neq 0 \tag{23}$$

$$\det[E + E^*] \neq 0\,. \tag{24}$$

If $G(\jmath\omega) + G(\jmath\omega)^* > 0$ for all $\omega \in \mathbb{R}$ then the above conditions are equivalent to the following conditions.

$$\lim_{|\omega|\to\infty} \omega^{2\rho} \det[G(\jmath\omega) + G(\jmath\omega)^*] > 0 \tag{25}$$

$$\lim_{|\omega|\to\infty} \left[L(\jmath\omega) + L(\jmath\omega)^*\right] > 0 \tag{26}$$

$$E + E^* > 0 \tag{27}$$

B. Proof of Lemma 1

Necessity. Suppose G is SPR and regular. Then there exists $\epsilon > 0$ such that G is analytic in a region containing the set $\{s \in \mathbb{C} : \Re(s) \geq -\epsilon\}$ and

$$G(\jmath\omega-\epsilon) + G(\jmath\omega-\epsilon)^* \geq 0 \qquad \text{for all} \qquad \omega \in \mathbb{R}\,. \tag{28}$$

Using properties of analytic functions and the fact that G is regular, one can show that G is analytic in the region $\{s \in \mathbb{C} : \Re(s) > -\epsilon\}$ and $G(s) + G(s)^* > 0$ for all s in this region; in particular, we must have

$$G(\jmath\omega) + G(\jmath\omega)^* > 0 \tag{29}$$

for all $\omega \in \mathbb{R}$.

To complete the proof of necessity, we will show that

$$E + E^* > 0 \tag{30}$$

which, using Lemma 4, implies the desired result (2). It follows from the SPR condition (28) and part (i) of Lemma 3 that

$$L_\epsilon(\jmath\omega) + L_\epsilon(\jmath\omega)^* \geq 0 \tag{31}$$

for all nonzero ω. Since G is PR (recall (29)) and regular, Corollary 1 tells us that $\tilde{G}_1$ is hermitian; also $\tilde{G}_1 > 0$ for $\rho > 0$. Considering limits as $|\omega| \to \infty$ in (31) and recalling part (ii) of Lemma 3 we obtain that

$$E_\epsilon + E_\epsilon^* \geq 0\,. \tag{32}$$

If $\rho = 0$ then $D + D^* > 0$ and $E = D$; hence (30) holds. Consider now $\rho = m$. In this case, $E_\epsilon = -G_2 - \epsilon G_1 = E - \epsilon \tilde{G}_1$ and (32) reduces to

$$E + E^* \geq 2\epsilon \tilde{G}_1 .$$

Since $\tilde{G}_1 > 0$, we have the desired result (30).

Now consider the remaining case in which $0 < \rho < m$. Partition $M := E + E^*$ in accordance with the partition of E as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} .$$

Then inequality (32) can be written as

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} - 2\epsilon \tilde{G}_1 \end{pmatrix} \geq 0 \tag{33}$$

Since $\epsilon > 0$ and $\tilde{G}_1 > 0$, there exists $\epsilon > 0$ such that $2\epsilon \tilde{G}_1 \geq \epsilon I$; thus inequality (33) yields that

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} - \epsilon I \end{pmatrix} \geq 0 .$$

The above inequality implies that $U^*(D+D^*)U = M_{11} \geq 0$. Since the columns of U span the range of $D+D^*$, we obtain that $D + D^* \geq 0$. Since the intersection of the range of U and the null space of $D+D^*$ is the zero vector we must have $M_{11} = U^*(D + D^*)U > 0$. It now follows from Lemma 5 in the Appendix that $M > 0$, that is, $E + E^* > 0$, which is the desired result.

Sufficiency. Clearly hypothesis (b) implies that G is regular.

