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Optimal Controller for Uncertain Stochastic Polynomial Systems with Deterministic Disturbances

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***Abstract*— This paper presents the optimal quadratic-Gaussian controller for uncertain stochastic polynomial systems with unknown coefficients and matched deterministic disturbances over linear observations and a quadratic criterion. As intermediate results, the paper gives closed-form solutions of the optimal regulator, controller, and identifier problems for stochastic polynomial systems with linear control input and a quadratic criterion. The original problem for uncertain stochastic polynomial systems with matched deterministic disturbances is solved using the integral sliding mode algorithm.**

I. Introduction

Although the optimal quadratic-Gaussian controller problem for linear systems was solved in 1960s, based on the solutions to the optimal filtering [1] and optimal regulator [2], [3] problems, the optimal controller for nonlinear systems has to be determined using the nonlinear filtering theory (see [4], [5], [6]) and the general principles of maximum [2] or dynamic programming [7], which do not provide an explicit form for the optimal control in most cases. There is a long tradition of the optimal control design for nonlinear systems (see, for example, [8]–[13]) and the optimal closed-form filter design for nonlinear [14], [15], [16], and in particular, polynomial ([17]–[20]) systems, as well as the robust filter design for stochastic nonlinear systems (see, for example, [21]–[23]). However, the optimal quadratic-Gaussian controller problem for nonlinear, in particular, polynomial, systems with unknown parameters has not even been consistently treated. Indeed, the optimal solution is not defined, if some parameters are undetermined. The problem becomes even more complicated, if the plant is affected by deterministic disturbances. Nonetheless, the problem statement starts making sense, if unknown parameters are modeled and deterministic disturbances are matched, in other words, belong to a controllable subspace. Taking into account the stochastic Gaussian specifics of the optimal quadratic-Gaussian problem, the unknown parameters are represented as stochastic Wiener processes. The extended state vector consists of the real unmeasured states and unknown parameters, and the obtained extended state equations are polynomial with respect to the extended state vector. The integral sliding mode algorithm for unmeasured states (see [25] for the original version and [26] for a modification) is used for compensating the matched deterministic disturbances optimally with respect to the observations. Other recent developments in the sliding mode theory and applications can be found in [27]–[32].

This paper presents solution to the optimal quadratic-Gaussian controller problem for uncertain stochastic polynomial systems with unknown coefficients and matched deterministic disturbances over linear observations and a quadratic criterion. First, the paper recalls the optimal solution to the quadratic-Gaussian controller problem for incompletely measured stochastic polynomial states with linear control input and a quadratic criterion [33]. Next, the paper provides the optimal solution to the quadratic-Gaussian controller problem with unknown parameters, which is based on the preceding result. Finally, the integral sliding mode algorithm yields a solution to the original quadratic-Gaussian controller problem for uncertain stochastic polynomial systems with deterministic disturbances, that is conditionally optimal with respect to the observations.

II. Optimal Controller Problem

A. Problem statement

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq t_0$, and let $(W_1(t), F_t, t \geq t_0)$ and $(W_2(t), F_t, t \geq t_0)$ be independent Wiener processes. The F_t-measurable random process $(x(t), y(t))$ is described by a nonlinear differential equation with a polynomial drift term including an unknown vector parameter $\theta(t)$ for the system state

$$dx(t) = f(x, \theta, t)dt + B(t)u(t)dt + h(t)dt + b(t)dW_1(t),$$
$$x(t_0) = x_0, \qquad (1)$$

and a linear differential equation for the observation process

$$dy(t) = (A_0(t) + A(t)x(t))dt + G(t)dW_2(t). \qquad (2)$$

Here, $x(t) \in R^n$ is the state vector, $u(t) \in R^l$ is the control input, and $y(t) \in R^m$ is the linear observation vector, $m \leq n$, and $\theta(t) \in R^p$ is the vector of unknown parameters. The initial condition $x_0 \in R^n$ is a Gaussian vector such that x_0, $W_1(t) \in R^r$, and $W_2(t) \in R^q$ are independent. The observation matrix $A(t) \in R^{m\times n}$ is not supposed to be invertible or even square. It is assumed that $G(t)G^T(t)$ is a positive definite matrix, therefore, $m \leq q$. All coefficients in (1)–(2) are deterministic functions of appropriate dimensions. The system (1),(2) is assumed to be uniformly controllable and observable; the definitions of uniform controllability and observability for nonlinear systems can be found in [34]. The plant operates under deterministic disturbances

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$h(t)$ and stochastic noises $dW_1(t)$ and $dW_2(t)$ represented as weak mean square derivatives (see [24]) of the Wiener processes, that is, white Gaussian noises. The function $h(t) \in R^n$ represents matched disturbances such that $h(t) = B(t)\gamma(t)$, $\gamma \in R^l$, and the norm $\|\gamma(t)\|$ is bounded by

$$\|\gamma(t)\| \le q_a(t), \quad q_a(t) > 0, \tag{3}$$

where $q_a(t)$ is a finite time-dependent function; $\|x\| = \sqrt{(x^T x)}$ denotes the Euclidean 2-norm of a vector $x \in R^l$.

