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LMI Characterizations of Positive Realness and Static Output Feedback Positive Real Control of Discrete-time Systems

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***Abstract*— This paper is concerned with the problems of positive real analysis and control synthesis for discrete-time systems. New linear matrix inequality (LMI) characterizations of positive realness are derived, which enable one to check the positive realness by using parameter-dependent Lyapunov function. The relationship between the proposed characterizations and the existing ones are clarified, which shows that our new results are of less conservatism for characterizing the positive realness of discrete-time systems with polytopic uncertainty. In addition, sufficient conditions for static output feedback positive real controller design are given in terms of solutions to a set of linear matrix inequalities. Numerical examples are included for illustration.**

I. Introduction

An important concept in the analysis of control system is positive realness, which is an essential property exhibited in many real-world systems such as linear circuits [6], [5] [15] [3] [21] [23]. Positive realness has found application in the analysis of the properties of immittance or hybrid matrices of various classes of networks, inverse problem of linear optimal control, stability of Lure's systems and so on. Recently, positive realness has also been generalized to time-delay systems [9] [11], descriptor systems [11] [26], and other systems. The problem of positive real control is to design a controller which render the resulting closed-loop system is stable with its transfer function being positive real [3] [14] [25]. The motivation for studying the positive real control problem stems from robust and nonlinear control [8].

The most pertinent work to this paper is [3], where Zhou et al proposed a characterization of positive realness for discrete-time systems via the projection lemma. As a consequence, the product between the Lyapunov matrix and the system matrices are decoupled. This method directly leads to less conservatism results for characterizing the positive realness and designing state feedback positive real controller for uncertain systems. In fact, this paper stems from the following motivations. First, we want to know whether there exist other positive realness characterizations of linear discrete-time systems, and if so, what are the relationships between them and the results reported at [3]. Secondly, in many practical problems, only output of the system is available to be used for constructing a positive real control law [24] [16], however, the relevant results are rare.

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As a development of the results stated above, this paper considers the positive real analysis and synthesis problems for linear discrete-time systems with/without polytopic uncertainties. The main contributions include: I) New characterizations of positive realness of linear discrete-time systems are proposed. II) The relationships between the proposed characterizations and the existing results are clarified, which show that our new results are of less conservatism for characterizing the positive realness of discrete-time systems with polytopic uncertainty. III) Instead of state feedback, static output feedback positive real controller design methods are developed. Numerical examples are also given to illustrate the validation and effectiveness of the proposed methods.

The rest of this paper is organized as follows. The problem formulations and preliminaries are given in Section 2. Section 3 is dedicated to derive the new positive realness characterizations and static output feedback positive controller designs for linear discrete-time systems. Numerical examples are included in Section 4 for illustration.

II. Preliminaries

Consider a nominal linear time-invariant discrete-time system described by:

$$\Sigma_0: \quad \begin{aligned} x(k+1) &= Ax(k) + B_1\omega(k) + B_2u(k) \\ z(k) &= C_1x(k) + D_{11}\omega(k) + D_{12}u(k) \\ y(k) &= C_2x(t) \end{aligned} \tag{1}$$

and an uncertain linear discrete-time system described by

$$\Sigma_\Delta: \quad \begin{aligned} x(k+1) &= \mathscr{A}x(k) + \mathscr{B}_1\omega(k) + \mathscr{B}_2u(k) \\ z(k) &= \mathscr{C}_1x(k) + \mathscr{D}_{11}\omega(k) + \mathscr{D}_{12}u(k) \\ y(k) &= \mathscr{C}_2x(t) \end{aligned} \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^q$ is the measured output, $z(k) \in \mathbb{R}^p$ is the regulated output, $w(k) \in \mathbb{R}^p$ is the exogenous input, and $u(k)$ is the control input The matrices

