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Stability of Feedback Switched Systems with State and Switching Delays

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Abstract—**We study stability of state feedback switched systems in which time delays are present in both the feedback state and the switching signal of the controller. For switched linear systems with average dwell-time switching signals, we provide a condition in terms of upper bounds on the delays and a lower bound on the average dwell-time to guarantee asymptotic stability of the closed loop. Our approach employs multiple Lyapunov functions and the merging switching signal technique. We then apply our stability results in switched systems to consensus networks with asymmetric time-varying delays and switching topologies.**

I. Introduction

Switched systems are dynamical systems represented by a family of subsystems and a switching signal that indicates the active subsystem at every time (see, e.g., [1] for further background and references on switched systems). In this paper, we address stability of *feedback switched systems with delays*. By a feedback switched system, we mean a switched plant connected in a closed loop with a switched feedback controller.

In the ideal case, the controller has instant access to both the plant's state and the switching signal. In such cases, the controller's switching and the plant's switching are synchronized, and the closed loop can be represented by a single switched system, for which various tools for analyzing stability are available (e.g., [2], [3]). However, when delays exist between the plant and the controller (for example, when the plant and the controller communicate via a communication channel), there could be both state delays and switching delays, resulting in the closed loop system having asynchronous switching signals (one from the plant and one from the controller) as well as delayed states (in the controller). It is then of interest in control research to find conditions on the original switched system and the delays to guarantee stability of the closed loop.

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Stability of switched systems with delays is a fairly new research topic within the control systems community and only recently has been treated [4], [5], [6], [7]. Most of the results assume state delays only, without switching delays. The work [8] has considered stabilization of feedback switched systems with switching delays but without state delays and with dwell-time switching. In this work, we consider both state delays and switching delays as well as more general classes of switching signals (average dwell-time switching signals).

Our contribution is to show that stability of feedback switched systems with average dwell-time switching is robust with respect to delays in both the state and the switching signal, and we provide an explicit quantification of such robustness in terms of bounds on the average dwell-time, the chatter bound, and the delays. Another contribution is to provide a multiple Lyapunov functions technique for analyzing stability of switched systems with delays, building upon the technique to deal with state delays in [9], the small-gain technique in [9], [10], and the average dwell-time switching concept [2].

The type of feedback switched systems with delays described here could find application, for example, in consensus networks (see Section IV), or in multi-modal control systems where controller selection takes a finite amount of time (e.g., controller selection is carried out by human operators).

The notations in this paper are fairly standard. Denote by $|z|$ the Euclidean norm of a real vector z. For a matrix M, denote by $\|M\|$ the induced matrix norm. Define $\|f\|_{\mathcal{D}} := \sup_{s\in\mathcal{D}} |f(s)|$, where $\mathcal{D} \subseteq [0,\infty)$.

II. Problem formulation

A switched linear control system is of the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $\sigma : [0,\infty) \to \mathcal{P}$ is the switching signal mapping time to some finite index set $\mathcal{P}$, and $A_p \in \mathbb{R}^{n\times n}, B_p \in \mathbb{R}^{n\times m}$, $p \in \mathcal{P}$, are the state and input matrices. The *switching signal* σ is a piecewise-constant continuous-from-the-right function taking values in the index set

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$\mathcal{P}$. The discontinuities of σ are called *switching times* or *switches*. As often assumed in the switched system literature, no state jump occurs at a switching time, and there are finitely many switches in every finite interval (i.e., no Zeno behavior).

Assumption 1 *(A_p, B_p) are stabilizable $\forall p \in \mathcal{P}$.*

Let K_p be matrices such that $A_p + B_p K_p$ is Hurwitz $\forall p \in \mathcal{P}$. The ideal switched state feedback controller is

$$u = K_\sigma x, \tag{2}$$

and the closed loop is $\dot{x} = (A_\sigma + B_\sigma K_\sigma)x$. Recall [2] that a switching signal σ is an *average dwell-time signal* if the number of switches in any interval $[t_0, t)$, denoted by $N_\sigma(t, t_0)$, satisfies

$$N_\sigma(t, t_0) \leqslant N_0 + \frac{t - t_0}{\tau_{\mathbf{a}}} \quad t \geqslant t_0$$

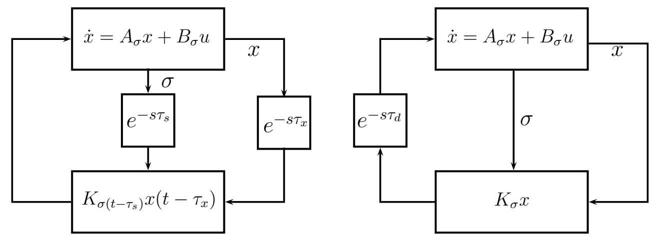
for some constant $N_0 \geqslant 1$; N_0 is called a *chatter bound*. Denote by $\mathcal{S}_{\text{ave}}[\tau_{\mathbf{a}}, N_0]$ the class of switching signals with average dwell-time $\tau_{\mathbf{a}}$ and chatter bound N_0. It is well-known [2] that if the switching signal σ has an average dwell-time $\tau_{\mathbf{a}}$, then there exists a positive number $\tau_{\mathbf{a}}^*$ (which depends on $A_p + B_p K_p$) such that the switched system $\dot{x} = (A_\sigma + B_\sigma K_\sigma)x$ is asymptotically stable for all $\tau_{\mathbf{a}} \geqslant \tau_{\mathbf{a}}^*$.

