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LMI-based $\mathbf{H}_-/\mathbf{H}_\infty$ Observer Design in Low Frequency Domain with Application in Fault Detection

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Abstract— **This paper deals with the robust fault detection(FD) problem in low frequency domain for linear time-delay systems. The $\mathbf{H}_\infty$ norm and $\mathbf{H}_-$ index are employed to measure the robustness to unknown inputs and the sensitivity to faults, respectively. The main results include derivation of a sufficient condition for the existence of a robust fault detection observer and a construction of it based on the linear matrix inequality(LMI) solution parameters. Finally, numerical simulation demonstrates the effectiveness of the presented methodology.**

I. INTRODUCTION

The research and application of robust FD in automated processes have received considerable attention during last decades and a great number of results have been achieved, see[1-4] and references therein. The main challenge in robust FD is to distinguish faults from other disturbances. There have been a number of results using $\mathbf{H}_\infty$ control theory to solve this problem, e.g., the $\mathbf{H}_\infty/\mathbf{H}_\infty$ approach[4], $\mathbf{H}_\infty$ filter approach[16], and recently developed $\mathbf{H}_-/\mathbf{H}_\infty$ approach[7-12].

However, most of those works were considered in whole frequency spectrum. In practice, however, faults usually emerge in low frequency domain, e.g., for an incipient signal, the fault information is contained within a low frequency band as the fault development is slow[1], and the actuator stuck failures which occur in flight control systems just belong to low frequency domain[14]. This motivated the fault detection observer design for LTI systems in finite frequency domain[12, 15, 23, 24].

On the other hand, time delays are frequently encountered in industry and are often the source of performance degradation of a system. So the main objective of this presented paper focuses on the fault detection observer design in low frequency domain for linear continuous-time delay systems with unknown inputs. The proposed design methodology of this paper is based on the following idea: combining the new results in[6] and $\mathbf{H}_-/\mathbf{H}_\infty$ observer approach, the fault detection problem is converted into a detection observer design problem in low frequency domain, and LMI-based design conditions are derived.

Depending on whether delay is given or not, the existing works can be classified two types: delay given ones[17-19] and delay unknown ones[4,20]. In this paper, we consider the case of a known positive constant time-delay.

This paper is organized as follows: in Section II the problem is formulated, in Section III LMI conditions are outlined and the main results are stated. Section IV gives a numerical example supporting the effectiveness of the proposed approach and some conclusions end this paper in Section V.

The following notations are used throughout this paper. For a matrix A, A^* denotes its complex conjugate transpose. The Hermitian part of a square matrix A is denoted by $\mathbf{He}(A) := A + A^*$. The symbol $\mathbf{H_n}$ stands for the set of $n \times n$ Hermitian matrices. I denotes the identity matrix with an appropriate dimension. For matrices Φ and $\mathbf{P}$, $\Phi \otimes \mathbf{P}$ means the Kronecker product. For matrices $G \in \mathbf{C^{n\times m}}$ and $\Pi \in \mathbf{H_{n+m}}$, a function σ: $\mathbf{C^{n\times m}} \times \mathbf{H_{n+m}} \to \mathbf{H_m}$ is defined by

$$\sigma(G,\Pi) := \begin{bmatrix} G \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I_m \end{bmatrix}.$$

II. PROBLEM FORMULATION

Firstly, we give the definition of $\mathbf{H}_-$ index of a transfer function matrix. Consider a linear time-invariant system described by the following model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

where $x \in R^n$ is the state space vector, $u \in R^{n_u}$ denotes the control input vector, $y \in R^{n_y}$ denotes the measurement output vector, and A, B, C and D are known constant matrices of appropriate dimensions. The transfer function matrix is

$$G(s) = C(sI - A)^{-1}B + D \tag{2}$$

Definition 1[5] The $\mathbf{H}_-$ index of a transfer function matrix $G(s)$ is defined as

$$\|G(s)\|_-^\Omega := \inf_{\omega\in\Omega} \underline{\sigma}[G(j\omega)]$$

where $\underline{\sigma}$ denotes the minimum singular value, Ω is a subset of real numbers as shown in Table I[5].