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$\mathscr{A}, \mathscr{B}_1, \mathscr{B}_2, \mathscr{C}_1, \mathscr{D}_{12}, \mathscr{C}_2$ of uncertain system Σ_Δ (2) belong to the following uncertainty polytope:

$$\mathcal{P} = \left\{ \left[\begin{array}{c|c|c} \mathscr{A} & \mathscr{B}_1 & \mathscr{B}_2 \\ \hline \mathscr{C}_1 & \mathscr{D}_{11} & \mathscr{D}_{12} \\ \hline \mathscr{C}_2 & \mathscr{D}_{21} & \end{array} \right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c|c} A_i & B_{1i} & B_{2i} \\ \hline C_{1i} & D_{11i} & D_{12i} \\ \hline C_{2i} & & \end{array} \right] \right\} \tag{3}$$

where $\alpha_i > 0$, $\sum_{i=1}^{N} \alpha_i = 1$. The nominal and uncertain unforced discrete-time systems of (1) and (2) are given by

$$\Sigma_0^*: \qquad \begin{aligned} x(k+1) &= Ax(k) + Bw(k) \\ z(k) &= Cx(k) + Dw(k) \end{aligned} \tag{4}$$

and

$$\Sigma_\Delta^*: \qquad \begin{aligned} x(k+1) &= \mathscr{A}x(k) + \mathscr{B}w(k) \\ z(k) &= \mathscr{C}x(k) + \mathscr{D}w(k) \end{aligned} \tag{5}$$

respectively.

Throughout this paper, we shall adopt the following concept of positive realness.

Definition 1. *(Lee and Chen [10])*
Let $G(z)$ be a square real rational transfer matrix in z.
(a). *System Σ_0 is said to be positive real (PR) if its transfer function $G(z)$ is analytic in $\|z\| > 1$ and satisfies $G(z) + G^*(z) \geq 0$ for $\|z\| > 1$.*
(b). *System Σ_0 is said to be strictly positive real (SPR) if its transfer function $G(z)$ is analytic in $\|z\| \geq 1$ and satisfies $G(z) + G^*(z) > 0$ for $\|z\| \geq 1$.*

Next, some existing results about positive realness of nominal unforced system (4) and uncertain unforced system (5) are presented.

Lemma 1.*(Haddad and Bernstein [6]). The following statements are equivalent:*
(a) The unforced nominal system Σ_0^* (4) is stable with SPR.
(b) There exist matrices $P = P^T > 0$ such that

$$\begin{bmatrix} A^TPA - P & A^TPB - C^T \\ B^TPA - C & B^TPB - (D + D^T) \end{bmatrix} < 0 \tag{6}$$

Lemma 2.*(Zhou et al [3]). The following statements are equivalent:*
(a) The unforced nominal system Σ_0^* (4) is stable with SPR.
(b) There exist matrices P such that

$$\begin{bmatrix} -P & PA^T & PC^T \\ AP & -P & -B \\ CP & -B^T & -(D + D^T) \end{bmatrix} < 0 \tag{7}$$

Lemma 3.*(Zhou et al [3]). The following statements are equivalent:*
(a) The unforced nominal system Σ_0^* (4) Σ_0^* is stable with SPR.
(b) There exist matrices P and G such that

$$\begin{bmatrix} -\frac{G+G^T}{2} & * & * & * \\ AG & -P & * & * \\ CG & -B^T & -He(D)) & * \\ \frac{G^T+2G-2P}{2} & -G^TA^T & -G^TC^T & -He(G) \end{bmatrix} < 0 \tag{8}$$

Lemma 4.*(Zhou et al [3]). The unforced uncertain system Σ_Δ^* (5) is robustly stable with SPR if there exist matrices P_i and G such that for all $i = 1, ..., n$,*

$$\begin{bmatrix} -\frac{G+G^T}{2} & * & * & * \\ A_iG & -P_i & * & * \\ C_iG & -B_i^T & -He(D_i)) & * \\ \frac{G^T+2G-2P_i}{2} & -G^TA_i^T & -G^TC_i^T & -He(G) \end{bmatrix} < 0 \tag{9}$$

Remark 1. *The conditions of Lemma 3 is exactly the same with the conditions of theorem 2 (Zhou [3]) provided that the linear fractional uncertainties are vanished, see [3] for details.*

The objective of this paper is to develop new characterizations of positive realness for the unforced systems (4) and (5), and give relationships between the new derived characterizations and the above ones. Moreover, we also interest to design a static output feedback controller

$$u(k) = Ky(k) \tag{10}$$

for the systems (1) and (2) such that the resulting closed-loop systems (11) and (12):

$$\Sigma_0^c: \qquad \begin{aligned} x(k+1) &= Ax(k) + B_1\omega(k) + B_2u(k) \\ z(k) &= (C_1 + D_{12}KC_2)x(k) + D_{11}\omega(k) \end{aligned} \tag{11}$$

and

$$\Sigma_\Delta^c: \qquad \begin{aligned} x(k+1) &= \mathscr{A}x(k) + \mathscr{B}_1\omega(k) + \mathscr{B}_2u(k) \\ z(k) &= (\mathscr{C}_1 + \mathscr{D}_{12}K\mathscr{C}_2)x(k) + \mathscr{D}_{11}\omega(k) \end{aligned} \tag{12}$$

are (robust) stable with SPR.

