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Switched System Based Approach to Analysis and Synthesis of Discrete-Time Linear Systems With Time-Varying State Delay

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***Abstract*— This paper establishes a framework for stability analysis and memoryless state-feedback control synthesis of linear discrete-time systems with time varying state delay. The underlying idea is converting the considered system into an equivalent constrained switched system, leading to *sufficient* and *necessary* condition for stability analysis. Based on this switched system modeling, available information about the time-varying delay such as known bounds of variation rate can be incorporated into the analysis in a natural way. Moreover, sufficient conditions for memoryless state-feedback control design are obtained in a similar style. The analysis and synthesis conditions are given in terms of solvability of a set of linear matrix inequalities (LMIs). Numerical examples are included for illustration.**

I. Introduction

Systems with time-delay exist in many engineering fields such as chemical processes, communication systems and networked control systems. Analysis and Synthesis of time-delay systems have been received extensive attention in recent years. (see, e.g., [1]-[18], survey papers [26] [27] and the references therein).

The fundamental problem in the research area of systems with state delay is the stability analysis in the presence of available information about the delay. General characteristics capturing real situations of delay is that it is time-varying within known bounds and It's varying behavior obeys some known limitations such as bounded variation rate. Many researchers have been tried their best for deriving stability conditions and many results have been reported. In the continuous-time setting, both range-dependent [1] [2] [3] and range-derivative-dependent [4] [5] [6] [7] [8] [9] conditions have been obtained. It should be noted that the latter ones are generally of less conservatism since more information are taken into account. In the discrete-time context, besides bounds of variation rate, more types of the known limitations imposed on the varying behavior of delay may be exist.A typical example can be found in networked control systems(NCSs). For discrete-time NCSs, the existing literature (such as [20] [21] [22] [23] [24] [25]) usually suppose that if the NCSs experience time delay or packet dropout, the latest available control inputs will be used, which means that $d(k+1) = d(k) + 1$ (where $d(k)$ is the network-induced sensor-to-actuator delay), on the other hand, if the newly generated control inputs arrive at the actuator on time, we have $d(k+1) \leq d(k)$. However, to our best knowledge, only range-dependent conditions [10] [11] [12] [13] [14] [15] [16] [17] [18] are available under the discrete-time setting in the existing literatures. It should be noticed that almost all the above mentioned conditions are derived by constructing an appropriate Lyapunov function and adopting some bounding techniques. Therefore, there are two challenging and interesting tasks remained to be further investigated for linear discrete-time systems with time-varying delay. One is to provide a way such that exact stability analysis of the discrete-time system with time-varying delay can be achieved since all the existing stability conditions are sufficient ones. The other is to establish connections between the stability and the known limitations about the varying behavior of the delay.

In this paper, we are dedicated to solve the problems of stability and memoryless state-feedback stabilization for linear discrete-time systems with time-varying state delay. Our approach is based on converting the considered system into an equivalent constrained switched system $(\mathcal{A}, \mathcal{M})$. Where $\mathcal{A}$ is a *vertex set* that consists of a group of subsystems which are in augmented dimension and delay free, $\mathcal{M}$ is an *adjacency matrix* that representing the known information about the varying behaviors such as the bounds of variation rate of the delay. Thus, the equivalence enables us to address the considered complicated problems in a simple manner. Consequently, *sufficient* and *necessary* condition for stability analysis is derived Based on the constrained switched system and multiple Lyapunov functions. Moreover, it is shown that the problem of memoryless state-feedback stabilization is cast to design a *robust* static output feedback controller for the equivalent constrained switched system, and sufficient conditions for (H_∞) controller synthesis are given by using LMI techniques.

The rest of the paper is organized as follows. In section 2, the system and problems under consideration are given and some graph theory preliminaries are presented. Section 3 shows the modeling procedures that converting the studied

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system into a constrained switched system. The stability and stabilization conditions are given in Section 4 and Section 5 respectively. Several examples are given in Section 6 and some conclusions are drawn in Section 7.

II. Problem Statements and Preliminaries

A. System description

Consider the following discrete-time system with time-varying state delay

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-d(k)) + B_2 u(k) + B_1 \omega(k) \\ z(k) &= C_1 x(k) + D_{12} u(k) \\ y(k) &= x(k) \\ x(k) &= \phi(k) \qquad k = -d_M, -d_M+1, ..., 0. \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input. $z(k) \in \mathbb{R}^q$ is the controlled output. $A, A_d, B_2, B_1, C_1, D_{12}$ are system matrices with compatible dimensions, $d(k)$ is a time-varying state delay in the state.