Faults considered in this paper are assumed to be in low frequency domain, i.e., $\omega \in \Omega = [-\varpi, \varpi]$, where ϖ is a positive scalar.

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We now concentrate our attention on the fault detection problem for linear time-delay systems. The system model under consideration is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + B_f f(t) + B_d d(t), \\ y(t) &= Cx(t) + D_f f(t) + D_d d(t), \end{aligned} \tag{3}$$

where $x(t) \in R^n$ is the state space vector, $y(t) \in R^{n_y}$ denotes the measurement output vector , $d(t) \in R^{n_d}$ is the unknown input vector satisfying $d(t) \in L_2$, $f(t) \in R^{n_f}$ denotes the fault to be detected. A, A_d, B_f, B_d C, D_f, D_d are known matrices with appropriate dimensions and τ is a known constant time-delay. Without loss of generality, we assume (A, C) is observable and control input is omitted.

We propose to use the following fault detection observer:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + A_d \hat{x}(t-\tau) + H(y - \hat{y}) \\ \hat{y(t)} &= C\hat{x}(t) \\ r(t) &= y - \hat{y}, \end{aligned} \tag{4}$$

where $\widehat{x} \in R^n$ and $\widehat{y} \in R^{n_y}$ represent the state and output estimation vectors, respectively. $r(t) \in R^{n_r}$ is the so-called residual signal. The design parameter is observer gain matrix H.

Remark 1: The disturbance considered in the observer design is assumed to be in the same frequency range as that of fault since disturbances that belong to the high frequency domain can be decoupled by designing a low-pass filter after the residual outputs.

Denoting $e(t) = x(t) - \hat{x}(t)$ and augmenting the model of system (3) to include the states of fault detection observer (4), we obtain the following augmented system:

$$\begin{aligned} \dot{e}(t) &= \bar{A}e(t) + A_d e(t-\tau) + \bar{B_d} d(t) + \bar{B_f} f(t) \\ r(t) &= Ce(t) + D_d d(t) + D_f f(t) \end{aligned} \tag{5}$$

where $\bar{A} = A - HC, \bar{B_d} = B_d - HD_d, \bar{B_f} = B_f - HD_f$.

Then, the FD observer design problem can now be formulated as follows:

i: system (5) is asymptotically stable,

ii: $\| Grf(j\omega) \|_{-}^{[-\varpi,\varpi]} > \beta_1$,

iii: $\| Grd(j\omega) \|_{\infty}^{[-\varpi,\varpi]} < \beta_2$, where

$$G_{rf}(s) = C(sI - \bar{A} - e^{-ds}A_d)^{-1}\bar{B_f} + D_f \tag{6}$$

$$G_{rd}(s) = C(sI - \bar{A} - e^{-ds}A_d)^{-1}\bar{B_d} + D_d \tag{7}$$

and β_1, β_2 are two given positive scalars.

Remark 2: Here the $\mathbf{H}_-$ index is used to guarantee the worst-case sensitivity of the residual to faults(see ii) and (iii) is another important performance index called $\mathbf{H}_\infty$ norm which is used to attenuate the disturbance effect.

III. Fault Detection Observer Design

In this section, a solution to FD observer is provided.

The results for linear time-delay systems[6] give a sufficient condition for the transfer function to satisfy a required frequency domain property over restricted frequency ranges in terms of LMI conditions, so these results can be applied to state the above $\mathbf{H}_-$ index and $\mathbf{H}_\infty$ norm of system(5). Then we introduce the main results about[6]. Given a linear time-delay system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + Bd(t), \\ y(t) &= Cx(t) + Dd(t), \end{aligned} \tag{8}$$

where $x(t) \in R^n$ is the state space vector, $y(t) \in R^{n_y}$ denotes the measurement output vector , $d(t) \in R^{n_d}$ is the disturbance input vector. A, A_d, B, C, and D are known matrices with appropriate dimensions and τ is a constant time-delay. The transfer function matrix $G(\lambda)$ from d to y is denoted by