We end this section by giving three lemmas which will be used in the later development.

Lemma 5. *(Finsler' Lemma) Letting that $\xi \in \mathbb{R}^N$, $\mathcal{P} = \mathcal{P}^T \in \mathbb{R}^{N\times N}$, and $\mathcal{H} \in \mathbb{R}^{M\times N}$ such that $rank(\mathcal{H}) = R < N$, then the following statements are equivalent:*
(a) $\xi^T\mathcal{P}\xi < 0$, *for all* $\xi \neq 0$, $\mathcal{H}\xi = 0$;
(b)$\exists \mathcal{M} \in \mathbb{R}^{N\times M}$ *such that* $\mathcal{P} + He(\mathcal{M}\mathcal{H}) < 0$.

Lemma 7. *(Geromel and Korogui [2]). If the symmetric matrices $V_{ij} \in R^{n\times n}$ are such that*

$$\begin{aligned} &V_{ij} + V_{ji} \geq 0, && 1 \leq j < i \leq N \\ &\sum_{i=1}^{N} (V_{ij} + V_{ji}) \leq 0, && j = 1, ..., N \end{aligned} \tag{13}$$

then the following inequlity

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\alpha_j V_{ij} \leq 0 \qquad \forall\alpha\in\Lambda \tag{14}$$

holds, where Λ is the simplex $\Lambda := \left\{\alpha_i \geq 0,\ \sum_{i=1}^{N}\alpha_i = 1\right\}$

III. Main Results

A. New characterizations of positive realness

In this subsection, several new characterizations of positive realness of the unforced systems (4) and (5) are presented. First, lets focus on the unforced nominal system (4)

Theorem 1. *The unforced nominal system Σ_0^* (4) is side to be stable with SPR if there exist matrices $P = P^T > 0$, G and F such that*

$$\begin{bmatrix} P-G-G^T & G^TA^T-F & G^TC^T \\ AG-F^T & AF+F^TA^T-P & F^TC^T-B \\ CG & CF-B^T & -D-D^T \end{bmatrix} < 0 \tag{15}$$

Proof. Only basic lines of the proof are sketched here for the reasons of space.

Defining the following matrices and vector

$$\mathcal{P} = \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & -I & 0 \end{bmatrix} \quad \mathcal{M} = \begin{bmatrix} G^T & 0 \\ F^T & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix}$$

$$\mathcal{H} = \begin{bmatrix} -I & A^T & 0 & C^T \\ 0 & B^T & -I & D^T \end{bmatrix}$$

$$\xi(k) = \begin{bmatrix} \tilde{x}(k+1)^T & \tilde{x}(k)^T & \tilde{z}(k)^T & \tilde{\omega}(k)^T \end{bmatrix}^T$$

Applying the Finsler's lemma and combining the duality property, Theorem 1 can be obtained

Remark 2.

a) Let us look into the above three matrices $\mathcal{P},\mathcal{M},\mathcal{H}$, the matrix $\mathcal{P}$ represents the desired performance (i.e. positive realness), the matrix $\mathcal{H}$ is composed by system matrices, and $\mathcal{M}$ is a multiplier which decoupling the system matrices and the Lyapunov matrix P, and we call the matrices G,F in $\mathcal{M}$ auxiliary variables throughout of this paper.

c) Introducing auxiliary variables for system analysis or synthesis is known as the parameter dependent Lyapunov function (PDLF) method. In [1], Oliveira et al derived a relaxed LMI stability condition for discrete-time system by introducing one auxiliary variable (i.e. G). In [20], Peaucelle et al proposed further less conservative robust stability conditions for continuous-time and discrete-time systems by introducing two auxiliary matrix variables (i.e. G and F). The PDLF method has been extended and applied on many system analysis and design problems. For example, in [22], Duan et al given an improved robust filter design by proposing a proper structure of the auxiliary variables. Then, a new characterization was given for SPR of continuous-time systems and used to consider robust convergence of a new class of nonlinear systems in [23]. The characterization of SPR given by Theorem 1 can be viewed as discrete-time counterpart of the characterization of SPR given by Lemma 1 of [23]. Here, the two auxiliary variables G and F were introduced just by employing the Finsler lemma.