Now suppose that there is a delay $\tau_{\mathbf{x}}$ between the plant's output (which is assumed to be the exact state in this paper) and the state available to the controller, i.e. the controller receives the state $x(t - \tau_{\mathbf{x}})$ at time t. Because the controller is also switching, there are two scenarios here: 1) the switching signal available to the controller is synchronized with the switching signal σ of the plant, or 2) the controller's switching signal is a delayed version of the plant's switching signal. The first case is possible, for example, when the switching signal is generated by timing, and the plant and the controller use the same clock. The second case occurs, for example, when information about the switching signal of the plant has to be sent to the controller over a communication channel with delay (see Fig 1a). In the first case, the control signal going into the plant is $u(t) = K_{\sigma(t)} x(t - \tau_{\mathbf{x}})$, and in the second case, $u(t) = K_{\sigma(t-\tau_{\mathbf{s}})} x(t - \tau_{\mathbf{x}})$, where $\tau_{\mathbf{x}}$ is the state delay, and $\tau_{\mathbf{s}}$ is the switching delay. Another type of delay is input delay (see Fig 1b), in which case the control signal going into the plant is $u(t) = K_{\sigma(t-\tau_d)} x(t - \tau_d)$.

In general, the control signal going into the plant is of the following form:

$$u(t) = K_{\sigma(t-\tau_{\mathbf{s}})} x(t - \tau_{\mathbf{x}}) \tag{3}$$

for some non-negative constants $\tau_{\mathbf{s}}$ and $\tau_{\mathbf{x}}$. The formula (3) also covers the case where input, output, and switching delays are all present (i.e. superimposing Fig. 1a on Fig. 1b), in which case $u(t) = K_{\sigma(t-\tau_{\mathbf{s}}-\tau_d)} x(t - \tau_{\mathbf{x}} - \tau_d)$ and, hence, is also of the form (3).



a. Output delay b. Input delay

Fig. 1. Feedback switched systems with delays

For the closed loop system consisting of (1) and (3), for every initial state $x_0 : [-\tau_{\mathbf{x}}, 0] \to \mathbb{R}^n$, piecewise continuous input $u : [0, \infty) \to \mathbb{R}^m$, and switching signal $\sigma : [-\tau_{\mathbf{s}}, \infty] \to \mathcal{P}$, a solution (or trajectory) x exists for all time in $[0, \infty)$ and is unique in the Caratheodory sense.

For $\tau_{\mathbf{x}} = \tau_{\mathbf{s}} = 0$, we have asymptotic stability of the closed-loop switched system under average dwell-time switching, and for large enough $\tau_{\mathbf{x}}$ and $\tau_{\mathbf{s}} = 0$, we may have instability. Intuitively, for $\tau_{\mathbf{s}} = 0$, one could expect to find an upper bound on $\tau_{\mathbf{x}}$ to guarantee closed-loop stability[1] (although, rigorously, even the existence of such a bound is not apparent for switched systems and has to be proved). Stability of the closed loop is even more challenging when there are switching delays, i.e. $\tau_{\mathbf{s}} \neq 0$. The main problem is to determine and quantify if upper bounds on $\tau_{\mathbf{x}}$ and $\tau_{\mathbf{s}}$ for closed loop stability indeed exist and for which type of switched systems.

Problem: *For the switched system* (1) *with the control* (3)*, find upper bounds on $\tau_{\mathbf{x}}$ and $\tau_{\mathbf{s}}$ and classes of the switching signal σ to guarantee asymptotic stability of the closed loop.*

Compared to the case without delays [2], the difficulties in the case with delays are due to: 1) the mismatch between the states $x(t)$ and $x(t - \tau_{\mathbf{x}})$ in the control law, and 2) the mismatch between the indices of $\sigma(t)$ and $\sigma(t - \tau_{\mathbf{s}})$ when t is near switching times.

III. Main result

Our main theorem characterizes the relationship between the delays and the average dwell-time of the

[1] An upper bound is a sufficient condition only; it is possible to have stable non-switched linear systems with arbitrarily large delays.

plant's switching signal to guarantee asymptotic stability.