To demonstrate the existence of $\epsilon > 0$ such that (1) holds, we first show that there exists ϵ_1 and $\omega_1 > 0$ with $0 < \epsilon_1 < \beta$ such that

$$G(\jmath\omega - \epsilon) + G(\jmath\omega - \epsilon)^* > 0 \text{ for } 0 \leq \epsilon \leq \epsilon_1 \text{ and } |\omega| \geq \omega_1 . \tag{34}$$

To see this consider the function

$$F(\epsilon, \eta) = \begin{cases} L_\epsilon(\frac{\jmath}{\eta}) + L_\epsilon(\frac{\jmath}{\eta})^* & \text{for} \quad \eta \neq 0 \\ E_\epsilon + E_\epsilon^* & \text{for} \quad \eta = 0 \end{cases}$$

Since G is PR, the matrix $\tilde{G}_1$ is hermitian and it follows from part (ii) of Lemma 3 that F is continuous. Recall from Lemma 4 that, under hypothesis (b), hypothesis (c) is equivalent to $E + E^* > 0$, that is $F(0, 0) > 0$. Since F is continuous, there exists ϵ_1 and $\eta_1 > 0$ such that $0 < \epsilon_1 < \beta$ and $F(\epsilon, \eta) > 0$ for $|\epsilon| \leq \epsilon_1$ and $|\eta| \leq \eta_1$. Letting $\omega_1 = 1/\eta_1$ yields the desired result (34).

We now show that there exists ϵ_2 with $0 < \epsilon_2 < \beta$ such that

$$\det[G(\jmath\omega - \epsilon) + G(\jmath\omega - \epsilon)^*] > 0 \text{ for } 0 \leq \epsilon \leq \epsilon_1 \, , |\omega| \leq \omega_1 . \tag{35}$$

To achieve this introduce the continuous function

$$f(\epsilon) = \min\{\det[G(\jmath\omega - \epsilon) + G(\jmath\omega - \epsilon)^*] : |\omega| \leq \omega_1\}$$

where $|\epsilon| < \beta$. It follows from hypothesis (b) that $f(0) > 0$. Hence there exists ϵ_2 such that $0 \leq \epsilon_2 < \beta$ and $f(\epsilon) > 0$ for $0 \leq \epsilon \leq \epsilon_2$. This yields (35).

If we let $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$ we obtain the desired result that (1) holds with $\epsilon = \epsilon_3$ and G is analytic in the region for which $\Re(s) \geq -\epsilon_3$.

■

APPENDIX

C. Proof of Lemma 2

PROOF. Consider any non-zero $u \in \mathbb{C}^m$ for which $u(D + D^*)u = 0$ and let g be the scalar-valued transfer function given by $g(s) = u^* \hat{G}(s) u$ where $\hat{G}(s) = G(s) + \frac{1}{2}(D^* - D)$. Then, $g(s) + g(s)^* = u^*[G(s) + G(s)^*]u$ and, since G is positive real, we must have $g(\jmath\omega) + g(\jmath\omega)^* \geq 0$ for all $\omega \in \mathbb{R}$. Since g is analytic for $\Re(s) \geq 0$, we must also have

$$g(s) + g(s)^* \geq 0 \qquad \text{for} \qquad \Re(s) \geq 0 \tag{36}$$

The power series expansion (5) for G yields

$$g(s) = \frac{g_1}{s} + \frac{g_2}{s^2} \cdots \tag{37}$$

where $g_n = u^* G_n u$ for $n = 1, 2, \cdots$.

Consider now any function g which satisfies (36) and which has a power series of the form

$$g(s) = \frac{g_n}{s^n} + \frac{g_{n+1}}{s^{n+1}} + \cdots . \tag{38}$$

We will show that if $n = 1$, g_1 is real and $g_1 \geq 0$ and if $n > 1$, $g_n = 0$. To this end, consider any real ρ and θ with $\rho \geq 0$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $\Re(s) \geq 0$ where $s = \rho e^{-\jmath\theta}$ and it follows from inequality (36) that

$$\rho^n [g(\rho e^{-\jmath\theta}) + g(\rho e^{-\jmath\theta})^*] \geq 0 .$$

Considering the power series expansion (38) we see that

$$\lim_{\rho \to \infty} \rho^n g(\rho e^{-\jmath\theta}) = e^{\jmath n\theta} g_n .$$