The nonlinear function $f(x,\theta,t)$ is considered polynomial of n variables, components of the state vector $x(t) \in R^n$, with time-dependent coefficients. Since $x(t) \in R^n$ is a vector, this requires a special definition of the polynomial for $n > 1$. In accordance with [19], a p-degree polynomial of a vector $x(t) \in R^n$ is regarded as a p-linear form of n components of $x(t)$

$$f(x,t) = a_0(\theta,t) + a_1(\theta,t)x + a_2(\theta,t)xx^T + \ldots + a_s(\theta,t)x \ldots_{s\ times} \ldots x,$$

where a_0 is a vector of dimension n, a_1 is a matrix of dimension $n \times n$, a_2 is a 3D tensor of dimension $n \times n \times n$, a_s is an $(s+1)$D tensor of dimension $n \times \ldots_{(s+1)\ times} \ldots \times n$, and $x \times \ldots_{s\ times} \ldots \times x$ is a pD tensor of dimension $n \times \ldots_{s\ times} \ldots \times n$ obtained by p times spatial multiplication of the vector $x(t)$ by itself. Such a polynomial can also be expressed in the summation form

$$f_k(x,t) = a_{0\ k}(\theta,t) + \sum_i a_{1\ ki}(\theta,t)x_i(t) + \sum_{ij} a_{2\ kij}(\theta,t)x_i(t)x_j(t) + \ldots + \sum_{i_1 \ldots i_s} a_{s\ ki_1 \ldots i_s}(\theta,t)x_{i_1}(t) \ldots x_{i_s}(t), \quad k,i,j,i_1 \ldots\ i_s = 1,\ldots,n.$$

The dependence of $a_0(\theta,t)$, $a_1(\theta,t),\ldots,a_s(\theta,t)$ on θ means that those structures contain unknown components $a_{0_i} = \theta_k(t)$, $k = 1,\ldots,p_1 \le n$, $a_{1\ ki} = \theta_k(t)$, $k = p_1+1,\ldots,p_2 \le n \times n + n, \ldots, a_{s\ ki_1 \ldots i_s} = \theta_k(t)$, $k = p_s + 1,\ldots,p \le n + n^2 +,\ldots,+n^s$, as well as known components $a_{0_i}(t), a_{1\ ki}(t),\ldots,a_{s\ ki_1 \ldots i_s}(t)$, whose values are known functions of time.

It is considered that there is no useful information on values of the unknown parameters $\theta_k(t)$, $k = 1,\ldots,p$. In other words, the unknown parameters can be modeled as F_t-measurable Wiener processes

$$d\theta(t) = \beta(t)dW_3(t), \tag{4}$$

with unknown initial conditions $\theta(t_0) = \theta_0 \in R^p$, where $(W_3(t), F_t, t \ge t_0)$ is a Wiener process independent of x_0, $W_1(t)$, and $W_2(t)$, and $\beta(t) \in R^{p \times p}$ is an intensity function.

The quadratic cost function J to be minimized is defined as follows

$$J = \frac{1}{2}E[x^T(T)\Phi x(T) + \int_{t_0}^T u^T(s)R(s)u(s)ds + \int_{t_0}^T x^T(s)L(s)x(s)ds], \tag{5}$$

where R is positive definite and Φ, L are nonnegative definite symmetric matrices, $T > t_0$ is a certain time moment, the symbol $E[f(x)]$ means the expectation (mean) of a function f of a random variable x, and a^T denotes transpose to a vector (matrix) a.

The optimal controller problem is to find the control $u^*(t)$, $t \in [t_0,T]$, that minimizes the criterion J along with the unobserved trajectory $x^*(t)$, $t \in [t_0,T]$, generated upon substituting $u^*(t)$ into the state equation (1).

B. Problem Reduction

To deal with the stated controller problem, the equations (1) and (4) should be rearranged. For this purpose, a vector $\alpha_0(t) \in R^{(n+p)}$, matrix $\alpha_1(t) \in R^{(n+p)\times(n+p)}$, cubic tensor $\alpha_2(t) \in R^{(n+p)\times(n+p)\times(n+p)}$, and other $k+1$-dimensional tensors $\alpha_k(t) \in R^{(n+p)\times \cdots_{(k+1)\ times} \cdots \times(n+p)}$, $k = 0,\ldots,s+1$, are introduced as follows.