Before presenting the theorem, we define some notation and variables. Let $\bar{A}_{p,q} := A_p + B_p K_q$ and let $\mathcal{P}^2 := \mathcal{P} \times \mathcal{P}$. Let $\mathcal{P}_s^2$ be the set of all indices in $\mathcal{P}^2$ such that $\exists$ positive definite quadratic functions $V_{p,q} : \mathbb{R}^n \to [0, \infty)$ such that

$$\frac{\partial V_{p,q}}{\partial x}(\bar{A}_{p,q}x + v) \leqslant -\lambda_s V_{p,q} + \gamma |v|^2 \quad \forall v \in \mathbb{R}^n \tag{4}$$

for all $(p,q) \in \mathcal{P}_s^2$ and for some $\lambda_s, \gamma > 0$. Let $\mathcal{P}_u^2 := \mathcal{P}^2 \setminus \mathcal{P}_s^2$. From linear system theory, there always exist positive definite quadratic functions $V_{p,q} : \mathbb{R}^n \to [0, \infty)$ such that

$$\frac{\partial V_{p,q}}{\partial x}(\bar{A}_{p,q}x + v) \leqslant \lambda_u V_{p,q} + \gamma |v|^2 \quad \forall v \in \mathbb{R}^n \tag{5}$$

for all $\forall (p,q) \in \mathcal{P}_u^2$ and for some $\lambda_u, \gamma > 0$. Because $V_{p,q}$ are quadratic positive definite, there exist $\alpha_1, \alpha_2 > 0$, and $\mu \geqslant 1$ such that

$$\alpha_1 |x|^2 \leqslant V_{p,q}(x) \leqslant \alpha_2 |x|^2 \quad \forall (p,q) \in \mathcal{P}^2 \tag{6a}$$

$$V_{p_1,q_1}(x) \leqslant \mu V_{p_2,q_2}(x) \quad \forall (p_1,q_1),(p_2,q_2) \in \mathcal{P}^2. \tag{6b}$$

It is always possible to set $\mu = \alpha_2/\alpha_1$, but a smaller μ may exist, depending on the particular V_j. Define $c_B := \sup_{(p,q)\in\mathcal{P}^2} \|B_p K_q\|$ and $c_A := \sup_{p\in\mathcal{P}} \|A_p\|$. Let $c_1 := c_A + c_B$. Following the notation in [9], define $x_d(t) := \|x\|_{[t-2\tau_{\mathbf{x}},t]}$. We have the following result (see the sketch of the proof in the Appendix).

Theorem 1 *Consider the switched system* (1) *with the controller* (3). *Suppose that the set $\mathcal{P}_s^2 \neq \emptyset$, where $\mathcal{P}_s^2$ is as in* (4), *and $\sigma \in \mathcal{S}_{ave}[\tau_{\mathbf{a}}, N_0]$. Let*

$$\bar{\tau}_{\mathbf{a}} := \tau_{\mathbf{a}}/2, \quad \overline{N}_0 := 2N_0 + \tau_{\mathbf{s}}/\tau_{\mathbf{a}}. \tag{7}$$

If all of the following conditions hold,

$$\bar{\tau}_{\mathbf{a}} > \frac{\ln \mu}{\lambda} \tag{8a}$$

$$\tau_{\mathbf{s}} < \frac{\lambda_s - \lambda}{\lambda_s + \lambda_u}\tau_{\mathbf{a}} \tag{8b}$$

$$(\tau_{\mathbf{x}} c_1 c_B)^2 \kappa < \lambda - \ln \mu / \bar{\tau}_{\mathbf{a}} \tag{8c}$$

for some $\lambda \in (0, \lambda_s)$, where $\kappa := \mu^{\overline{N}_0} \exp((\lambda_s + \lambda_u)(N_0 + 1)\tau_{\mathbf{s}})(a_2\gamma/a_1)$, then we have

$$|x_d(t)|^2 \leqslant (g_0 e^{2\lambda' \tau_{\mathbf{x}}} e^{-\lambda'(t-t_0)} + g_1(\tau_{\mathbf{x}}))\, |x_d(t_0)|^2 \quad \forall t \geqslant t_0 \geqslant \tau_{\mathbf{x}} \tag{9}$$

for some constants $\lambda', g_0 > 0$ and function $g_1 : [0,\infty) \to [0,\infty)$ such that $g_1(\tau_{\mathbf{x}}) \to 0$ as $\tau_{\mathbf{x}} \to 0$.