Similar with Theorem 1, the following results are presented without proof.

Corollary 1. *The following statements are equivalent:*
(a) The unforced nominal system Σ_0^* (4) is stable with SPR.
(b) There exist matrices P and G such that

$$\begin{bmatrix} P-G-G^T & G^TA^T & G^TC^T \\ AG & -P & -B \\ CG & -B^T & -D-D^T \end{bmatrix} < 0 \tag{16}$$

Theorem 2. *The unforced uncertain system Σ_Δ^* (5) is robustly stable with SPR if there exist matrices $P_i = P_i^T > 0$, G and F such that for all $i = 1,...,n$,*

$$\begin{bmatrix} P_i-G-G^T & G^TA_i^T-F & G^TC^T \\ A_iG-F^T & A_iF+F^TA_i^T-P_i & F^TC_i^T-B_i \\ C_iG & C_iF-B_i^T & -D_i-D_i^T \end{bmatrix} < 0 \tag{17}$$

Corollary 2. *The unforced uncertain system Σ_Δ^* (5) is robustly stable with SPR if there exist matrices P_i, G and F such that for all $i = 1,...,n$,*

$$\begin{bmatrix} P_i-G-G^T & G^TA_i^T & G^TC_i^T \\ A_iG & -P_i & -B_i \\ C_iG & -B_i^T & -D_i-D_i^T \end{bmatrix} < 0 \tag{18}$$

B. Relationships between the Characterizations

In this subsection, we are dedicated to understand the relationships between the characterizations presented in the above subsection.

There are five characterizations (i.e. Lemma 1, Lemma 2, Lemma 3, Theorem 1, Corollary 1) of positive realness for the unforced nominal system (4) given above. Obviously, those conditions are equivalent to each other as each one is a sufficient and necessary condition

Now, lets turn attention to the three characterizations (i.e. Lemma 4, Theorem 2, Corollary 2) of positive realness for the unforced uncertain system (5). The following result will play an important role in going into the relationships between the characterizations

Lemma 8. *For any given matrices A, B, C, D with appropriate dimension, if there exist matrices P and G such that*

$$\begin{bmatrix} -\frac{G+G^T}{2} & * & * & * \\ AG & -P & * & * \\ CG & -B^T & -He(D) & * \\ \frac{G^T+2G-2P}{2} & -G^TA^T & -G^TC^T & -He(G) \end{bmatrix} < 0 \tag{19}$$

then there exist matrices P and G such that the following inequality holds

$$\begin{bmatrix} P-G-G^T & G^TA^T & G^TC^T \\ AG & -P & -B \\ CG & -B^T & -D-D^T \end{bmatrix} < 0 \tag{20}$$

Proof. The proof are omitted here for the reasons of space. One can find the proof in [28], the journal version of this paper.

Now, we will clarify the relationships between the robust SPR characterizations (i.e. Lemma 4, Theorem 2, Corollary 2) .

Theorem 3. *Consider the uncertain system (5), the following statements hold.*
(a): *If the SPR condition in Corollary 2 is feasible, then the SPR condition in Theorem 2 is feasible.*
(b): *If the SPR condition in Lemma 4 is feasible, then the SPR conditions in Theorem 2 and Corollary 2 are all feasible.*
Proof.
(a). It is immediate by letting the auxiliary variable $F=0$ in the LMI condition (17).
(b). It is immediate from Lemma 8.