$$G(s) = C(sI - A - e^{-\tau s}A_d)^{-1}B + D \tag{9}$$

Given a Hermitian matrix Π, the specification can be described by

$$\sigma(G(\lambda), \Pi) < 0 \quad \forall \lambda \in \overline{\Lambda}(\Phi, \Psi) \tag{10}$$

where

$$\Lambda(\Phi, \Psi) := \{\lambda \in \mathbf{C} | \sigma(\lambda, \Phi) = 0, \sigma(\lambda, \Psi) \geq 0\} \tag{11}$$

and $\overline{\Lambda} := \Lambda$ if Λ is bounded and $\overline{\Lambda} := \Lambda \bigcup \{\infty\}$ if unbounded.

Lemma 1[6]: Let matrices $A \in \mathbf{C}^{\mathbf{n}\times\mathbf{n}}$, $A_d \in \mathbf{C}^{\mathbf{n}\times\mathbf{n}}$, $B \in \mathbf{C}^{\mathbf{n}\times\mathbf{n}_{\bar{\omega}}}$, $C \in \mathbf{C}^{\mathbf{n_y}\times\mathbf{n}}$, $D \in \mathbf{C}^{\mathbf{n_y}\times\mathbf{n}_{\bar{\omega}}}$, $\Pi \in H_{n_y+n_{\bar{\omega}}}$, and $\Phi, \Psi \in H_2$ be given and define Λ by (11). Suppose Λ represents curves on the complex plane. Then $\sigma(G(\lambda), \Pi) < 0$ holds for all $\lambda \in \overline{\Lambda}(\Phi, \Psi)$ if there exist $P = P^*$, $Q = Q^* > 0$ and $X = X^*$ such that

$$\begin{bmatrix} A & B & A_d \\ I & 0 & 0 \end{bmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{bmatrix} A & B & A_d \\ I & 0 & 0 \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & -X \end{bmatrix} < 0 \tag{12}$$

Remark 3 In the rest of this paper we choose $\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\Psi = \begin{bmatrix} -1 & 0 \\ 0 & \varpi^2 \end{bmatrix}$, then $\lambda \in \overline{\Lambda}(\Phi, \Psi)$ is equivalent to $\omega \in [-\varpi, \varpi]$, where $\lambda = j\omega$.

Furthermore, for the later development, the following Lemmas are required also:

Lemma 2(Finsler's Lemma): Let $\xi \in \mathbf{C}^n$, $\mathscr{P} \in \mathbf{C}^{n\times n}$ and $\mathscr{H} \in \mathbf{C}^{n\times m}$. Let $\mathscr{H}^\perp$ be any matrix such that $\mathscr{H}^\perp \mathscr{H} = 0$. The following statement are equivalent:

i) $\xi^* \mathscr{P} \xi < 0$, $\forall \mathscr{H}^* \xi = 0$, $\xi \neq 0$,

ii) $\mathscr{H}^\perp \mathscr{P} \mathscr{H}^{\perp^*} < 0$,

iii) $\exists \mu \in R : \mathscr{P} - \mu \mathscr{H}\mathscr{H}^* < 0$,

iv) $\exists \mathscr{X} \in R^{m\times n} : \mathscr{P} + \mathscr{H}\mathscr{X} + \mathscr{X}^* \mathscr{H}^* < 0$.

Lemma 3(Elimination Lemma)[21] Let Γ, Λ and $\Theta = \Theta^*$ be given matrices. There exists a matrix F to solve the matrix inequality

$$\Gamma F \Lambda + (\Gamma F \Lambda)^* + \Theta < 0$$

if and only if the following conditions are satisfied

$$\Gamma^\perp \Theta \Gamma^{\perp^*} < 0$$

$$\Lambda^{*\perp} \Theta \Lambda^{*\perp^*} < 0.$$

A. Fault Sensitivity Condition

In this section, the fault sensitivity condition is considered. Let $d(t) = 0$ in (5), we have

$$\begin{aligned} \dot{e}(t) &= \bar{A}e(t) + A_d e(t-\tau) + \bar{B}_f f(t) \\ r(t) &= Ce(t) + D_f f(t). \end{aligned} \tag{13}$$

If we choose $\Pi = \begin{bmatrix} -I & \\ & \beta_1^2 I \end{bmatrix}$ and Φ, Ψ as given in Remark 3 , then for system (13), the performance (10) becomes $\| G_{rf}(j\omega) \|_{-}^{[-\varpi,\varpi]} > \beta_1,\ \omega \in [-\varpi,\varpi]$.