Remark 1 *The inequality* (4) *is equivalent to saying that the systems $\dot{x} = \bar{A}_{p,q}x$ are asymptotically stable for all $(p,q) \in \mathcal{P}_s^2$, for which one can have $V_{p,q} = x^T P_{p,q} x$, where $P_{p,q}$ is the solution to the Lyapunov equation $(\bar{A}_{p,q} + \frac{\lambda_s}{2}I)^T P_{p,q} + P_{p,q}(\bar{A}_{p,q} + \frac{\bar{\lambda}_s}{2}I) = -I$, and $-\bar{\lambda}_s/2$ is the smallest real part of the eigenvalues of $\bar{A}_{p,q}, (p,q) \in \mathcal{P}_s^2$, $0 < \lambda_s < \bar{\lambda}_s$. The set $\mathcal{P}_s^2$ will contain at least all the elements (p,p), $\forall p \in \mathcal{P}$, but it can contain other pairs of the form (p,q) where $p \neq q$. The existence of a common λ_s and a common γ in* (4) *and the existence of a common λ_u and a common γ in* (5) *follow from the fact that $\mathcal{P}$ is finite. Similarly, the existence of a common γ in both* (4) *and* (5) *also follows from the fact that $\mathcal{P}$ is finite.*

Remark 2 *The condition* (8a) *gives a lower bound on $\tau_{\mathbf{a}}$. The condition* (8b) *gives an upper bound on $\tau_{\mathbf{s}}$ for a fixed $\tau_{\mathbf{a}}$ and a lower bound on $\tau_{\mathbf{a}}$ for a fixed $\tau_{\mathbf{s}}$. The condition* (8b) *can be rewritten as $\tau_{\mathbf{s}}/\tau_{\mathbf{x}} < \lambda_s/(\lambda_s + \lambda_u)$, which verbally means that the fraction of the switching delay compared to the average dwell-time must be less than the ratio of the stable pole (absolute value) and the total of the stable pole and the unstable pole (absolute value). The condition* (8c) *gives an upper bound on the state delay $\tau_{\mathbf{x}}$ for fixed $\tau_{\mathbf{s}}$ and $\tau_{\mathbf{a}}$.*

Remark 3 *For time-varying delays, i.e. $\tau_{\mathbf{x}}$ and $\tau_{\mathbf{s}}$ are functions of time, we still have the result claimed in Theorem 1, in which $\tau_{\mathbf{x}}$ is replaced by a bound $\bar{\tau}_{\mathbf{x}}$ on $\tau_{\mathbf{x}}$ such that $\tau_{\mathbf{x}}(t) \leqslant \bar{\tau}_{\mathbf{x}}$ for all t, and $\tau_{\mathbf{s}}$ is replaced by $\bar{\tau}_{\mathbf{s}}$ such that $\tau_{\mathbf{s}}(t) \leqslant \bar{\tau}_{\mathbf{s}}$ for all t.*

Remark 4 *The result in this section remains the same if* (4), (5), *and* (6) *hold for all $x \in \Omega$ for some set Ω, instead of requiring* (4), (5), *and* (6) *hold for all $x \in \mathbb{R}^n$.*

A. Stability

The following result shows robustness of feedback switched systems with respect to delays.

Corollary 1 *Consider the switched system* (1) *with the controller* (3). *Suppose that the set $\mathcal{P}_s^2 \neq \emptyset$, where $\mathcal{P}_s^2$ is as in* (4), *and $\sigma \in \mathcal{S}_{ave}[\tau_{\mathbf{a}}, N_0]$. Suppose that*

$$\tau_{\mathbf{a}} > 2\frac{\ln \mu}{\lambda_s}. \tag{10}$$

There exist positive numbers $\bar{\tau}_{\mathbf{x}}$ and $\bar{\tau}_{\mathbf{s}}$ such that for all $\tau_{\mathbf{x}} < \bar{\tau}_{\mathbf{x}}$ and $\tau_{\mathbf{s}} < \bar{\tau}_{\mathbf{s}}$, we have

$$|x_d(t)|^2 \leqslant (g_2 e^{-\lambda'(t-t_0)} + g_3)|x_d(t_0)|^2 \quad \forall t \geqslant t_0 \geqslant \tau_{\mathbf{x}} \tag{11}$$

for some $g_2, \lambda' > 0$ and $g_3 \in (0,1)$.

Remark 5 *The inequality* (11) *implies asymptotic stability such that x is Lyapunov stable and $|x(t)| \to 0$ as $t \to \infty$. To see this result, let $\epsilon > 0$ be a number such that $g_3 + \epsilon < 1$, and let $T > 0$ be a number such that $g_2 e^{-\lambda T} \leqslant \epsilon$. Then $|x_d(T + t_0)|^2 \leqslant (g_3 + \epsilon)|x_d(t_0)|^2$. Because* (11) *is true for all $t \geqslant t_0 + \tau_{\mathbf{x}}$ and $T \geqslant \tau_{\mathbf{x}}$, we also have $|x_d(2T + t_0)|^2 \leqslant (g_3 + \epsilon)^2 |x_d(T + t_0)|^2$. It follows that $|x_d(kT + t_0)|^2 \leqslant (g_3 + \epsilon)^k |x_d(t_0)|^2$ for all $k = 1, \ldots$. Because $g_3 + \epsilon < 1$, we have that $|x_d(kT + t_0)| \to 0$ as $k \to \infty$. Because $|x_d(kT + t_0 + t)|^2 \leqslant (g_3 + g_2)|x_d(kT + t_0)|^2$ for all $t \in [0, T)$, it follows that $x_d(t)$ is bounded for all $t \geqslant t_0$, and $|x_d(t)| \to 0$ as $t \to \infty$. In view of the fact that $|x(t)| \leqslant |x_d(t)|$, we conclude that the system is asymptotically stable.*