So, we must have

$$e^{\jmath n\theta} g_n + e^{-\jmath n\theta} g_n^* \geq 0 . \tag{39}$$

Considering $\theta = \pi/2n$ and $\theta = -\pi/2n$ yields $\jmath(g_n - g_n^*) \geq 0$ and $-\jmath(g_n - g_n^*) \geq 0$, respectively. Hence $\jmath(g_n - g_n^*) = 0$ and so we must have $g_n^* = g_n$, that is, g_n is real. We now show that $g_n \geq 0$. Considering $\theta = 0$ in (39) yields $2g_n \geq 0$, that is, $g_n \geq 0$. Considering $\theta = \pi/n$ in (39) for $n \geq 2$, results in $-2g_n \geq 0$, that is, $g_n \leq 0$. So, we must have $g_n = 0$ when $n \geq 2$.

Returning now to (37) where $g_n = u^* G_n u$, we obtain that $u^* G_1 u$ is real and $u^* G_1 u \geq 0$.

Suppose now that $u^* G_1 u = 0$. From the above it should be clear that this results in $u^* G_n u = 0$ for all n and, hence, $u^* \hat{G}(s) u \equiv 0$. Since $u^*[\hat{G}(s) + \hat{G}(s)^*]u = u^*[G(s) + G(s)^*]u$, we obtain that $u^* G_1 u = 0$ implies that $u^*[G(\jmath\omega) + G(\jmath\omega)^*]u \equiv 0$. So, when $u^*[G(\jmath\omega) + G(\jmath\omega)^*]u$ is not identically zero, we must have $u^* G_1 u > 0$.

■

D. Proof of Corollary 1

Suppose V is $m \times m_2$ and consider any nonzero $w \in \mathbb{C}^{m_2}$. Since G is PR and $(Vw)^*(D+D^*)Vw = 0$, it follows from Lemma 2 that $w^*V^*G_1Vw$ is real and $w^*V^*G_1Vw \geq 0$. Hence V^*G_1V is hermitian and $V^*G_1V \geq 0$.

Suppose G is regular and PR. We claim that for any nonzero $u \in \mathbb{C}^m$, it must be the case that $u^*[G(\jmath\omega) + G(\jmath\omega)^*]u$ is not identically zero for all $\omega \in \mathbb{R}$. To see this, consider any $\omega \in \mathbb{R}$ for which $u^*[G(\jmath\omega) + G(\jmath\omega)^*]u$ is zero. Since $G(\jmath\omega) + G(\jmath\omega)^* \geq 0$, we must have $[G(\jmath\omega) + G(\jmath\omega)^*]u = 0$ and, hence, $\det[G(\jmath\omega) + G(\jmath\omega)^*] = 0$. Since G is regular, we conclude that for any nonzero $u \in \mathbb{C}^m$, $u^*[G(\jmath\omega) + G(\jmath\omega)^*]u$ is not identically zero for all $\omega \in \mathbb{R}$. Suppose V has full column rank and consider $u = Vw$ where w is any nonzero vector in $\mathbb{C}^{m_2}$. Then $u \neq 0$, and using Lemma 2, we obtain that $w^*VG_1Vw > 0$. Since this holds for any non-zero $w \in \mathbb{C}^{m_2}$, we have $V^*G_1V > 0$. ■

E. Proof of Lemma 3

(i) Consider any $\omega \in \mathbb{R}$. For $\rho = 0$, we have $L_\epsilon(\jmath\omega) = G(\jmath\omega - \epsilon)$. Thus,

$$L_\epsilon(\jmath\omega) + L_\epsilon(\jmath\omega)^* = G(\jmath\omega - \epsilon) + G(\jmath\omega - \epsilon)^*$$

and clearly, the result holds.