Assumption 1:

i) *It is assumed that the range of time-varying delay $d(k)$ is bounded and known, i.e.,*

$$d_m \leq d(k) \leq d_M$$

where d_m and d_M are constant positive scalars representing the lower and upper delays, respectively.

ii) *There are information about the varying behavior of $d(k)$ such as the bound of variation rate*

$$d_v \leq |\Delta d(k)| \leq d_V$$

available. where d_v and d_V are constant positive scalars representing the lower and upper delays, respectively.

Remark 1.

i) The bounded range assumption about the time varying delay is a natural one, a great deal of delay range-dependent stability analysis conditions have been reported both in continuous-time setting and discrete-time setting

ii) For deriving less conservative results, the bound of variation rate $\dot{d}(t)$ have been take into consideration in continuous-time domain [4] -[9]. However, to our best knowledge, there is no existing result tackling the stability analysis with the consideration about $|\Delta d(k)|$ in discrete-time domain.

iii) Besides the variation rate $|\Delta d(k)|$, other types of restriction about the varying behavior of $d(k)$ may be exist. For example, in the networked control system, it is known that $d(k) = d(k)+1$ during the waiting period and $d(k+1) \leq d(k)$ while the newly control input arrived at $k+1$ instant (See [20]-[25]).

B. Problem statements

In this paper, we consider questions of stability analysis and stabilizing controller synthesis for system (1).

P1): With $u(k)$ and $\omega(k)$ identically zero, i.e., consider the following system:

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-d(k)) \\ x(k) &= \phi(k) \qquad k = -d_M, -d_M+1, ..., 0. \end{aligned} \tag{2}$$

does the state of system (2) satisfy $\lim_{t\to\infty} x(k) = 0$ for every initial condition $\phi(0)$? If so, we say that system is asymptotically stable with the time-varying delay $d(k)$.

P2): Does there exist a memoryless state-feedback control law

$$u(k) = Kx(k) \tag{3}$$

such that the resulting closed-loop system:

$$\begin{aligned} x(k+1) &= (A + B_2 K)x(k) + A_d x(k-d(k)) + B_1 \omega(k) \\ z(k) &= C_1 x(k) + D_{12} u(k) \\ x(k) &= \phi(k) \qquad k = -d_M, -d_M+1, ..., 0. \end{aligned} \tag{4}$$

is asymptotically stable or simultaneously meets H_∞ performance bound? i.e.,$\|T_{\omega z}\|_\infty \leq \gamma$, where $T_{\omega z}$ is the closed-loop transfer function from ω to z.

C. Graph theory preliminaries and useful Lemma

In this section, we briefly present some basic concepts and notations in graph theory that will be used in this paper.

A graph $\mathcal{G}$ consists of a *vertex set* $\mathcal{V} = \{1, 2, ..., N\}$ and an *edge set* $\mathcal{E} = \{(i,j) : i, j \in \mathcal{V}\}$, where an *edge* is an unordered or ordered pair of distinct vertices of $\mathcal{V}$. If $i, j \in \mathcal{V}$, and $(i,j) \in \mathcal{E}$, then we say that i and j are adjacent or that j is a neighbor of i, and denote this by writing $j \sim i$. A graph is called *complete* if every pair of vertices are adjacent. The $valence$ of a vertex v of $\mathcal{G}$ is defined as the number of edges of $\mathcal{G}$ which are $incident$ with v if one of the two vertices of the edge is v. A path of length r form i to j in a graph is a sequence of $r+1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is path between any two vertices of $\mathcal{G}$, the $\mathcal{G}$ is connected. The $adjacency\ matrix$ $M(\mathcal{G})$ of $\mathcal{G}$ is an $n \times n$ matrix of whose ijth entry is 1 if (i, j) is one of $\mathcal{G}'s$ edges and 0 if it is not. Any undirected graph can be represented by its adjacency matrix $M(\mathcal{G})$, which is a matrix with 0-1 elements.