B. Special case: switching delay and no state delay

Corollary 2 *Consider the switched system* (1) *with the controller* (3)*. Suppose that $\tau_{\mathbf{x}} = 0$. Suppose that the set $\mathcal{P}_s^2 \neq \emptyset$, where $\mathcal{P}_s^2$ is as in* (4)*, and $\sigma \in \mathcal{S}_{ave}[\tau_{\mathbf{a}}, N_0]$. Let $\bar{\tau}_{\mathbf{a}}$ be as in* (7)*. If* (8a) *and* (8b) *hold for some $\lambda \in (0, \lambda_s)$, then we have*

$$|x(t)|^2 \leqslant g_1 e^{-\lambda'(t-t_0)} |x(t_0)|^2 \quad \forall t \geqslant t_0 \geqslant 0 \tag{12}$$

for some positive constants λ', g_1.

Remark 6 *If there is no unstable mode even in the case of switching mismatch, then $\lambda_u = 0$. Thus, if $\tau_{\mathbf{s}} = 0$, a strict upper bound on $\tau_{\mathbf{s}}$ can be as large as $\tau_{\mathbf{a}}$, and as $\tau_{\mathbf{a}} \to \infty$, we can have $\tau_{\mathbf{s}} \to \infty$.*

C. Special case: state delay and no switching delay

For the special case $\tau_{\mathbf{s}} = 0$, with a little tweak in the proof of Theorem 1, we have the following stronger result (compare (13a) with (10)).

Corollary 3 *Consider the switched system* (1) *with the controller* (3)*. Suppose that $\tau_{\mathbf{s}} = 0$. Suppose that the set $\mathcal{P}_s^2 \neq \emptyset$, where $\mathcal{P}_s^2$ is as in* (4)*, and $\sigma \in \mathcal{S}_{ave}[\tau_{\mathbf{a}}, N_0]$. If*

$$\tau_{\mathbf{a}} > \frac{\ln \mu}{\lambda} \tag{13a}$$

$$(\tau_{\mathbf{x}} c_1 c_B)^2 \kappa < \lambda - \ln \mu / \tau_{\mathbf{a}} \tag{13b}$$

for some $\lambda \in (0, \lambda_s)$, where $\kappa := \mu^{N_0} a_2 \gamma / a_1$, then we have

$$|x_d(t)|^2 \leqslant (g_1 e^{-\lambda'(t-t_0)} + g_2)|x_d(t_0)|^2 \quad \forall t \geqslant t_0 \geqslant \tau_{\mathbf{x}} \tag{14}$$

for some $g_1, \lambda' > 0$ and $g_2 \in (0,1)$.

Note that (14) implies asymptotic stability (see Remark 5). Corollary 3 shows that with no switching delay, asymptotic stability of switched linear systems with average dwell-time switching [2] (which uses exactly the same condition as (13a)) is robust to small state delays.

Remark 7 *When a common Lyapunov function exists among $\bar{A}_{p,q}$, $(p,q) \in \mathcal{P}_s'$, where $\bar{A}_{p,q}$ are as in* (4)*, then $\mu = 1$, and the average dwell-time $\tau_{\mathbf{a}}$ can be arbitrary small (the righthand side of* (13a) *is zero).*

Remark 8 *The inequality* (13b) *implies that $\tau_{\mathbf{x}} \to 0$ as $\tau_{\mathbf{a}} \to \ln \mu / \lambda$, and $\tau_{\mathbf{x}} \to {\tau_{\mathbf{x}}}^*$ as $\tau_{\mathbf{a}} \to \infty$, where ${\tau_{\mathbf{x}}}^* := 1/(c_1 c_B \sqrt{\mu^{N_0} a_2 \gamma / (a_1 \lambda)})$. The inequality $(\tau_{\mathbf{x}} c_1 c_B)^2 \kappa < \lambda - \ln \mu / \tau_{\mathbf{a}}$ can be rewritten as $\lambda_s (\tau_{\mathbf{x}} / \tau_{\mathbf{x}}^*)^2 + \ln \mu / \tau_{\mathbf{a}} < \lambda$, from which the relationship between $\tau_{\mathbf{x}}$ and $\tau_{\mathbf{a}}$ can be visualized as the shaded area in Fig. 2 (in the figure, $\tau_{\mathbf{a}}^* = \lambda / \ln \mu$). We can calculate that in order to have $\tau_{\mathbf{x}} \geqslant (1 - \epsilon)\tau_{\mathbf{x}}^*$, we need $\tau_{\mathbf{a}} > \tau_{\mathbf{a}}^* / \epsilon$.*