Theorem 1 Consider system (13), let a symmetric matrix $\Pi_1 = \begin{bmatrix} -I & \\ & \beta_1^2 I \end{bmatrix} \in \mathbf{R}^{(\mathbf{n_r}+\mathbf{n_f})\times(\mathbf{n_r}+\mathbf{n_f})}$ and $\Phi, \Psi, \beta_1 > 0$ be given. Suppose $R_1 \in \mathbf{C}^{\mathbf{n}\times(\mathbf{4n}+\mathbf{n_f}+\mathbf{n_r})}$ satisfies

$$YT \begin{bmatrix} \Phi\otimes P_1 + \Psi\otimes Q_1 & 0 & 0 & 0 \\ 0 & \Pi_1 & 0 & 0 \\ 0 & 0 & X_1 & 0 \\ 0 & 0 & 0 & -X_1 \end{bmatrix} T^* Y^* - \mu_1 Y R_1^* R_1 Y^* < 0 \tag{14}$$

$$Y := \begin{bmatrix} A^* - C^* H^* & I & C^* & I & 0 & 0 \\ B_f^* - D_f^* H^* & 0 & D_f^* & 0 & I & 0 \\ A_d^* & 0 & 0 & 0 & 0 & I \\ I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

where $\mu_1 > 0$ is a real scalar and T is the permutation matrix such that

$$[M_1, M_2, M_3, M_4, M_5, M_6]T = [M_1, M_2, M_3, M_5, M_4, M_6] \tag{16}$$

for arbitrary matrices M_1, M_2, M_3, M_4, M_5, and M_6 with column dimensions n, n, n_r, n, n_f and n, respectively. If there exist $P_1 = P_1^*$, $Q_1 = Q_1^* > 0$, $X_1 = X_1^*$, W, V_{f1}, V_{f2}, and $\mathscr{K}$ such that the following inequality

$$T \begin{bmatrix} \Phi\otimes P_1 + \Psi\otimes Q_1 & 0 & 0 & 0 \\ 0 & \Pi_1 & 0 & 0 \\ 0 & 0 & X_1 & 0 \\ 0 & 0 & 0 & -X_1 \end{bmatrix} T^* < \mathbf{He} \begin{bmatrix} WR_1 \\ V_{f1} \\ V_{f2} \\ -A^* W R_1 + C^* \mathscr{K} R_1 - V_{f1} - C^* V_{f2} \\ -B_f{}^* W R_1 + D_f^* \mathscr{K} R_1 - D_f^* V_{f2} \\ -A_d^* W R_1 \end{bmatrix} \tag{17}$$

holds, then there is a fault detection observer satisfying $\| G_{rf}(j\omega) \|_{-}^{[-\varpi,\varpi]} > \beta_1$. In this case, the observer gain matrix is given by

$$\mathscr{K} := H^* W. \tag{18}$$

Proof: By Lemma 1, the performance $\| G_{rf}(j\omega) \|_{-}^{[-\varpi,\varpi]} > \beta_1$ is satisfied if the following inequality

$$\begin{bmatrix} \Xi & I \end{bmatrix} T\Delta T^* \begin{bmatrix} \Xi & I \end{bmatrix}^* < 0, \tag{19}$$

$$\Xi := \begin{bmatrix} A^* - C^* H^* & I & C^* \\ B_f^* - D_f^* H^* & 0 & D_f^* \\ A_d^* & 0 & 0 \end{bmatrix}$$

$$\Delta := \begin{bmatrix} \Phi\otimes P_1 + \Psi\otimes Q_1 & 0 & 0 & 0 \\ 0 & \Pi_1 & 0 & 0 \\ 0 & 0 & X_1 & 0 \\ 0 & 0 & 0 & -X_1 \end{bmatrix} \tag{20}$$

holds, where T is defined by (16) .