The equation for the i-th component of the state vector is given by

$$dx_i(t) = (a_{0\ i}(\theta,t) + \sum_k a_{1\ ik}(\theta,t)x_k(t) + \sum_{jk} a_{2\ ijk}(\theta,t)x_j(t)x_k(t) + \ldots + \sum_{k_1 \ldots k_s} a_{s\ ik_1 \ldots k_s}(\theta,t)x_{k_1}(t) \ldots x_{k_s}(t))dt + \sum_j B_{ij}(t)u_j(t)dt + \sum_j B_{ij}(t)\gamma_j(t)dt + \sum_j b_{ij}(t)dW_{1_j}(t),$$

$i,j,k_1 \ldots\ k_s = 1,\ldots,n$, $x_i(t_0) = x_{0_i}$. Then:

1. If the variable $a_{0_i}(t)$ is a known function, then the i-th component of the vector $\alpha_0(t)$ is set to this function, $\alpha_{0_i}(t) = a_{0_i}(t)$; otherwise, if the variable $a_{0_i}(t)$ is an unknown function, then the $(i, n+i)$-th entry of the matrix $\alpha_1(t)$ is set to 1.

2. If the variable $a_{1_{ij}}(t)$ is a known function, then the (i,j)-th component of the matrix $\alpha_1(t)$ is set to this function, $\alpha_{1_{ij}}(t) = a_{1_{ij}}(t)$; otherwise, if the variable $a_{1_{ij}}(t)$ is an unknown function, then the $(i, n+p_1+k, j)$-th entry of the cubic tensor $\alpha_2(t)$ is set to 1, where k is the number of this current unknown entry in the matrix $a_1(t)$, counting the unknown entries consequently by rows from the first to n-th entry in each row.

...

3. If the variable $a_{s\ ik_1 \ldots k_s}(t)$ is a known function, then the $(i,k_1,\ldots,k_s)$-th component of the sD tensor $\alpha_s(t)$ is set to this function, $\alpha_{s_{i,k_1,\ldots,k_s}}(t) = a_{s\ ik_1 \ldots k_s}(t)$; otherwise, if the variable $a_{s\ ik_1 \ldots k_s}(t)$ is an unknown function, then the $(i, n + p_s + k,\ldots,k_s)$-th entry of the $(s+1)$D tensor $\alpha_{s+1}(t)$ is set to 1, where k is the number of this current unknown entry in the tensor $a_s(t)$, counting the unknown entries consequently by rows from the first to s-th dimension and from the first to n-th entry in each row.

4. All other unassigned entries of the tensors $\alpha_k(t) \in R^{(n+p)\times \cdots_{(k+1)\ times} \cdots \times(n+p)}$, $k = 0,\ldots,s+1$, are set to 0.

Using the introduced notation, the state equations (1),(4) for the vector $z(t)=[x(t),\theta(t)]\in R^{n+p}$ can be rewritten as

$$dz(t)=g(z,t)dt+[B(t)\mid 0_{p\times l}]u(t)dt+[B(t)\gamma(t)\mid 0_{p\times l}]dt+$$
$$diag[b(t),\beta(t)]d[W_1^T(t),W_3^T(t)]^T,\ z(t_0)=[x_0,\theta_0], \qquad (6)$$

where the polynomial function $g(z,t)$ is defined as

$$g(z,t)=\alpha_0(t)+\alpha_1(t)z+\alpha_2(t)zz^T+$$
$$\ldots+\alpha_{s+1}(t)z\ldots_{(s+1)\ times}\ldots z.$$

Thus, the equation (6) is polynomial with respect to the extended state vector $z(t)=[x(t),\theta(t)]$.

C. Optimal Estimate Design

Let us replace the unmeasured bilinear state $z(t)=[x(t),\theta(t)]$, satisfying (6), with its optimal estimate $m(t)=[\hat{x}(t),\hat{\theta}(t)]$ over linear observations $y(t)$ (2), which is obtained using the following optimal filter for polynomial states over linear observations (see [19] for the corresponding filtering problem statement and solution)