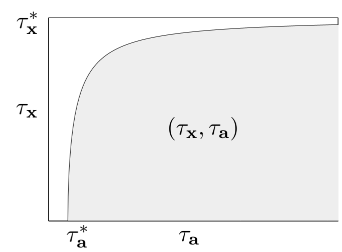


Fig. 2. Relationship between $\tau_{\mathbf{x}}$ and $\tau_{\mathbf{a}}$.

IV. Application

Consensus networks with switching topology and delay

Consider a network of n agents with an undirected communication topology G (G is represented by a graph), whose agents' dynamics are $\dot{x}_i = u_i, i = 1, \ldots, n$, and without loss of generality, assume that $x_i \in \mathbb{R}$. Suppose that each agent employs the consensus protocol $u_i = \sum_{j \in \mathcal{N}_i} x_j - x_i$, where $\mathcal{N}_i$ is the neighborhood of the node i of the graph G (which is the set of nodes of G that have edges with the node i). The reader is referred to, for example, [11] for background on graph theory. If a time-varying communication delay $\tau_{\mathbf{x}}(t)$ is present when information is transmitted between two agents, the actual control signals are

$$u_i(t) = \sum_{j \in \mathcal{N}_i} x_j(t - \tau_{\mathbf{x}}(t)) - x_i(t) \tag{15}$$

(there is no state delay for information from the same agent). The work [12] considers the network $u_i(t) =$

$\sum_{j\in\mathcal{N}_i} x_j(t-\tau_{\mathbf{x}})-x_i(t-\tau_{\mathbf{x}})$ with the delay $\tau_{\mathbf{x}}$ but without topology switching, under which the collective dynamics are a non-switched linear system with delays and can be analyzed using well-known techniques (Nyquist criterion) for time-delay linear systems. Such an approach is not applicable when the topology is time-varying.

With the protocol (15), the collective dynamics will be of the form of a switched system with delay as we shall see. Let $x = (x_1, \ldots, x_n)$ be the collective state of the network. The dynamics of the network are

$$\dot{x}(t) = -D_{\sigma(t)}x(t) + A_{\sigma(t)}x(t-\tau_{\mathbf{x}}),$$

where A_p is the adjacency matrix of G_p, D_p is the degree matrix of G_p, and $\sigma(t) := i : G(t) = G_p, \quad p = 1, \ldots, m$. Define $\delta(t) := x(t) - \frac{1}{n}(\mathbf{1}^T x(t))\mathbf{1} =: F(x(t))$, where $\mathbf{1} \in \mathbb{R}^{\mathbf{n}}$ is the vector of all ones. Then $\mathbf{1}^T\delta = 0$. The vector δ is known as the *disagreement vector* [12].

Assumption 2 *For every $p = 1, \ldots, m$, G_p is k_p-regular[2] for some k_p.*

For a k_p-regular graph G_p, we have $\mathbf{1}^T A_p = k_p\mathbf{1}^T$ and $A_p\mathbf{1} = k_p\mathbf{1}$, and so $(\mathbf{1}^T A_p x)\mathbf{1} = A_p(\mathbf{1}^T x)\mathbf{1} = k_p(\mathbf{1}^T x)\mathbf{1}$. The foregoing equality implies that $F(A_{\sigma(t)}x(t-\tau_{\mathbf{x}})) = A_{\sigma(t)}F(x(t-\tau_{\mathbf{x}}))$. Then

$$\dot{\delta}(t) = -k_{\sigma(t)}\delta(t) + A_{\sigma(t)}\delta(t-\tau_{\mathbf{x}}) \tag{16}$$

in view of $\delta(t-\tau_{\mathbf{x}}) = F(x(t-\tau_{\mathbf{x}}))$. The system (16) can be cast as the feedback switched system

$$\begin{aligned} \dot{\delta} &= -k_\sigma\delta + u \\ u &= A_\sigma\delta(t-\tau_{\mathbf{x}}). \end{aligned} \tag{17}$$