Remark 3. *Theorem 3 shows that the robust SPR condition given in Theorem 1 is not more conservative than that in Corollary 2 and Lemma 4, moreover, the robust SPR condition given in Corollary 2 is not more conservative than that in Lemma 4. Noticing that there is one auxiliary variable G employed in Corollary 2, which is same as the case of Lemma 4, then one can conclude that the Corollary 2 provides a less conservative characterization without adding computational complexity. In addition, it should be pointed out that the LMI condition (20) can be reduced to the standard one (7) by letting (P,G)=(P,P), while the LMI condition (19) does not have this property .*

C. Positive real control

Without loss of generality, assume that the output matrix C_2 of the nominal system is of full row rank, then there exist nonsingular transformation matrix T such that

$$C_2T = \begin{bmatrix} I & 0 \end{bmatrix} \tag{21}$$

Remark 4. *For any given C_2, the corresponding T generally are not unique. A special T can be obtained by following formula,*

$$T = \begin{bmatrix} C_2^T(C_2C_2^T)^{-1} & C_2^{\perp} \end{bmatrix} \tag{22}$$

Theorem 4. *If there exist a scalar λ, symmetric positive matrix P, and matrices G, Y with the following structure*

$$G = \begin{bmatrix} G^{11} & 0 \\ G^{21} & G^{22} \end{bmatrix} \quad L = \begin{bmatrix} Y_1 & 0 \end{bmatrix} \tag{23}$$

satisfying the following LMI:

$$\begin{bmatrix} P-TG-G^TT^T & * & * \\ ATG+B_2Y & -P & * \\ C_1TG+D_{12}Y & -B_1^T & -D_{11}-D_{11}^T \end{bmatrix} < 0 \tag{24}$$

or satisfying the following LMI:

$$\begin{bmatrix} P-TGT^T-TG^TT^T & * & * \\ ATGT^T+B_2YT^T & -P & * \\ C_1TGT^T+D_{12}YT^T & -B_1^T & -D_{11}-D_{11}^T \end{bmatrix} < 0 \tag{25}$$

then the static output feedback controller (10) with $K = Y_1G_{11}^{-1}$ renders the the closed-loop system (11) SPR.
Proof. From the structure of G, Y and (3), (10), we can obtain

$$\begin{aligned} Y &= \begin{bmatrix} KG_{11} & 0 \end{bmatrix} = \begin{bmatrix} K & 0 \end{bmatrix} \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix} \\ &= K \begin{bmatrix} I & 0 \end{bmatrix} T^{-1}TG = KC_2TG \end{aligned} \tag{26}$$

Substituting Y for KC_2TG in (24) and (25), then (24) and (25) can be rewritten as

$$\begin{bmatrix} P-He(TG) & * & * \\ (A+B_2KC_2)TG & -P & * \\ (C_1+D_{12}KC_2)TG & -B_1^T & -He(D_{11}) \end{bmatrix} < 0 \tag{27}$$

and

$$\begin{bmatrix} P-He(TGT^T) & * & * \\ (A+B_2KC_2)TGT^T & -P & * \\ (C_1+D_{12}KC_2)TGT^T & -B_1^T & -He(D_{11}) \end{bmatrix} < 0 \tag{28}$$

respectively, which are equivalent to

$$\begin{aligned} &\begin{bmatrix} P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & -I & 0 \end{bmatrix} \\ &+ He\left(\begin{bmatrix} G^TT^T & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} -I & 0 \\ A_{cl} & B_1 \\ 0 & -I \\ C_{cl} & D_{11} \end{bmatrix}^T \right) < 0 \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\begin{bmatrix} P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & -I & 0 \end{bmatrix} \\ &+ He\left(\begin{bmatrix} TG^TT^T & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} -I & 0 \\ A_{cl} & B_1 \\ 0 & -I \\ C_{cl} & D_{11} \end{bmatrix}^T \right) < 0 \end{aligned} \tag{30}$$

respectively, where $A_{cl} = A+B_2KC_2, C_{cl} = C_1+D_{12}KC_2$. From Finsler's Lemma and combining the proof of Theorem 1, we can obtain that the closed-loop system (11) is stable with SPR. Moreover, form (24) and (25), we can deduce that the matrix G is positive-definite (not necessarily symmetric) which implies that the matrix G, and implicitly G_{11} is invertible. Then $G_{11}K = Y_1$ admits the solution of the controller gain $K = G_{11}^{-1}Y_1$. Thus, the proof is complete. □

Remark 5.

a): Theorem 4 presents sufficient conditions for designing static output feedback positive real controllers for discrete-time system (1). The conditions (24) and (25) are parallelled with each other instead of equivalent, then there may exist some cases that the condition (24) is feasible while the condition (25) is infeasible and vise versa, using the two conditions (24) and (25) together will increasing the possibility of solvability for a given static output feedback positive real control problem.

b): Noticing that there is only one auxiliary variable G involved in the conditions (24) and (24), then the full advantage of the new proposed positive realness characterization (i.e. Theorem 1) may not be utilized since there are two auxiliary variables G, F in the condition (15). A possible way to further reduce the conservatism of Theorem 4 is letting the multiplier $\mathcal{M}$ as follows:

$$\mathcal{M} = \begin{bmatrix} G^T T^T & 0 \\ F^T T^T & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} \quad F = \begin{bmatrix} \lambda G_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix} \tag{31}$$

where λ is scalar. However, for adding the auxiliary variable F (explicitly F_{21} and F_{22}), one should rely on the scalar variable λ to render the problem convex and use line searches to obtain the final result.