$$dm(t)=E(g(z,t)\mid F_t^Y)dt+[B(t)\mid 0_{p\times l}]u(t)dt+$$
$$[B(t)\gamma(t)\mid 0_{p\times l}]dt+P(t)[A(t),0_{m\times p}]^T\times \qquad (7)$$
$$(G(t)G^T(t))^{-1}(dy(t)-(A_0(t)+A(t)\hat{x}(t))dt).$$
$$m(t_0)=[E(x(t_0)\mid F_t^Y),E(\theta(t_0)\mid F_t^Y)],$$
$$dP(t)=(E((z(t)-m(t))(g(z,t))^T\mid F_t^Y)+$$
$$E(g(z,t)(z(t)-m(t))^T)\mid F_t^Y)+ \qquad (8)$$
$$diag[b(t),\beta(t)]diag[b(t),\beta(t)]^T-P(t)[A(t),0_{m\times p}]^T\times$$
$$(G(t)G^T(t))^{-1}[A(t),0_{m\times p}]P(t))dt,$$
$$P(t_0)=E((z(t_0)-m(t_0))(z(t_0)-m(t_0))^T\mid F_t^Y),$$

where $0_{m\times p}$ is the $m\times p$ - dimensional zero matrix; $P(t)$ is the conditional variance of the estimation error $z(t)-m(t)$ with respect to the observations $Y(t)$.

Recall that $\hat{z}(t)=m(t)=[\hat{x}(t),\hat{\theta}(t)]$ is the optimal estimate for the state vector $z(t)=[x(t),\theta(t)]$, based on the observation process $Y(t)=\{y(s),t_0\le s\le t\}$, that minimizes the Euclidean 2-norm $H=E[(z(t)-\hat{z}(t))^T(z(t)-\hat{z}(t))\mid F_t^Y]$ at every time moment t. Here, $E[\xi(t)\mid F_t^Y]$ means the conditional expectation of a stochastic process $\xi(t)=(z(t)-\hat{z}(t))^T(z(t)-\hat{z}(t))$ with respect to the σ - algebra F_t^Y generated by the observation process $Y(t)$ in the interval $[t_0,t]$. As known [24], this optimal estimate is given by the conditional expectation $\hat{z}(t)=m(t)=E(z(t)\mid F_t^Y)$ of the system state $z(t)$ with respect to the σ - algebra F_t^Y generated by the observation process $Y(t)$ in the interval $[t_0,t]$. As usual, the matrix function $P(t)=E[(z(t)-m(t))(z(t)-m(t))^T\mid F_t^Y]$ is the estimation error variance.

Remark 1. The equations (7) and (8) do not form a closed system of equations due to the presence of polynomial terms depending on x, such as $E(g(z,t)\mid F_t^Y)$, and $E((z(t)-m(t))g^T(z,t))\mid F_t^Y)$, which are not expressed yet as functions of the system variables, $m(t)$ and $P(t)$. However, as shown in [17]-[20], the closed system of the filtering equations can be obtained for any polynomial state (6) over linear observations (2), using the technique of representing of superior moments of the conditionally Gaussian random variable $z(t)-m(t)$ as functions of only two its lower conditional moments, $m(t)$ and $P(t)$ (see [17]-[20] for more details of this technique). Apparently, the polynomial dependence of $g(z,t)$ and $(z(t)-m(t))g^T(x,t)$ on z is the key point making this representation possible.

D. Optimal control problem solution: Measured state

To handle the optimal control problem for the designed optimal estimate (8), let us first give the solution to the general optimal control problem for a polynomial system with linear control input and a quadratic cost function.

Consider a polynomial system with linear control input

$$dx(t)=f(x,t)dt+B(t)u(t)dt+b(t)dW_1(t),\ x(t_0)=x_0, \qquad (11)$$

where $x(t)\in R^n$ is the state vector, $u(t)\in R^l$ is the control input, the polynomial drift function $f(x,t)$ is defined by

$$f(x,t)=a_0(t)+a_1(t)x+a_2(t)xx^T+\ldots+a_s(t)x\ldots_{s\ times}\ldots x,$$

and the assumptions made for the system (1) hold. The quadratic cost function J to be minimized is defined by (5).

The optimal control problem is to find the control $u^*(t)$, $t\in[t_0,T]$, that minimizes the criterion J along with the trajectory $x^*(t)$, $t\in[t_0,T]$, generated upon substituting $u^*(t)$ into the state equation (11). The solution to the stated optimal control problem is given by the following theorem [33].