Lemma 1. *(Finsler' Lemma) Letting that $\xi \in \mathbb{R}^N$, $\mathcal{P} = \mathcal{P}^T \in \mathbb{R}^{N\times N}$, and $\mathcal{H} \in \mathbb{R}^{M\times N}$ such that $rank(\mathcal{H}) = R < N$, then the following statements are equivalent:*
1) $\xi^T \mathcal{P} \xi < 0$, *for all* $\xi \neq 0$, $\mathcal{H}\xi = 0$;
2)$\exists \mathcal{X} \in \mathbb{R}^{N\times M}$ *such that* $\mathcal{P} + \mathcal{X}\mathcal{H} + \mathcal{H}^T\mathcal{X}^T < 0$.

III. Switched System Modeling

To avoid dealing with the time-varying term $d(k)$, the augmentation technique will be adopted here. Defining an augmented signal $\eta(k)$ as

$$\eta(k) = \left[\begin{array}{cccc} x^T(k) & x^T(k-1) & \cdots & x^T(k-d_M) \end{array}\right]^T$$

Then, we can obtain an equivalent switched system model:

$$\begin{aligned} \eta(k+1) &= \mathbf{A}_{\delta(k)}\eta(k) + \mathbf{B}_{2\delta(k)}u(k) + \mathbf{B}_{1\delta(k)}\omega(k) \\ z(k) &= \mathbf{C}_{1\delta(k)}\eta(k) + \mathbf{D}_{12\delta(k)}u(k) \\ y(k) &= \mathbf{C}_{2\delta(k)}\eta(k) \end{aligned} \tag{5}$$

The mapping $\delta(k) : \mathbb{R}^n \times [0,\infty) \to \{1,...,N\}$ is so-called switching signal that orchestrating the switching between the subsystems $\mathcal{S}_{si} : (\mathbf{A}_i, \mathbf{B}_{2i}, \mathbf{B}_{1i}, \mathbf{C}_{1i}, \mathbf{D}_{12i}, \mathbf{C}_{2i})$, where

$$\begin{aligned} &(\mathbf{A}_i, \mathbf{B}_{2i}, \mathbf{B}_{1i}, \mathbf{C}_{1i}, \mathbf{D}_{12i}, \mathbf{C}_{2i}) \\ &\in \{\mathscr{S} : (\mathcal{A}, \mathcal{B}_2, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_{12}, \mathcal{C}_2)\} \\ &= \{(\mathbf{A}_1, \mathbf{B}_{21}, \mathbf{B}_{11}, \mathbf{C}_{11}, \mathbf{D}_{121}, \mathbf{C}_{21}), ..., \\ &\qquad (\mathbf{A}_N, \mathbf{B}_{2N}, \mathbf{B}_{1N}, \mathbf{C}_{1N}, \mathbf{D}_{12N}, \mathbf{C}_{2N})\} \end{aligned} \tag{6}$$

and

$$\begin{aligned} \mathbf{A}_i &= \left[\begin{array}{c} \mathbf{A}_i^u \\ \mathbf{A}_i^l \end{array}\right] \\ \mathbf{A}_i^l &= \left[\begin{array}{cc} I_{nd_M} & \mathbf{0} \end{array}\right], \qquad \text{for } i \in \{1,...,N\} \\ \mathbf{A}_i^u &= \left[\begin{array}{cccc} A & \mathbf{0} & \cdots & \mathbf{0} \end{array}\right] + \left[\begin{array}{cccc} Z_i^0 & Z_i^1 & \cdots & Z_i^{d_M} \end{array}\right] \\ Z_i^j &= \begin{cases} A_d & \text{if } \; j = d_m + i - 1 \\ \mathbf{0} & else \end{cases} \\ \mathbf{B}_{2i} &= \left[\begin{array}{c} B_2 \\ \mathbf{0} \end{array}\right] \quad \mathbf{B}_{1i} = \left[\begin{array}{c} B_1 \\ \mathbf{0} \end{array}\right] \\ \mathbf{C}_{1i} &= \left[\begin{array}{cc} C_1 & \mathbf{0} \end{array}\right] \quad \mathbf{D}_{12i} = \left[\begin{array}{cc} D_{12} & \mathbf{0} \end{array}\right] \\ \mathbf{C}_{2i} &= \left[\begin{array}{cc} I_{n\times n} & \mathbf{0} \end{array}\right] \end{aligned} \tag{7}$$