Now, for every $i \in \{1, \ldots, m\}$, we have $A_p - k_pI = -L_i$, where L_i is the Laplacian of the graph G_p. For any undirected graph G, the graph Laplacian L_G has the following property (see, e.g., [12]): $\delta^T L_G\delta \geqslant \lambda_2(L_G)|\delta|^2 \quad \forall\delta : \mathbf{1}^T\delta = 0$, where $\lambda_2(L_G)$ is the smallest nonzero eigenvalue of L_G. Then the quadratic function $V = \delta^T\delta$ satisfies the following property: for every $i \in \{1, \ldots, m\}$, along the trajectory of $\dot{\delta} = -L_i\delta + v$, we have $\dot{V} = -2\delta^T(L_i\delta + v) \leqslant -(2\lambda_2(L_{G_p})-\epsilon)V + 1/\epsilon|v|^2$ for some $\epsilon \in (0, 2\lambda_2(L_{G_p}))$. The foregoing inequality shows that V satisfies the condition (4) with $\lambda_s := \min_i 2\lambda_2(L_{G_p})-\epsilon$ and $\gamma = 1/\epsilon$ for some $\epsilon \in (0, \min_i 2\lambda_2(L_{G_p}))$. Therefore, the set $\mathcal{P}_s^2$ as in (4) is nonempty because $\mathcal{P}_s^2$ contains (p,p) for all $p = 1, \ldots, m$.

From Corollary 3 and Remark 5, if $\sigma \in \mathcal{S}_{\text{ave}}[\tau_{\mathbf{a}}, N_0]$ and $\tau_{\mathbf{a}} > \ln\mu/\lambda_s$, then $\bar{\tau}_{\mathbf{x}}$ exists such that for all initial states and all delays $\tau_{\mathbf{x}} < \bar{\tau}_{\mathbf{x}}$, we have $\delta(t) \to 0$ as $t \to \infty$. From the definition of δ, we have $x_i(t) \to \frac{1}{n}\sum_{j=1}^n x_j(t)$ as $t \to \infty$ for all i, and hence, network consensus is asymptotically achieved. This result is true for arbitrary small $\tau_{\mathbf{a}}$ because $V = \delta^T\delta$ is a common Lyapunov function, and so $\mu = 1$ (see Remark 7).

[2]Recall that a graph is $k-$regular if every node has degree k.

Theorem 2 *Consider a multi-agent network with the protocol* (15) *and a switching topology $G : [0,\infty) \to \{G_1, \ldots, G_m\}$, where G_p are k_p-regular undirected graphs for some k_p, $p = 1, \ldots, m$. For every $\tau_{\mathbf{a}} > 0$ and $N_0 > 0$, there exists a number $\bar{\tau}_{\mathbf{x}} > 0$ such that if the switching signal of the switching topology belongs to $\mathcal{S}_{ave}[\tau_{\mathbf{a}}, N_0]$, and the delay $\tau_{\mathbf{x}}(t) < \bar{\tau}_{\mathbf{x}}$ $\forall t$, then the network of agents will asymptotically reach a consensus for all initial states.*

V. Conclusions

In this work, we addressed stability of feedback switched systems with state and switching delays. We provided conditions in terms of upper bounds on the delays and lower bounds on the average dwell-time of the plant's switching signal to guarantee asymptotic stability. We applied our results in switched systems to study stability of multi-agent dynamics networks with delays and switching topologies. Future work aims to extend the results here to the output feedback case, to switched nonlinear systems, and to other classes of slow switching signals (as in [13]).

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APPENDIX

Due to the space limitation, the proofs of the lemmas below are omitted, and the reader is referred to the full version of this paper [14].

Merging switching signals

A key technique used in this paper to deal with mismatched switching signals σ and σ_c is to merge those switching signals. The idea is to create a (virtual) new switching signal $\sigma' : [0, \infty) \to \mathcal{P} \times \mathcal{P}$ as follows:

$$\sigma'(t) := (\sigma(t), \sigma_c(t)). \tag{18}$$

The merging action is denoted by $\oplus$ such that $\sigma' = \sigma \oplus \sigma_c$.

Lemma 1 *If $\sigma_1 \in \mathcal{S}_{ave}[\tau_{a_1}, N_1]$ and $\sigma_2 \in \mathcal{S}_{ave}[\tau_{a_2}, N_2]$, then $\sigma_1 \oplus \sigma_2 \in \mathcal{S}_{ave}[\tau_{\mathbf{a}}, N_1 + N_2]$, where $\tau_{\mathbf{a}} := (1/\tau_{a_1} + 1/\tau_{a_2})^{-1}$.*

Delayed average dwell-time switching signals

In the case $\sigma_c = \sigma(t - \tau_{\mathbf{s}})$ for some constant $\tau_{\mathbf{s}}$, if $\sigma \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$, then it follows immediately that $\sigma_c \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$. For the case of time varying $\tau_{\mathbf{s}}$, using Lemma 1, we have the following lemma.