Theorem 1. The optimal regulator for the polynomial system (11) with linear control input with respect to the quadratic criterion (5) is given by the control law

$$u^*(t)=R^{-1}(t)B^T(t)[Q(t)x(t)+p(t)], \qquad (12)$$

where the matrix function $Q(t)$ is the solution of the Riccati equation

$$\dot{Q}(t)=L(t)-[a_1(t)+2a_2(t)x(t)+3a_3(t)x(t)x^T(t)+\ldots \qquad (13)$$
$$+sa_s(t)x(t)\ldots_{s-1\ times}\ldots x(t)]^TQ(t)-$$
$$Q(t)[a_1(t)+a_2(t)x(t)+a_3(t)x(t)x^T(t)+\ldots$$
$$+a_s(t)x(t)\ldots_{s-1\ times}\ldots x(t)]-Q(t)B(t)R^{-1}(t)B^T(t)Q(t),$$

with the terminal condition $Q(T)=-\psi$, and the vector function $p(t)$ is the solution of the linear equation

$$\dot{p}(t)=-Q(t)a_0(t)-[a_1(t)+2a_2(t)x(t)+3a_3(t)x(t)x^T(t)+\ldots$$
$$+sa_s(t)x(t)\ldots_{s-1\ times}\ldots x(t)]^Tp(t)- \qquad (14)$$
$$Q(t)B(t)R^{-1}(t)B^T(t)p(t),$$

with the terminal condition $p(T)=0$. The optimally controlled state of the polynomial system (11) is governed by the equation

$$dx(t)=f(x,t)dt+B(t)R^{-1}(t)B^T(t)[Q(t)x(t)+p(t)]dt+$$
$$+b(t)dW_1(t),\quad x(t_0)=x_0. \qquad (15)$$

E. Optimal controller problem solution: Unmeasured state

Based on the result of Theorem 1, the solution of the optimal controller problem for the polynomial state (11) over linear observations (2) with a quadratic criterion (5) is given as follows [33]. The corresponding optimal control law takes the form

$$u^*(t) = R^{-1}(t)B^T(t)[Q(t)\hat{x}(t) + p(t)], \tag{16}$$

where $\hat{x}(t) = E(x(t) \mid F_t^Y)$, the matrix function $Q(t)$ is the solution of the Riccati equation

$$\dot{Q}(t) = L(t) - [c_1(t) + 2c_2(t)\hat{x}(t) + 3c_3(t)\hat{x}(t)\hat{x}^T(t) + \dots \tag{17}$$
$$+ sc_s(t)\hat{x}(t)\dots_{s-1\ times}\dots\hat{x}(t)]^T Q(t) -$$
$$Q(t)[c_1(t) + c_2(t)\hat{x}(t) + c_3(t)\hat{x}(t)\hat{x}^T(t) + \dots$$
$$+ c_s(t)\hat{x}(t)\dots_{s-1\ times}\dots\hat{x}(t)] - Q(t)B(t)R^{-1}(t)B^T(t)Q(t),$$

with the terminal condition $Q(T) = -\psi$, and the vector function $p(t)$ is the solution of the linear equation

$$\dot{p}(t) = -Q(t)c_0(t) - [c_1(t) + 2c_2(t)\hat{x}(t) + 3c_3(t)\hat{x}(t)\hat{x}^T(t) + \dots$$
$$+ sc_s(t)\hat{x}(t)\dots_{s-1\ times}\dots\hat{x}(t)]^T p(t) - \tag{18}$$
$$Q(t)B(t)R^{-1}(t)B^T(t)p(t),$$

with the terminal condition $p(T) = 0$, where $c_0(t), c_1(t), \dots, c_s(t)$ are the coefficients in the representation of the term $E(f(x,t) \mid F_t^Y)$ as a polynomial of $\hat{x}$, that is,

$$E(f(x,t) \mid F_t^Y) = c_0(t) + c_1(t)\hat{x} + c_2(t)\hat{x}\hat{x}^T +$$
$$\dots + c_s(t)\hat{x}\dots_{s\ times}\dots\hat{x}.$$

Upon writing down the optimal estimate equation for the polynomial state (11) over the linear observations (2) (see [20] and also the equations (7),(8)) and substituting the optimal control (16) into its right-hand side, the following optimally controlled state estimate equation is obtained

$$d\hat{x}(t) = E(f(x,t) \mid F_t^Y)dt + B(t)R^{-1}(t)B^T(t)[Q(t)\hat{x}(t) + p(t)]dt \tag{19}$$
$$+ P(t)A^T(t)(G(t)G^T(t))^{-1}(dy(t) - (A_0(t) + A(t)\hat{x}(t))dt).$$

with the initial condition $\hat{x}(t_0) = E(x(t_0) \mid F_t^Y)$.

Thus, the optimally controlled state estimate equation (19), the gain matrix constituent equations (17) and (18), the optimal control law (16), and the corresponding variance equation give the complete solution to the optimal controller problem for polynomial systems with linear control input and a quadratic cost function. This solution is not yet written in a closed form due to non-closeness of the filtering equations in the general situation; however, as noted in Remark 1, the closed-form solution can be obtained for any specific form of the polynomial drift $f(x,t)$ in the equation (11).