It is set that the evolution of $\delta(k)$ is synchronized with the varying delay $d(k)$ by letting $\delta(k) = d(k) - d_m + 1$. The consequence is that the switched system $\mathcal{S}_s$ (5) is totally equivalent with the system $\mathcal{S}_d$ (1) since the equality

$$\eta(k) = \left[\begin{array}{cccc} x^T(k) & x^T(k-1) & \cdots & x^T(k-d_M) \end{array}\right]^T$$

holds for any $k \in [0,\infty)$. As the constraint on the transition of $\delta(k)$ is inherited form the constraint of $d(k)$, the adjacency matrix $\mathcal{M}$ of them will be identical. For the extreme case that there exist no any restriction except the bounded range of $d(k)$, i.e., $d_m \le d(k) \le d_M, |\Delta d(k)| \le d_M - d_m$, $\mathcal{M}$ is just the identity matrix $\mathbf{1}$. While some additional constraints such as bounded variation rate are imposed on $d(k)$, the corresponding entries of $\mathcal{M}$ will becomes zero. In the sequel, we will use $(\mathscr{S}, \mathcal{M})$ to describe the considered constrained switched system $\mathcal{S}_s$ (5).

IV. Switched System Method Based Stability

In this section, we are concerned with the asymptotically stability of linear discrete-time systems with time-varying delay $\mathcal{S}_d$ (1). Based on the modeling way of last section, we will only focus on the stability analysis of the equivalent constrained switched system $\mathcal{S}_s : (\mathscr{S}, \mathcal{M})$ (5).

Definition 1. Let Ω be the set of every infinite sequence in {1,...,N}; each element of Ω shall be called a *switching sequence*. Let L be a nonnegative integer and Ω^{L+1} denote the set of every sequence with L-paths in $\{1,...,N\}^{L+1}$, and if $\mathbf{i} = (i_0, ..., i_L)$ with $i_0, ..., i_L \in 1, ..., N$, then $\mathbf{i}$ is called a *switching path* of length L. Let Ω_a be a nonempty subset of Ω, then Ω_a is called *admissible set* if every switching sequence in it is generated according $\mathcal{M}$, then we can write

$$\Omega_a = \{\delta(\cdot) : \delta(k) \to \delta(k+1) \;\text{ in }\; \mathcal{M} \;\text{ for all }\; k \ge 0\}$$

where $i \to j$ reads that there exists a directed edge form node i to node j. Similarly, a nonempty subset Ω_a^{L+1} of Ω^{L+1} is said to be an *admissible set of L-paths* if, for each $(i_0, ..., i_L) \in \Omega_a^{L+1}$, there exist an integer $K \ge L$ and a switching path $(i_{K-L}, ..., i_K)$ such that $(i_{K-L}, ..., i_L) = (i_0, ..., i_L)$ and $(i_k, ..., i_{k+L}) \in \Omega_a^{L+1}$ for all $k \in \{0, ..., K-L\}$.

Theorem 1. The constrained switched system $\mathcal{S}_s : (\mathscr{S}, \mathcal{M})$ (5) is asymptotically stable if and only if there exists a nonnegative integer L and symmetric positive definite matrices $P_i \in \mathbb{R}^{n(d_M+1)\times n(d_M+1)}, i \in \{1,...,N\}$, such that the following LMIs

$$(\mathbf{A}_{i_{L-1}} ... \mathbf{A}_{i_1}\mathbf{A}_{i_0})^T \mathbf{P}_{i_L} (\mathbf{A}_{i_{L-1}} ... \mathbf{A}_{i_1}\mathbf{A}_{i_0}) - \mathbf{P}_{i_0} < 0 \tag{8}$$

hold, where $(i_0, i_1, ..., i_L) \in \Omega_a^{L+1}$ is an admissible switching path of length L.

Proof. The sufficiency part of the proof is as follows: suppose that there exist $P_i > 0, i \in 1, ..., N$ such that (8) holds for all admissible $(i_1, i_2, ..., i_L) \in \Omega_a^L$. Let us consider the following Lyapunov function candidate

$$V(k) = \eta^T(k)\mathbf{P}_i\eta(k), \quad \text{if} \quad \delta(k) = i, \quad i \in \{1,...,N\} \tag{9}$$