We let $\mathscr{P} := T\Delta T^*$, $\mathscr{H}^{\perp^*} := \begin{bmatrix} \Xi & I \end{bmatrix}^*$ and $\mathscr{H} = \begin{bmatrix} I \\ -\Xi \end{bmatrix}$, then condition (19) is equivalent to the item ii) of Lemma 2 and the following inequality

$$T \begin{bmatrix} \Phi\otimes P_1 + \Psi\otimes Q_1 & 0 & 0 & 0 \\ 0 & \Pi_1 & 0 & 0 \\ 0 & 0 & X_1 & 0 \\ 0 & 0 & 0 & -X_1 \end{bmatrix} T^* < \mathbf{He}\left(\begin{bmatrix} I \\ -\Xi \end{bmatrix} \mathscr{X} \right) \tag{21}$$

is equivalent to the item iv) of Lemma 2, where $\mathscr{X}$ is a multiplier. So by Lemma 2, condition (19) is equivalent to condition (21).

However, in Lemma 2, we should notice the equivalence between the item iv) and other items needs the structure of $\mathscr{X}$ in iv) has no any constraint, once we add additional constraint to $\mathscr{X}$, then iv) will be sufficient condition for other items.

In order to make the problem tractable, similar to that of [13], we restrict the class of multiplier $\mathscr{X}$ to be

$$\mathscr{X} := \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} W R_1 + \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} V_f \tag{22}$$

where $W \in \mathbf{C}^{\mathbf{n}\times\mathbf{n}}$, $\det(W) \neq 0$, $V_f \in \mathbf{C}^{(\mathbf{n}+\mathbf{n_r})\times(\mathbf{4n}+\mathbf{n_f}+\mathbf{n_r})}$ and $R_1 \in \mathbf{C}^{\mathbf{n}\times(\mathbf{4n}+\mathbf{n_f}+\mathbf{n_r})}$ is a multiplier to be chosen. Then (19) will be held if

$$T \begin{bmatrix} \Phi\otimes P_1 + \Psi\otimes Q_1 & 0 & 0 & 0 \\ 0 & \Pi_1 & 0 & 0 \\ 0 & 0 & X_1 & 0 \\ 0 & 0 & 0 & -X_1 \end{bmatrix} T^* < \mathbf{He}\left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ -A^* + C^* H^* & -I & -C^* \\ -B_f^* + D_f^* H^* & 0 & -D_f^* \\ -A_d^* & 0 & 0 \end{bmatrix} \begin{bmatrix} W R_1 \\ V_f \end{bmatrix} \right) \tag{23}$$

holds. Defining $\mathscr{K} = H^* W$ and $V_f := \begin{bmatrix} V_{f1} \\ V_{f2} \end{bmatrix}$, with some matrix manipulations, we have that (23) is equivalent to (17), then we have that condition (17) provides a sufficient condition for performance index $\| G_{rf}(j\omega) \|_{-}^{[-\varpi,\varpi]} > \beta_1$, which completes the proof.

Remark 4 As pointed out in [6] and [13], we can choose R_1 to satisfy (14) with Y defined by (15). If R_1 is given, condition (17) is an LMI in P_1, Q_1, X_1, W, V_{f1}, V_{f2}, and $\mathcal{K}$.

Remark 5 By the condition (16), we can get

$$T = \begin{bmatrix} I_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_5 & 0 \\ 0 & 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_6 \end{bmatrix}$$

where $I_n (n = 1, 2 \cdots 6)$ denote identity matrices with appropriate dimensions.

B. Robustness Condition

Here, we study the robustness requirement of system (5). Let $f(t) = 0$ in (5), then we have

$$\begin{aligned} \dot{e}(t) &= \bar{A}e(t) + A_d e(t-\tau) + \bar{B}_d d(t) \\ r(t) &= Ce(t) + D_d d(t). \end{aligned} \tag{24}$$

To attenuate the disturbance influence, we give the following theorem.