For $\rho = m$, we have $L_\epsilon(\jmath\omega) = \omega^2 G(\jmath\omega - \epsilon)$. Thus,

$$L_\epsilon(\jmath\omega) + L_\epsilon(\jmath\omega)^* = \omega^2[G(\jmath\omega - \epsilon) + G(\jmath\omega - \epsilon)^*]$$

and clearly, the result holds.

When $0 < \rho < m$, we have $L_\epsilon(\jmath\omega) = S^*G(\jmath\omega - \epsilon)S$ where $S = (U \;\; \jmath\omega V)$. Hence

$$L_\epsilon(\jmath\omega) + L_\epsilon(\jmath\omega)^* = S^*\left[G(\jmath\omega - \epsilon) + G(\jmath\omega - \epsilon)^*\right]S\,.$$

When ω is nonzero, S is invertible and

$$G(\jmath\omega - \epsilon) + G(\jmath\omega - \epsilon)^* = S^{-*}\left[L_\epsilon(\jmath\omega) + L_\epsilon(\jmath\omega)^*\right]S^{-1}$$

These last two identities yield the desired result.

(ii) Suppose $\tilde{G}_1$ is hermitian and $\{(\omega_k, \epsilon_k)\}_{k=1}^{\infty}$ is any sequence with $\lim_{k\to\infty}\epsilon_k = \epsilon$ and $\lim_{k\to\infty}|\omega_k| = \infty$. Recall that, for $|s|$ sufficiently large,

$$G(s) = D + \frac{1}{s}G_1 + \frac{1}{s^2}G_2 + \frac{1}{s^3}G_3 \cdots\,. \qquad (40)$$

When $\rho = 0$, we have $L_\epsilon(\jmath\omega) = G(\jmath\omega - \epsilon)$ and $E_\epsilon = D$. It follows from the above power series expansion that

$$\lim_{k\to\infty}[G(\jmath\omega_k - \epsilon_k) + G(\jmath\omega_k - \epsilon_k)^*] = D + D^*\,, \qquad (41)$$

that is, the desired result (17) holds.

When $\rho = m$, we have $L_\epsilon(\jmath\omega) = \omega^2 G(\jmath\omega - \epsilon)$, $E_\epsilon = -\epsilon G_1 - G_2$ and $\tilde{G}_1 = G_1$. Thus G_1 is hermitian and we need to show that

$$\lim_{k\to\infty}\omega_k^2[G(\jmath\omega_k - \epsilon_k) + G(\jmath\omega_k - \epsilon_k)^*] = -2\epsilon G_1 - G_2 - G_2^*\,. \qquad (42)$$

Since $D + D^* = 0$ and $G_1^* = G_1$,

$$G(s) + G(s)^* = \left(\frac{1}{s} + \frac{1}{\bar{s}}\right)G_1 + \frac{1}{s^2}G_2 + \frac{1}{\bar{s}^2}G_2^* + \frac{1}{s^3}G_3 + \frac{1}{\bar{s}^3}G_3^* + \cdots$$

where $\bar{s} = s^*$. When $s = s_k := \jmath\omega_k - \epsilon_k$,

$$\frac{1}{s_k} = \frac{1}{\jmath\omega_k - \epsilon_k} = \frac{-\jmath\omega_k - \epsilon_k}{\omega_k^2 + \epsilon_k^2}\,;$$

hence

$$\left(\frac{1}{s_k} + \frac{1}{\bar{s}_k}\right)G_1 = \frac{-2\epsilon_k}{\omega_k^2 + \epsilon_k^2}G_1$$

which results in

$$\lim_{k\to\infty}\omega_k^2\left(\frac{1}{s_k} + \frac{1}{\bar{s}_k}\right)G_1 = -2\epsilon G_1\,.$$

Noting that

$$\lim_{k\to\infty}\frac{\omega_k^2}{(\jmath\omega_k - \epsilon_k)^2} = \lim_{k\to\infty}\frac{\omega_k^2}{(-\jmath\omega_k - \epsilon_k)^2} = -1$$

we obtain the desired result (42).