The L-step difference of $V(k)$ is:

$$\begin{aligned} &V(k+L) - V(k) \\ &= \eta^T(k+L)\mathbf{P}_{\delta(k+L)}\eta(k+L) - \eta^T(k)\mathbf{P}_{\delta(k)}\eta(k) \\ &= \eta^T(k)\Xi\eta(k) \end{aligned} \tag{10}$$

where

$$\Xi = \mathbf{A}_{\delta(k+L-1)} \cdots \mathbf{A}_{\delta(k)})^T \mathbf{P}_{\delta(k+L)} (\mathbf{A}_{\delta(k+L-1)} \cdots A_{\delta(k)}) - \mathbf{P}_{\delta(k)}$$

It can be concluded that $V(k+L) - V(k) < 0, k \in \{0, \infty\}$, which means $\lim_{k\to\infty} V(k) = 0$, i.e., $\lim_{k\to\infty} \eta(k) = 0$.

To show necessity, suppose that the constrained switched system $\mathcal{S}_s : (\mathscr{S}, \mathcal{M})$ is asymptotically stable, i.e., $\eta(k)_{k\to\infty} = 0$ for all admissible $\delta(\cdot)$. In the sequel it is assumed that at least one of the sets of LMI conditions in (8) does not satisfied for any finite L, and let C_u denotes this condition and $\mathbf{i}_u$ denotes switching path of length L induced form C_u. Since $\mathcal{M}$ is inreducible, there always exists an admissible switching sequence $\delta(\cdot)$ which is recurrent, such that the sequence $\mathbf{i}_u \in \Omega_a^L$ occurs infinitely many times in $\delta(\cdot)$. Then, for any positive function $V(k) = \eta^T(k)Q\eta(k)$, where Q is a any given symmetric positive matrix, which will never vanish to zero. Along with $V(k)$, $\eta(k)$ will never vanish to zero for any $\eta(0) \ne 0$. This is contradict with $x(k)_{k\to\infty} = 0$ as the system is assumed to asymptotically stable. Thus, the proof is complete. □

Remark 2. Theorem 1 provide a framework for checking the stability of discrete-time systems with time-varying delay exactly. It is given in a union of increasing family of linear matrix inequality conditions which can be effectively solved via LMI toolbox [30] by applying the following simple algorithm.

Algorithm 1

1. Set L=0.
2. Solve the feasibility problem (8) for all admissible $\mathbf{i}^L \in \Omega_a^L$.
3. If not feasible and $L < L_{max}$, increment L to $L+1$ and go to step 1, where L_{max} is a prescribed constant according with the requirement of exactness on stability analysis. else terminate the algorithm.

Remark 3. The conditions given by Theorem 1 is presented in a semi-definite fashion, i.e., the path length L is not known a priori. Therefore, the main disadvantage of the proposed stability analysis technique is that, in the worst case ($\mathcal{M} = \mathbf{1}$), the complexity of verifying the asymptotically stability of system (1) increases exponentially with the path length L. However, this is the price one has to pay for nonconservatism. In fact, the value of L required for stability is usually very small, so the technique is expected to find wide-ranging application-especially. Moreover, the adjacency matrix may be a sparse one for the cases that there are constraints such as $|\Delta d(k)|$ imposed on the varying behaviors of $d(k)$, which directly leads to reduction of the number of LMI conditions involved and the corresponding computational burden. It should be noticed that the size of matrices $\mathcal{A}_i$ will be increased with the upper bound of $d(k)$, this brings limitation for applying the LMI method provided here as the limited available computing power nowadays.

V. Controller Synthesis

The objective of this section is to design a memoryless H_∞ state-feedback controller (3) for the time-delay systems $\mathcal{S}_d$ (1). It is easy to convert it as a problem that design a static output feedback controller

$$u(k) = Ky(k) \tag{11}$$

for the equivalent constrained switched system $\mathcal{S}_s : (\mathscr{S}, \mathcal{M})$ (5). The resulting closed-loop system can be described as:

$$\begin{array}{r}\eta(k+1) = \mathbf{A}_i^{cl}\eta(k) + \mathbf{B}_i^{cl}\omega(k) \\ z(k) = \mathbf{C}_i^{cl}\eta(k) + \mathbf{D}_i^{cl}\omega(k)\end{array} \qquad i \in \{1, ..., N\} \tag{12}$$

where

$$\left\{\begin{array}{l}\mathbf{A}_i^{cl} = \mathbf{A}_i + \mathbf{B}_{2i}K\mathbf{C}_{2i} \\ \mathbf{B}_i^{cl} = \mathbf{B}_{1i} \\ \mathbf{C}_i^{cl} = \mathbf{C}_{1i} + \mathbf{D}_{12i}K\mathbf{C}_{2i} \\ \mathbf{D}_i^{cl} = \mathbf{0}\end{array}\right. \qquad i \in \{1, ..., N\} \tag{13}$$