For the uncertain system (2), the following theorem presents a robust static output feedback positive real controller design method.

Theorem 5. *If there exist symmetric positive matrices P_{ij}, V_{ij}, and matrices G_{ij}, Y with the following structure*

$$\begin{aligned} G_{ij} &= \begin{bmatrix} G^{11} & 0 \\ G_{ij}^{21} & G_{ij}^{22} \end{bmatrix} \quad Y = \begin{bmatrix} Y_1 & 0 \end{bmatrix} \\ V_{ij} &= \begin{bmatrix} V_{ij}^{11} & * & * \\ V_{ij}^{21} & V_{ij}^{22} & * \\ V_{ij}^{31} & V_{ij}^{32} & V_{ij}^{33} \end{bmatrix} \end{aligned} \tag{32}$$

satisfying the following LMIs:

$$\begin{bmatrix} \Theta_{ij11} & * & * & * \\ \Theta_{ij21} & \Theta_{ij22} & * & * \\ \Theta_{ij31} & \Theta_{ij32} & \Theta_{ij33} & * \\ \Theta_{ij41} & \Theta_{ij42} & \Theta_{ij43} & \Theta_{ij44} \end{bmatrix} < 0, \ 1 \le i \le j \le N \tag{33}$$

$$\begin{aligned} & V_{ij} + V_{ji} \ge 0, && 1 \le j < i \le N \\ & \sum_{i=1}^{N} (V_{ij} + V_{ji}) \le 0, && j = 1, ..., N \end{aligned} \tag{34}$$

where

$$\begin{aligned} \Theta_{ij11} &= -T_i^T G_i^T - T_i G_i \\ \Theta_{ij21} &= A_i T_i G_i + B_{2i} Y \\ \Theta_{ij31} &= C_{1i} T_i G_i + D_{12i} \\ \Theta_{ij41} &= P_j \\ \Theta_{ij22} &= -P_i + V_{ij}^{11} \qquad \Theta_{ij33} = -D_{11i} - D_{11i}^T + V_{ij}^{22} \\ \Theta_{ij32} &= -B_i^T + V_{ij}^{21} \qquad \Theta_{ij43} = V_{ij}^{32} \\ \Theta_{ij42} &= V_{ij}^{32} \qquad \Theta_{ij44} = -P_j + V_{ij}^{33} \end{aligned}$$

where the transformation matrices $T_i, i = 1, ..., N$ are selected similar with (21)i.e.

$$T_i = \begin{bmatrix} C_{2i}^T (C_{2i} C_{2i}^T)^{-1} & C_{2i}^{\perp} \end{bmatrix}$$

then the static output feedback controller (10) with $K = Y_1 G_{11}^{-1}$ renders the the closed-loop system (12) robust SPR.

Proof. The proof can be accessed in [28] and omitted here.

IV. Numerical Examples

Example 1. (*Positive realness*)

Consider an uncertain system Σ_Δ^1 described by (2) and (3) with the following parameter matrices:

$$\Sigma_\Delta^1 : \left\{ \begin{array}{l} \left[\begin{array}{c|c} A_1^1 & B_1^1 \\ \hline C_1^1 & D_1^1 \end{array}\right] = \left[\begin{array}{cc|c} -0.5776 & -0.1874 & -0.3287 \\ 0.4297 & 0.3855 & 0.0970 \\ \hline 0.8456 & -0.6702 & 0.4613 \end{array}\right] \\[2ex] \left[\begin{array}{c|c} A_2^1 & B_2^1 \\ \hline C_2^1 & D_2^1 \end{array}\right] = \left[\begin{array}{cc|c} -0.1174 & 0.6299 & 0.7434 \\ 0.1815 & -0.7937 & -0.5528 \\ \hline 0.2592 & 0.8012 & 0.3409 \end{array}\right] \end{array} \right.$$

We are interested in investigating whether this uncertain system Σ_Δ^1 is of positive realness. Solving the LMI conditions given by Theorem 2, Corollary 2, and Lemma 4, we obtain the following results about feasibility. See Table 1.