F. Solution of optimal controller problem with deterministic disturbances

The next step is to give the solution to the optimal controller problem for a polynomial stochastic system with linear control input and a quadratic cost function, that operates under influence of matched deterministic disturbances (3).

Consider a polynomial system (11) with deterministic disturbances

$$dx(t) = f(x,t)dt + B(t)u(t)dt + B(t)\gamma(t)dt + b(t)dW_1(t),$$
$$x(t_0) = x_0, \tag{20}$$

where the assumptions made for the system (11) hold, and the quadratic cost function J to be minimized is defined by (5). The deterministic disturbance $\gamma(t)$ satisfies the condition (3). The optimal control problem is to find the control $u^*(t)$, $t \in [t_0, T]$, that minimizes the criterion J along with the trajectory $x^*(t)$, $t \in [t_0, T]$, generated upon substituting $u^*(t)$ into the state equation (11).

If the realization of the Wiener process $W_1(t)$, affecting the state (20), and the initial condition x_0 are exactly known, the optimal control $u^*(t)$ is represented as [25]

$$u^*(t) = u_0^*(t) + u_1^*(t).$$

Here, $u_0^*(t)$ is the optimal control for the nominal system (11), which is given by (12), and $u_1^*(t)$ is the integral sliding mode control $u_1^*(t) = -Ksign[s(t)]$, where the sliding mode manifold is defined as

$$s(t) = (B^T(t)B(t))^{-1}B^T(t)(x(t) -$$
$$[x_0 + \int_{t_0}^{t}(f(x,s) + B(s)u_0^*(s))ds + \int_{t_0}^{t} b(s)dW_1(s)]).$$

Note that $s(t) \in R^l$; therefore, $sign[s(t)]$ is defined as a vector: $\{sign[s_1(t)], \dots, sign[s_l(t)]\}$.

The key idea is as follows: the sliding mode control $u_1^*(t)$ leads the state trajectory to the sliding manifold $s(t)$, where the deterministic disturbances $h(t) = B(t)\gamma(t)$ are absent and the state is therefore regulated optimally by the control $u_0^*(t)$. If the current realization of the Wiener process $W_1(t)$ and the initial condition x_0 are exactly known, then there is no reaching phase and the sliding mode motion on the manifold $s(t)$ starts from the initial moment t_0 (see [25] for substantiation of the integral sliding mode technique).

If the current realization of the Wiener process $W_1(t)$ and the initial condition x_0 are unknown and the state $x(t)$ is not measurable directly but only through the observations (2), the best estimate of the sliding mode manifold $s(t)$ is given (see [24]) by its conditional estimate with respect to observations (2), i.e.,

$$\hat{s}(t) = E[s(t) \mid F_t^Y] = (B^T(t)B(t))^{-1}B^T(t)(\hat{x}(t) - \tag{21}$$
$$[\hat{x}(t_0) + \int_{t_0}^{t}(E(f(x,s) \mid F_s^Y) + B(s)u_{00}^*(s))ds]),$$

where $\hat{x}(t) = E[x(t) \mid F_t^Y]$, and $u_{00}^*(s)$ is the optimal control corresponding to the case of a directly unmeasurable state (20). Therefore, the optimal control is given by

$$u^*(t) = u_{00}^*(t) + u_{11}^*(t), \quad (22)$$

where $u_{00}^*(t)$ is the optimal control for the nominal unmeasurable system (20), which is given by (16), and $u_{11}^*(t) = -Ksign[\hat{s}(t)]$. More discussion of the integral sliding mode technique for unmeasurable systems can be found in [26].

Taking into account the preceding considerations, the following optimally controlled state estimate equation is obtained

$$d\hat{x}(t) = E(f(x,t) \mid F_t^Y)dt + B(t)(R(t))^{-1}B^T(t)\times$$
$$[Q(t)\hat{x}(t) + p(t)]dt - Bsign[\hat{s}(t)]dt - \quad (23)$$
$$P(t)A^T(t)(G(t)G^T(t))^{-1}(dy(t) - (A_0(t) + A(t)\hat{x}(t))dt).$$

with the initial condition $\hat{x}(t_0) = E(x(t_0) \mid F_t^Y)$, where $\hat{s}(t)$ is given by (21).