Theorem 3. *If there exist symmetric matrices* $\mathbf{P}_i \in R^{n(d_M+1)\times n(d_M+1)}$, *and matrices* $\mathbf{G} \in R^{n(d_M+1)\times n(d_M+1)}, \mathbf{F} \in R^{n(d_M+1)\times n(d_M+1)}$, $\mathbf{Y} \in R^{(H+1)m\times n}$ $i \in 1, ..., N$, *with the following structure*

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{0} \\ \mathbf{G}_{12} & \mathbf{G}_{22} \end{bmatrix} \mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 & \mathbf{0} \end{bmatrix} \mathbf{F} = \begin{bmatrix} \lambda\mathbf{G}_{11} & \mathbf{0} \\ \mathbf{F}_{12} & \mathbf{F}_{22} \end{bmatrix} \tag{14}$$

satisfying the following LMIs,

$$\begin{bmatrix} \mathbf{P}_j - \mathbf{G} - \mathbf{G}^T & * & * & * \\ 0 & -I & * & * \\ \mathbf{A}_i\mathbf{G} + \mathbf{B}_{2i}\mathbf{Y} - \mathbf{F}^T & \mathbf{B}_{1i} & He(\mathbf{A}_i\mathbf{F} + \mathbf{B}_{2i}\mathbf{Y}) - \mathbf{P}_i & * \\ \mathbf{C}_{1i}\mathbf{G} + \mathbf{D}_{12i}\mathbf{Y} & 0 & \mathbf{C}_{1i}\mathbf{F} + \mathbf{D}_{12}\mathbf{Y} & -\gamma^2 I \end{bmatrix} < 0 \quad \text{where} \quad (i,j) \in \Omega_a^2 \tag{15}$$

then the closed-loop system (12) *is asymptotically stable and the memoryless state-feedback controller (3) with* $K = \mathbf{Y}_1\mathbf{G}_{11}^{-1}$ *renders the* H_∞*-norm of the corresponding closed-loop system (12) less than* γ*, i.e.,*$\|T_{z\omega}\|_\infty < \gamma$.

Proof: From the structure of $\mathbf{G}, \mathbf{F}, \mathbf{Y}$ (14), $K = \mathbf{Y}_1\mathbf{G}_{11}^{-1}$, and (13), we have

$$\left\{\begin{array}{l}\mathbf{A}_i\mathbf{G} + \mathbf{B}_{2i}\mathbf{Y} = \mathbf{A}_{cl}^i\mathbf{G} \\ \mathbf{C}_{1i}\mathbf{G} + \mathbf{D}_{12i}\mathbf{Y} = \mathbf{C}_{cl}^i\mathbf{G} \\ \mathbf{A}_i\mathbf{F} + \mathbf{B}_{2i}\mathbf{Y} = \mathbf{A}_{cl}^i\mathbf{F} \\ \mathbf{C}_{1i}\mathbf{F} + \mathbf{D}_{12i}\mathbf{Y} = \mathbf{C}_{cl}^i\mathbf{F}\end{array}\right. \tag{16}$$

Substituting (16) with $\mathbf{D}_{21i} = 0$ in (15), it can be equivalently rewritten as:

$$\mathcal{P}_{ji} + \mathcal{X}\mathcal{H}_i + \mathcal{H}_i^T\mathcal{X}^T < 0 \tag{17}$$

where

$$\mathcal{P}_{ji} := \begin{bmatrix} \mathbf{P}_j & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -\mathbf{P}_i & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$
$$\mathcal{X} := \begin{bmatrix} \mathbf{G}^T & 0 \\ 0 & I \\ \mathbf{F}^T & 0 \\ 0 & 0 \end{bmatrix} \quad \mathcal{H}_i := \begin{bmatrix} -I & 0 \\ 0 & -I \\ \mathbf{A}_i^{cl} & \mathbf{B}_i^{cl} \\ \mathbf{C}_i^{cl} & \mathbf{D}_i^{cl} \end{bmatrix}^T \tag{18}$$