Now consider the remaining case in which $0 < \rho < m$. Recalling that the columns of V are in the null space of $D + D^*$, we have $(D + D^*)V = 0$; hence the power series expansion (40) yields that

$$\begin{aligned}
&\jmath\omega U^*G(\jmath\omega - \epsilon)V + \jmath\omega U^*G(\jmath\omega - \epsilon)^*V \\
&= \jmath\omega U^*(D + D^*)V \\
&+ \left(\frac{\jmath\omega}{\jmath\omega - \epsilon}\right)U^*G_1V + \left(\frac{\jmath\omega}{-\jmath\omega - \epsilon}\right)U^*G_1^*V \\
&+ \frac{\jmath\omega}{(\jmath\omega - \epsilon)^2}U^*G_2V + \frac{\jmath\omega}{(-\jmath\omega - \epsilon)^2}U^*G_2^*V + \cdots\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&\lim_{k\to\infty}[\jmath\omega_k U^*G(\jmath\omega_k - \epsilon_k)V + \jmath\omega_k U^*G(\jmath\omega_k - \epsilon_k)^*V] \\
&\qquad = U^*G_1V - U^*G_1^*V \qquad (43)
\end{aligned}$$

Since V^*G_1V is hermitian, we can use the same arguments as used in the case when $\rho = m$ to obtain that

$$\begin{aligned}
&\lim_{\omega_k\to\infty}\omega_k^2[V^*G(\jmath\omega_k - \epsilon_k)V + V^*G(\jmath\omega_k - \epsilon_k)^*V] \\
&\qquad = -2\epsilon V^*G_1V - V^*G_2V - V^*G_2^*V \qquad (44)
\end{aligned}$$

The desired result (17) now follows from (41) and (43)-(44).

(iii) For $\rho = 0$ and $\rho = m$, we have $L(s) = G(s)$ and $L(s) = -s^2G(s)$, respectively. So clearly the result holds in these cases. When $0 < \rho < m$, we can express $L(s)$ as

$$L(s) = \begin{pmatrix} I_{m_1} & 0 \\ 0 & -\bar{s}I_\rho \end{pmatrix}^* T^*G(s)T \begin{pmatrix} I_{m_1} & 0 \\ 0 & sI_\rho \end{pmatrix}$$

where m_1 is the rank of $D + D^*$ and $T = [U \;\; V]$. Hence $L(s) + L(-\bar{s})^*$ is given by

$$\begin{pmatrix} I_{m_1} & 0 \\ 0 & -\bar{s}I_\rho \end{pmatrix}^* T^*\left[G(s) + G(-\bar{s})^*\right]T \begin{pmatrix} I_{m_1} & 0 \\ 0 & sI_\rho \end{pmatrix}$$

and using properties of determinants, we see that $\det[L(s) + L(-\bar{s})^*]$ equals

$$\det(T^*T)(-s^2)^\rho \det\left[G(s) + G(-\bar{s})^*\right]\,.$$

Since $\det(T^*T) > 0$ we are done. ■.

F. A final lemma

Lemma 5: Suppose

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

is hermitian with M_{11} *square and positive definite and there exists* $\epsilon > 0$ *such that*

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} - \epsilon I \end{pmatrix} \geq 0\,. \tag{45}$$

Then, M *is positive definite.*

PROOF. Since $M_{11} > 0$, inequality (45) implies that

$$M_{22} - \epsilon I - M_{21} M_{11}^{-1} M_{12} \geq 0\,.$$

This inequality and $\epsilon > 0$ result in

$$M_{22} - M_{21} M_{11}^{-1} M_{12} > 0\,.$$

Since $M_{11} > 0$, the last inequality implies that $M > 0$. ■

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