G. Solution of optimal controller problem with deterministic disturbances and uncertain parameters

Based on the result of the preceding subsections, the solution of the original optimal controller problem for the polynomial state (1) with deterministic disturbances (3) and uncertain parameters θ over linear observations (2) with a quadratic criterion (5) is given as follows. The corresponding optimal control law takes the form (22), where the matrix function $Q(t)$ is the upper left corner of the matrix $\bar{Q}(t) = diag[Q(t), 0_{p\times p}]$, which is the solution of the Riccati equation

$$\dot{\bar{Q}}(t) = diag[L(t), 0_{p\times p}] - [\gamma_1(t) + 2\gamma_2(t)m(t) + 3\gamma_3(t)\times$$
$$m(t)m^T(t) + \ldots + (s+1)\gamma_{s+1}(t)m(t)\ldots_{s\ times}\ldots m(t)]^T\bar{Q}(t) - \quad (24)$$
$$\bar{Q}(t)[\gamma_1(t) + \gamma_2(t)m(t) + \gamma_3(t)m(t)m^T(t) + \ldots$$
$$+\gamma_{s+1}(t)m(t)\ldots_{s\ times}\ldots m(t)] -$$
$$\bar{Q}(t)[B(t) \mid 0_{p\times l}]R^{-1}(t)[B(t) \mid 0_{p\times l}]^T\bar{Q}(t),$$

with the terminal condition $\bar{Q}(T) = diag[-\Phi \mid 0_{p\times p}]$, and the vector function $p(t)$ is the upper subvector of the vector $\bar{p}(t) = [p(t) \mid 0_p]$, which is the solution of the linear equation

$$\dot{\bar{p}}(t) = -\bar{Q}(t)\gamma_0(t) - [\gamma_1(t) + 2\gamma_2(t)m(t) + 3\gamma_3(t)\times$$
$$m(t)m^T(t) + \ldots + (s+1)\gamma_{s+1}(t)m(t)\ldots_{s\ times}\ldots m(t)]^T\bar{p}(t) - \quad (25)$$
$$\bar{Q}(t)[B(t) \mid 0_{p\times l}]R^{-1}(t)[B(t) \mid 0_{p\times l}]^T(t)\bar{p}(t),$$

with the terminal condition $\bar{p}(T) = 0_{n+p}$, where $\gamma_0(t), \gamma_1(t), \ldots, \gamma_{s+1}(t)$ are the coefficients in the representation of the term $E(g(z,t) \mid F_t^Y)$ in the right-hand side of (6) as a polynomial of m, that is,

$$E(g(z,t) \mid F_t^Y) = \gamma_0(t) + \gamma_1(t)m + \gamma_2(t)mm^T +$$
$$\ldots + \gamma_{s+1}(t)m\ldots_{s+1\ times}\ldots m.$$

Upon substituting the optimal control (22) into the equation (7), the following optimally controlled state estimate equation is obtained

$$dm(t) = E(g(z,t) \mid F_t^Y)dt + [B(t) \mid 0_{p\times l}]R^{-1}(t)B^T(t)\times \quad (26)$$
$$[Q(t)\hat{x}(t) + p(t)]dt - [B(t) \mid 0_{p\times l}]sign[\hat{s}(t)]dt + P(t)\times$$
$$[A(t), 0_{m\times p}]^T(G(t)G^T(t))^{-1}(dy(t) - (A_0(t) + A(t)\hat{x}(t))dt).$$

with the initial condition $m(t_0) = E(x(t_0) \mid F_t^Y)$, where $\hat{s}(t)$ is given by (21).

Thus, the optimally controlled state estimate equation (26), the gain matrix constituent equations (24) and (25), the optimal control law (22), and the variance equation (8) give the complete solution to the optimal controller problem for an uncertain polynomial system (1) with deterministic disturbances (3) over linear observations (2) and a quadratic cost function (5). This solution is not yet written in a closed form due to non-closeness of the filtering equations (7),(8) in the general situation. In the next subsection, the closed-form optimal solution is obtained for the particular case of a third degree polynomial drift $g(z,t)$, which corresponds to a second degree polynomial function $f(x,t)$.

1) Optimal controller problem solution for third degree polynomial state: Let the function

$$g(z,t) = \alpha_0(t) + \alpha_1(t)z + \alpha_2(t)zz^T + \alpha_3(t)zzz^T \quad (27)$$

be a third degree polynomial, where z is an $(n+p)$-dimensional vector, $a_0(t)$ is an $(n+p)$-dimensional vector, $a_1(t)$ is a $(n+p)\times(n+p)$-dimensional matrix, $a_2(t)$ is a 3D tensor of dimension $(n+p)\times(n+p)\times(n+p)$, $a_3(t)$ is a 4D tensor of dimension $(n+p)\times(n+p)\times(n+p)\times(n+p)$. In this case, taking into account the representations for $E(g(z,t) \mid F_t^Y)$ and $E((z(t)-m(t))(g(z,t))^T \mid F_t^Y)$ as functions of $m(t)$ and $P(t)$ (see the results obtained in [17]-[19] for third degree polynomial functions), the following filtering equations for the optimal estimate $m(t)$ and the error variance $P(t)$ are obtained