Consider the following dual system of the constrained switched system (12)

$$\begin{array}{r}\eta'(k+1) = \mathbf{A}_i^{clT}\eta'(k) + \mathbf{C}_i^{clT}\omega'(k) \\ z'(k) = \mathbf{B}_i^{clT}\eta'(k) + \mathbf{D}_i^{clT}\omega'(k)\end{array}, \quad i \in \{1, ..., N\} \tag{19}$$

It is easy to verify that

$$\begin{bmatrix} -I & 0 \\ 0 & -I \\ \mathbf{A}_i^{cl} & \mathbf{B}_i^{cl} \\ \mathbf{C}_i^{cl} & \mathbf{D}_i^{cl} \end{bmatrix}^T \begin{bmatrix} \eta'(k+1) \\ z'(k) \\ \eta'(k) \\ \omega'(k) \end{bmatrix} = 0 \tag{20}$$

Applying the Finsler's Lemma, onecan obtain

$$\begin{bmatrix} \eta'(k+1) \\ z'(k) \\ \eta'(k) \\ \omega'(k) \end{bmatrix}^T \mathcal{P}_{ji} \begin{bmatrix} \eta'(k+1) \\ z'(k) \\ \eta'(k) \\ \omega'(k) \end{bmatrix} < 0, \quad (i,j) \in \Omega_a^2 \tag{21}$$

From (21), we can find that for any nonzero $\omega(k) \in l_2[0, N)$, the following inequality

$$J := \sum_{k=0}^{\infty} \left(z^T(k)z(k) - \gamma^2 \omega^T(k)\omega(k) \right) < 0 \tag{22}$$

holds with the aid of Lyapunov function $V(k) = \eta'^T(k)\mathbf{P}_{\delta(k)}\eta'(k), \delta(k) \in \{1, ..., N\}$. Moreover, form (15), we can deduce that the matrix $\mathbf{G}$ is positive-definite (not necessarily symmetric) which implicitly implies that the matrix $\mathbf{G}_{11}$ is invertible. Then $K\mathbf{G}_{11} = \mathbf{Y}_1$ admits the solution of the controller gain $K = \mathbf{Y}_1\mathbf{G}_{11}^{-1}$. Thus, the proof is complete. □

VI. Numerical Examples

Example 1. *Stability analysis with bounded variation rate on d(k)*

Consider the following discrete-time system with a time-varying state delay

$$x(k+1) = \begin{bmatrix} -0.12 & -0.07 \\ 0.86 & -0.16 \end{bmatrix} x(k) + \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.4 \end{bmatrix} x(k-d(k)) \tag{23}$$

Here, $d(k)$ represents the time-varying state delay. Now assume the lower delay bound of $d(k)$ is $d_m = 0$. We are interested in the upper delay bound d_M below which the above system is asymptotically stable for all $0 \leq d(k) \leq d_M$ under different varying rate $|\Delta d(k)|$. Via the stability results of this paper, we can obtain the following table that containing the details.

L	$\lvert\Delta d(k)\rvert \leq 1$	$\lvert\Delta d(k)\rvert \leq 2$	...	$\lvert\Delta d(k)\rvert \leq d_M$
L=1	9	6	...	5
L=2	9	7	...	5

TABLE I

Upper bound of $d(k)$ under different variation rate restrictions of Example 1

The numerical data of Table 1 shows the impact of different variation rate on the stability of the discrete-time system with time varying delay. This is a natural rule which is never investigated in the existing literature. It is also shows that increasing the length of path L in Theorem 1, one can obtain less conservative results.

Besides, it is found that the upper delay bound $d_M = 1$ by using the latest delay-bound dependent LMI method proposed by (Gao [18]), which is a very conservative result for this example.

Example 2. *Stability analysis with network-induced time-varying delay*

Consider the following system

$$x(k+1) = \begin{bmatrix} 1.08 & 0.10 \\ -0.06 & 0.70 \end{bmatrix} x(k) + \begin{bmatrix} 0.64 \\ 0.32 \end{bmatrix} u(k) \tag{24}$$

It is assumed that a state-feedback control law is given by

$$u(k) = \begin{bmatrix} -0.3161 & -0.4205 \end{bmatrix} x(k - d(k))$$

where $0 \leq d(k) \leq 5$ is the network-induced time-varying delay. The mechanism of network-based control is that the latest available control inputs will be used in the presence of sensor-to-actuator delay, i.e., $d(k+1) = d(k) + 1$. On the other hand, if the newly generated control inputs arrive at the actuator on time, we have $d(k+1) \leq d(k)$ (see [20]-[25]), then we can conclude that the adjacency matrix of equivalent switched system is:

$$\mathcal{T} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Using the LMI method of (Gao[18]), the stability of the closed-loop system cannot be verified. However, it can be confirmed that the closed-loop system is stable via our approach. Figure 1 shows the responses of the signals involved in this system.