Theorem 2 Consider system (24), let a symmetric matrix $\Pi_2 = \begin{bmatrix} I & \\ & -\beta_2^2 I \end{bmatrix} \in \mathbf{R}^{(\mathbf{n_r}+\mathbf{n_d})\times(\mathbf{n_r}+\mathbf{n_d})}$ and $\Phi, \Psi, \beta_2 > 0$ be given. Suppose $R_2 \in \mathbf{C}^{\mathbf{n}\times(\mathbf{4n}+\mathbf{n_d}+\mathbf{n_r})}$ satisfies

$$YT \begin{bmatrix} \Phi\otimes P_2 + \Psi\otimes Q_2 & 0 & 0 & 0 \\ 0 & \Pi_2 & 0 & 0 \\ 0 & 0 & X_2 & 0 \\ 0 & 0 & 0 & -X_2 \end{bmatrix} T^* Y^* - \mu_2 Y R_2^* R_2 Y^* < 0 \tag{25}$$

$$Y := \begin{bmatrix} A^* - C^* H^* & I & C^* & I & 0 & 0 \\ B_d^* - D_d^* H^* & 0 & D_d^* & 0 & I & 0 \\ A_d^* & 0 & 0 & 0 & 0 & I \\ I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

where $\mu_2 > 0$ is a real scalar and T is defined by (16). If there exist $P_2 = P_2^*$, $Q_2 = Q_2^* > 0$ and $X_2 = X_2^*$, W, V_{d1}, V_{d2}, and $\mathcal{K}$ such that the following inequality

$$T \begin{bmatrix} \Phi\otimes P_2 + \Psi\otimes Q_2 & 0 & 0 & 0 \\ 0 & \Pi_2 & 0 & 0 \\ 0 & 0 & X_2 & 0 \\ 0 & 0 & 0 & -X_2 \end{bmatrix} T^* < \mathbf{He}\begin{bmatrix} WR_2 \\ V_{d1} \\ V_{d2} \\ -A^*WR_2 + C^*\mathcal{K}R_2 - V_{d1} - C^*V_{d2} \\ -B_d{}^*WR_2 + D_d^*\mathcal{K}R_2 - D_d^*V_{d2} \\ -A_d^*WR_2 \end{bmatrix} \tag{27}$$

holds, then there is a fault detection observer satisfying $\| G_{rd}(j\omega) \|_\infty^{[-\varpi,\varpi]} < \beta_2$, where $G_{rd}(s) = C(sI - \bar{A} - e^{-ds}A_d)^{-1}\bar{B}_d + D_d$. In this case, the observer gain matrix is given by

$$\mathcal{K} := H^* W. \tag{28}$$

Proof: The proof is similar to Theorem 1, so we omitted.

C. Stability condition

Conditions (17) and (27) don't ensure a stable observer, so we wish to add an additional constraint to guarantee the stability of system (5).

Lemma 4 System (5) is asymptotically stable if there exist matrices $W, \mathcal{K}, P_3 = P_3^* > 0$ and $X_3 = X_3^* > 0$ such that

$$\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} (\Phi\otimes P_3) \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}^* + \begin{bmatrix} 0 & 0 & 0 \\ 0 & X_3 & 0 \\ 0 & 0 & -X_3 \end{bmatrix} < \mathbf{He}\left(\begin{bmatrix} W \\ -A^*W + C^*\mathcal{K} \\ -A_d^*W \end{bmatrix} \begin{bmatrix} -qI & pI & 0 \end{bmatrix} \right), \tag{29}$$

where $r := [p^* \;\; q^*] \in \mathbf{C^2}$ is an arbitrary fixed vector satisfying $r\Phi r^* < 0$.