Methods	Feasibility
Theorem 2	feasible
Corollary 2	feasible
Lemma 4 [3]	infeasible

TABLE I

Feasibilities of characterizations of positive realness for the system Σ_Δ^1

Consider another uncertain system Σ_Δ^2 described by (2) and (3) with the following parameter matrices:

$$\Sigma_\Delta^2 : \left\{ \begin{array}{l} \left[\begin{array}{c|c} A_1^2 & B_1^2 \\ \hline C_1^2 & D_1^2 \end{array}\right] = \left[\begin{array}{cc|c} 0.0245 & -0.2248 & -0.2061 \\ -0.4988 & 0.2404 & -0.0829 \\ \hline -0.2931 & 0.3310 & 0.0976 \end{array}\right] \\[2ex] \left[\begin{array}{c|c} A_2^2 & B_2^2 \\ \hline C_2^2 & D_2^2 \end{array}\right] = \left[\begin{array}{cc|c} -0.1241 & -0.1916 & 0.1142 \\ 0.4887 & 0.4778 & -0.2963 \\ \hline -0.1084 & -0.1096 & 0.0204 \end{array}\right] \end{array} \right.$$

Correspondingly, we obtain the following feasibility distribution results. See Table 2. The numerical results here

Methods	Feasibility
Theorem 2	feasible
Corollary 2	infeasible
Lemma 4 [3]	infeasible

TABLE II

Feasibilities of characterizations of positive realness for the system Σ_Δ^2

show that the condition of Theorem 2 is the least conservative one within the three robust SPR characterizations (Theorem 2, Corollary 2, Lemma 4 [3]), and the condition of Lemma 4 [3] is the most conservative one. Then, the correctness of Theorem 3 is validated.

Example 2. (*Positive real control*)
Consider an uncertain system (2) with the vertices given by follows:

$$\left[\begin{array}{c|c|c} A_1 & B_{11} & B_{21} \\ \hline C_{11} & D_{111} & D_{121} \\ \hline C_{21} & & \end{array}\right] = \left[\begin{array}{cc|c|c} 0.2398 & 0.2299 & 0.2177 & -0.1367 \\ -0.0485 & -0.1184 & 0.2948 & 0.2393 \\ \hline 0.3783 & 0.2178 & 0.0630 & 0.3881 \\ \hline -0.3718 & -0.1114 & & \end{array}\right]$$

$$\left[\begin{array}{c|c|c} A_2 & B_{12} & B_{22} \\ \hline C_{12} & D_{112} & D_{122} \\ \hline C_{22} & & \end{array}\right] = \left[\begin{array}{cc|c|c} 0.1629 & 0.3667 & -0.1834 & 0.3397 \\ -0.3112 & 0.3290 & -0.3048 & -0.3581 \\ \hline -0.3859 & 0.3529 & 0.3843 & 0.0454 \\ \hline -0.3750 & -0.1124 & & \end{array}\right]$$

When this system is open-loop (i.e. $u(k) = 0$), the test for positive realness is failed even using the characterization of Theorem 1. Therefore, it is emergent to design a static output feedback controller (10) to render the closed-loop system is robust SPR. By applying Theorem 5 without the slack variables V_{ij}, one cannot find a feasible solution. At the same time, the design problem can be solved by applying Theorem 5 with the slack variables V_{ij}, and a positive real controller can be obtained as:

$$u(k) = 2.5424y(k)$$

The computation results show that the design method given by Theorem 5 is less conservative than the one given by Theorem 5 without the slack variables V_{ij}.

V. Conclusions

Characterizations of positive realness as well as synthesis of static feedback positive real controller for linear discrete-time systems are investigated in this paper. The main contributions include: I) New characterizations of positive realness have been proposed. II) The relationships between the proposed characterizations and the existing results have been clarified, which show that our new results are of less conservatism for characterizing the positive realness of discrete-time systems with polytopic uncertainty. III) Static output feedback positive real controller design methods have also been developed. Numerical examples are also given to illustrate the validation and effectiveness of the results.

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