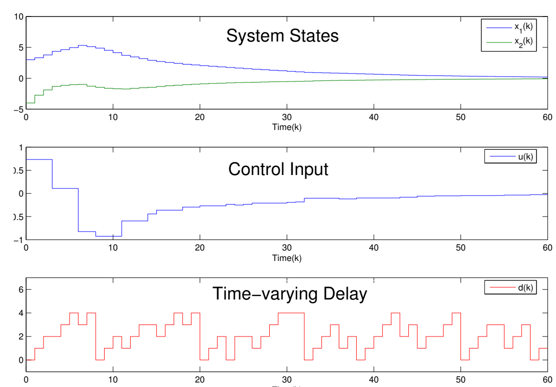

Fig. 1. Evolutions of the signals involved of Example 2

Example 3. *Memoryless H_∞ state-feedback stabilization*

Consider the following discrete-time system:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} -1.08 & 0.10 \\ -0.12 & 0.00 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k) \\ &+ \begin{bmatrix} -0.3 & 0.2 \\ 0.0 & 0.1 \end{bmatrix} x(k-d(k)) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \omega(k) \\ z(k) &= \begin{bmatrix} 1.00 & 1.00 \end{bmatrix} x(k) + [1]\,\omega(k) \end{aligned} \tag{25}$$

Where $0 \leq d(k) \leq 3$. The objective here is design a memoryless H_∞ state-feedback controller $u(k) = Kx(k)$ such that the close-loop system is stable and the H_∞-norm of $\|T_{z\omega}\|_\infty$ is optimized. By applying the LMI conditions given in Theorem 3, the following memoryless H_∞ state-feedback controllers can be obtained corresponding with different known variation rate $\Delta d(k)$.

For comparison, the obtained optimal H_∞ memoryless state-feedback controller using the LMI method [10] is given in Table 3. From the numerical results of Table 2, one can find that lower bound of rate of variation on $d(k)$ will leads to

Variation rate	Controller gain	γ_{opt}	λ
$\|\Delta d(k)\| \leq 1$	$K = \begin{bmatrix} 0.9643 & -0.0766 \end{bmatrix}$	3.6650	0.23
$\|\Delta d(k)\| \leq 2$	$K = \begin{bmatrix} 1.1380 & -0.0990 \end{bmatrix}$	6.0519	0.17
$\|\Delta d(k)\| \leq 3$	$K = \begin{bmatrix} 1.2366 & -0.1050 \end{bmatrix}$	8.1230	0.16

TABLE II

MEMORYLESS H_∞ STATE-FEEDBACK CONTROL DESIGN OF VIA OUR APPROACH FOR EXAMPLE 3

Method	Controller gain	γ_{opt}
Song [10]	$K = \begin{bmatrix} 1.6963 & -0.1544 \end{bmatrix}$	76.2892

TABLE III

MEMORYLESS H_∞ STATE-FEEDBACK CONTROL DESIGN OF VIA SONG [10] FOR EXAMPLE 3

less H_∞-norm bound of the closed-loop system. Comparing the results of Table 2 and Table 3, one can see the sharpness of our approach.

VII. CONCLUSION

In this paper, we have developed a framework for stability analysis of linear discrete-time systems with time-varying state delay. The underlying idea is to convert the considered system into a equivalent constrained switched system. Then, *necessary* and *sufficient* condition for stability is given by a union of increasing family of linear matrix inequality conditions. Based on the switched system model, the problem of memoryless state-feedback control design also discussed. It is shown that it is equivalent to solve a robust static output feedback one. Sufficient conditions for controller synthesis are given both for stabilization and H_∞ stabilization. Moreover, the proposed method allows us take the available knowledge such as the bounds of variation rate into consideration. Several numerical examples have been provided to illustrate the effectiveness and advantage of the proposed method.

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