Proof: From the Lyaponov stability conditions for time-delay systems, system (5) is stable if there exist symmetric matrices $P_3 > 0, X_3 > 0$ such that

$$\begin{bmatrix} \bar{A} & A_d \\ I & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} 0 & P_3 & 0 \\ P_3 & X_3 & 0 \\ 0 & 0 & -X_3 \end{bmatrix} \begin{bmatrix} \bar{A} & A_d \\ I & 0 \\ 0 & I \end{bmatrix} < 0. \tag{30}$$

Notice $\begin{bmatrix} \bar{A}^* & I & 0 \\ A_d^* & 0 & I \end{bmatrix}$ is the null space of $\begin{bmatrix} -I \\ \bar{A}^* \\ A_d^* \end{bmatrix}$, using Lemma 3, (30) will be held if the following inequality

$$\begin{bmatrix} 0 & P_3 & 0 \\ * & X_3 & 0 \\ * & * & -X_3 \end{bmatrix} < \mathbf{He} \begin{bmatrix} -I \\ \bar{A}^* \\ A_d^* \end{bmatrix} WR \tag{31}$$

holds, here we choose W as that in Theorem 1 and Theorem 2, and $R = \begin{bmatrix} -qI & pI & 0 \end{bmatrix}$, where $r := [p^* \;\; q^*] \in \mathbf{C^2}$ is an arbitrary fixed vector satisfying $r\Phi r^* < 0$. Let $\mathcal{K} := H^*W$, then Lemma 4 is completed.

Remark 6 By [6], if we choose $\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $Q = 0$, then we get the LMI conditions for fault detection in full frequency domain.

D. Detection Observer Design

Combining Theorem 1, Theorem 2 and Lemma 4, conditions i, ii, iii which stated in Section II will be satisfied if LMIs (17), (27),(29) hold simultaneously.

Theorem 3 System (5) is asymptotically stable and conditions $\| G_{rf}(j\omega) \|_-^{[-\varpi,\varpi]} > \beta_1$, $\| G_{rd}(j\omega) \|_\infty^{[-\varpi,\varpi]} < \beta_2$ are satisfied if there exist symmetric matrices $P_1, P_2, X_1, X_2, P_3 > 0, Q_1 > 0, Q_2 > 0, X_3 > 0$, and matrices $W, V_{f1}, V_{f2}, V_{d1}, V_{d2}, \mathcal{K}$ such that (17), (27), (29) hold, where $\mathcal{K} := H^*W$.

Given $\beta_2 > 0$, the observer gain matrix H can be determined through the following optimization:

$$\begin{aligned} & max \quad \beta_1 \\ & s.t. (17), (27), (29) \end{aligned} \tag{32}$$

IV. EXAMPLE

To illustrate the effectiveness of the proposed fault detection scheme, a numerical example is given in this section. Consider the linear time-delay system given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + B_f f(t) + B_d d(t), \\ y(t) &= Cx(t) + D_f f(t) + D_d d(t), \end{aligned} \tag{33}$$

with the following parameters

$$A = \begin{bmatrix} -0.9231 & 0.5422 \\ -0.9442 & -0.6764 \end{bmatrix}, \ A_d = \begin{bmatrix} 0.6264 & -0.7227 \\ 0.0117 & -0.1610 \end{bmatrix},$$

$$B_f = \begin{bmatrix} 0.4141 \\ -0.3287 \end{bmatrix}, \ B_d = \begin{bmatrix} 0.2093 \\ 0.1224 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.5432 & 0.4595 \end{bmatrix}, \ D_f = 0.7525, \ D_d = 0.0834.$$

The frequency range is restricted in $(-0.01, 0.01)$. Set $q = -1, p = 1$ and given $\beta_2 = 0.2$, furthermore we choose $R_1 = \begin{bmatrix} I_2 & I_2 & R_{13} & I_2 & R_{15} & 0 \end{bmatrix}$, $R_{13} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$, $R_{15} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $R_2 = \begin{bmatrix} R_{21} & R_{22} & 0 & I_2 & 0 & 0 \end{bmatrix}$, $R_{21} = R_{22} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and T as stated in Remark 5. Solving the optimization problem (32), we obtain the observer gain matrix $H_{low} = \begin{bmatrix} 0.6557 \\ -0.2478 \end{bmatrix}$ and $\beta_{1optlow} = 0.5297$. The actual achieved value of β_1 in low frequency domain is 0.6328.