$$dm(t) = (\alpha_0(t) + \alpha_1(t)m(t) + \alpha_2(t)m(t)m^T(t) + \alpha_2(t)P(t) + \quad (28)$$
$$3\alpha_3(t)P(t)m(t) + \alpha_3(t)m(t)m(t)m^T(t) + [B(t) \mid 0_{p\times l}]u(t))dt +$$
$$P(t)[A(t), 0_{m\times p}]^T(G(t)G^T(t))^{-1}(dy(t) - (A_0(t) + A(t)\hat{x}(t))dt),$$
$$dP(t) = (\alpha_1(t)P(t) + P(t)\alpha_1^T(t) + 2\alpha_2(t)m(t)P(t) +$$
$$2(\alpha_2(t)m(t)P(t))^T + 3(\alpha_3[P(t)P(t) + P(t)m(t)m^T(t)]) +$$
$$3(\alpha_3[P(t)P(t) + P(t)m(t)m^T(t)])^T + \quad (29)$$
$$diag[b(t), \beta(t)]diag[b(t), \beta(t)]^T -$$
$$P(t)[A(t), 0_{m\times p}]^T(G(t)G^T(t))^{-1}[A(t), 0_{m\times p}]P(t))dt,$$

with the same initial conditions as in (7),(8).

Taking into account the representation (24): $\gamma_0(t) = \alpha_0(t) + \alpha_2(t)P(t)$, $\gamma_1(t) = \alpha_1 + 3\alpha_3(t)P(t)(t)$, $\gamma_2(t) = \alpha_2(t)$, $\gamma_3(t) = \alpha_3(t)$, the equations (24) and (25) take the following particular forms in the case of a third degree polynomial (27)

$$\dot{\bar{Q}}(t) = diag[L(t), 0_{p\times p}] - [\alpha_1(t) + 3\alpha_3(t)P(t) +$$

$$2\alpha_2(t)m(t) + 3\alpha_3(t)m(t)m^T(t)]^T\bar{Q}(t) - \qquad (30)$$

$$\bar{Q}(t)[\alpha_1(t) + 3\alpha_3(t)P(t) + \alpha_2(t)m(t) + \alpha_3(t)m(t)m^T(t)] -$$

$$\bar{Q}(t)[B(t) \mid 0_{p\times l}]R^{-1}(t)[B(t) \mid 0_{p\times l}]^T(t)\bar{Q}(t),$$

with the terminal condition $\bar{Q}(T) = diag[-\Phi \mid 0_{p\times p}]$, and the vector function $\bar{p}(t)$ is the solution of the linear equation

$$\dot{\bar{p}}(t) = -\bar{Q}(t)(\alpha_0(t) + \alpha_2(t)P(t)) - \qquad (31)$$

$$[\alpha_1 + 3\alpha_3(t)P(t)(t) + 2\alpha_2(t)m(t) + 3\alpha_3(t)m(t)m^T(t)]^T\bar{p}(t) -$$

$$\bar{Q}(t)[B(t) \mid 0_{p\times l}]R^{-1}(t)[B(t) \mid 0_{p\times l}]^T(t)\bar{p}(t),$$

with the terminal condition $\bar{p}(T) = 0_{n+p}$.

The optimally controlled state estimate equation (26) takes the the following particular form

$$dm(t) = (\alpha_0(t) + \alpha_1(t)m(t) + \alpha_2(t)m(t)m^T(t) + \alpha_2(t)P(t) + \qquad (32)$$

$$3\alpha_3(t)P(t)m(t) + \alpha_3(t)m(t)m(t)m^T(t) + [B(t) \mid 0_{p\times l}]\times$$

$$R^{-1}(t)B^T(t)[Q(t)\hat{x}(t) + p(t)]dt - [B(t) \mid 0_{p\times l}]sign[\hat{s}(t)]dt -$$

$$P(t)[A(t), 0_{m\times p}]^T(G(t)G^T(t))^{-1}(dy(t) - (A_0(t) + A(t)\hat{x}(t))dt).$$

with the initial condition $m(t_0) = E(x(t_0) \mid F_t^Y)$, where $\hat{s}(t)$ is given by (21).

III. Conclusions

The optimal quadratic-Gaussian controller has been designed for uncertain stochastic polynomial systems with unknown parameters and deterministic disturbances over linear observations and a quadratic criterion, using the integral sliding mode algorithm for unmeasured states. The optimality of the obtained controller has been proved using the previous results in the optimal filtering for polynomial, in particular, third degree, states over linear observations and the optimal control theory for polynomial systems with linear control input and a quadratic criterion. The separation principle for polynomial systems with unknown parameters has been introduced and substantiated.

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