Lemma 2 *Let $\sigma_1 \in \mathcal{S}_{ave}[\tau_{\mathbf{a}}, N_0]$ and $\sigma_2(t) := \sigma_1(t - \tau_{\mathbf{s}}(t))$ for some positive function $\tau_{\mathbf{s}}$. Suppose that $\tau_{\mathbf{s}}(t) \leqslant \bar{\tau}_{\mathbf{s}}$ for all t, and $\bar{\tau}_{\mathbf{s}} < \tau_{\mathbf{a}}$. Then $\sigma_2 \in \mathcal{S}_{ave}[\tau_{\mathbf{a}}, N_0 + \bar{\tau}_{\mathbf{s}}/\tau_{\mathbf{a}}]$.*

Lemma 3 *Let $\sigma_1 \in \mathcal{S}_{ave}[\tau_{\mathbf{a}}, N_0]$ and $\sigma_2(t) := \sigma_1(t - \tau_{\mathbf{s}}(t))$ for some positive function $\tau_{\mathbf{s}}$. For an interval (t_0, t), let $m_{t_0,t}$ be the total time at which $\sigma_1(t) = \sigma_2(t)$, and $\bar{m}_{t_0,t} := t - t_0 - m_{t_0,t}$. Suppose that $\tau_{\mathbf{s}}(t) \leqslant \bar{\tau}_{\mathbf{s}}$ $\forall t$. If*

$$\bar{\tau}_{\mathbf{s}}(\lambda_m + \lambda_{\bar{m}}) \leqslant (\lambda_m - \lambda)\tau_{\mathbf{a}} \tag{19}$$

for some $\lambda_m > 0$, $\lambda_{\bar{m}} > 0$, and $\lambda \in [0, \lambda_m]$, then

$$-\lambda_m m_{t_0,t} + \lambda_{\bar{m}} \bar{m}_{t_0,t} \leqslant c_T - \lambda(t - t_0) \ \forall t \geqslant t_0, \tag{20}$$

where $c_T := (\lambda_m + \lambda_{\bar{m}})(N_0 + 1)\bar{\tau}_{\mathbf{s}}$.

SKETCH OF THE PROOF OF THEOREM 1

We outline the key ideas and steps behind the proof (for details, see [14]). The proof comprises five steps:

- **Step 1**: Obtain the closed loop as a switched system with a single switching signal, using the merging switching signal technique:
$$\dot{x}(t) = \bar{A}_{\sigma'(t)} x(t) + \bar{B}_{\sigma'(t)}(x(t - \tau_{\mathbf{x}}) - x(t)),$$
where $\bar{A}_{p,q} = A_p + B_p K_q$, $\bar{B}_{p,q} = B_p K_q$, and $\sigma'(t) := \sigma(t) \oplus \sigma(t - \tau_{\mathbf{s}})$.
- **Step 2**: Bound the difference between x and the delayed version of x in terms of $\bar{\tau}_{\mathbf{x}}$ and x: $|x(t_{s_{k+1}}) - x(t_{s_k})| \leqslant (t_{s_{k+1}} - t_{s_k})\|\dot{x}\|_{[t_{s_k}, t_{s_{k+1}})} \leqslant (t_{s_{k+1}} - t_{s_k})c_1\|x\|_{[t_{s_k} - \tau_{\mathbf{x}}, t_{s_{k+1}}]} \leqslant (t_{s_{k+1}} - t_{s_k})c_1\|x\|_{[t - 2\tau_{\mathbf{x}}, t]}$, where $c_1 = \sup_{p \in \mathcal{P}} \|A_p\| + \sup_{p,q} \|B_p K_q\|$.
- **Step 3**: Construct the candidate Lyapunov function $V(t) := V_{\sigma'(t)}(x(t))$, where V_j are as in (5). Then bound V using Lemma 2 and Lemma 3. The technique in this step is similar to the technique in the original average dwell-time paper [2] and in the extension [15] for mixed stable and unstable subsystems. The result in [15] is not directly applicable here because the dynamics in this paper are feedback systems, and there are state and switching delays, whereas in [15], the system is a switched system, and no delay is present.
- **Step 4**: Bound the Lyapunov function in Step 3 in terms of the initial state and the current sate, utilizing (8a), (8b), and Lemma 2:
$$\begin{aligned} V(T) &\leqslant \mu^{\overline{N}_0} e^{c_T} e^{-\lambda'(T - \bar{t}_0)} \alpha_2 |x(\bar{t}_0)|^2 \\ &+ (\mu^{\overline{N}_0} e^{c_T} \gamma(\tau_{\mathbf{x}} c_B c_1)^2/\lambda')\|x_d\|^2_{[\bar{t}_0, T)} \quad \forall T \geqslant \bar{t}_0. \end{aligned}$$
- **Step 5**: Bound the state using a small-gain technique, utilizing the condition (8c) in the theorem:
$$|x(T)|^2 \leqslant \left(g_0 e^{-\lambda'(T - \bar{t}_0)} + g_1(\tau_{\mathbf{x}})\right) |x_d(\bar{t}_0)|^2.$$