In the full frequency domain, the observer gain matrix $H_{full} = \begin{bmatrix} 1.3882 \\ 0.2243 \end{bmatrix}$ and $\beta_{1optfull} = 0.3578$. The actual achieved value of β_1 in full frequency domain is 0.3744.

The system is simulated with a stuck fault signal f(t) such that $f(t) = 5, t \geq 6s$ and $f(t) = 0$ elsewhere.

In the case of unknown input $d(t) = 0$ and $d(t) = sin(0.05t)$, the generated residuals are illustrated in Fig1 and Fig2, respectively.

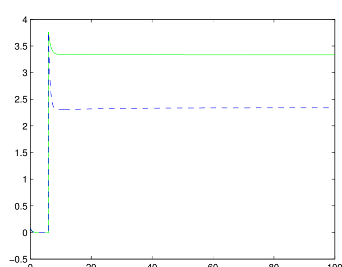

Fig. 1. Residual outputs of the low frequency method(solid lines) and full frequency method(dashed lines) with $d(t) = 0$

From Fig.1,Fig.2, we can conclude that the effects of faults in residual are less susceptible to the disturbance effects in low frequency domain than those in full frequency domain.

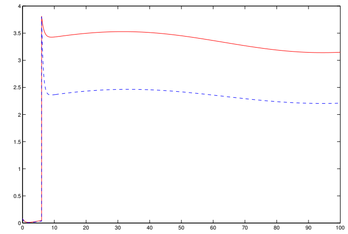

Fig. 2. Residual outputs of the low frequency method(solid lines) and full frequency method(dashed lines) with $d(t) = sin(0.05t)$

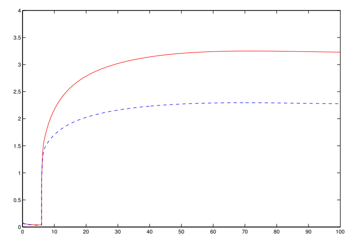

Fig. 3. Residual evaluation of the low frequency method(solid lines) and full frequency method(dashed lines) and the threshold(dash-dot lines)

After designing FD observer, the remaining important task is the evaluation of the generated residual. One of the widely adopted approaches is to choose a so-called threshold $J_{th} > 0$, and based on this, using the following logical relationship for fault detection:

$$J_r > J_{th} \Rightarrow with faults \Rightarrow alarm,$$
$$J_r \leq J_{th} \Rightarrow no faults,$$

where the so-called residual evaluation function J_r is determined by $J_r = \sqrt{\frac{1}{t}\int_0^t r^T(\tau) r(\tau) d\tau}$. Here, we set

$$J_{th} = \sup_{f=0, d \in L_2, \omega \in (-0.01, 0.01)} J_r.$$

The residual evaluation function J_r and threshold J_{th} are reported in Fig.3 and Fig.4 for the same disturbance and fault signal in Fig.2. Using MATLAB we obtain $J_{th} = 0.1469$. From Fig.4 we can conclude faults can be effectively detected by using finite frequency FD observer and Fig.3 illustrates finite frequency FD observer can receive better results than full frequency FD observer.

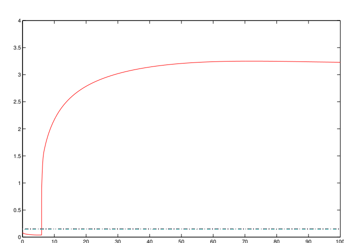

Fig. 4. Residual evaluation of the low frequency method(solid lines) and full frequency method(dashed lines) and the threshold(dash-dot lines)

V. Conclusions

In this paper, we have investigated the problem of fault detection for linear time-delay systems in low frequency domain. The $\mathbf{H}_\infty$ norm and $\mathbf{H}_-$ index have been employed to measure the robustness to unknown inputs and the fault sensitivity, respectively. A design method has been presented in terms of solutions to a set of LMIs and numerical example has been given to illustrate the effectiveness of the proposed method.

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