2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Stochastic H_∞ Control Problem with State-Dependent Noise for Multimodeling Systems

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***Abstract*— In this paper, stochastic H_∞ state feedback control with state-dependent noise for multimodeling systems is addressed. After establishing the asymptotic structure of the stochastic multi-modeling algebraic Riccati equation (SMARE), an iterative algorithm that is based on Newton's method is established. A high-order state feedback controller by means of the obtained iterative solution is given and the degradation of the H_∞ performance is investigated for the stochastic case for the first time. Finally, in order to demonstrate the efficiency of the proposed algorithm, numerical example is given for practical megawatt-frequency control problem.**

I. INTRODUCTION

When several small singular perturbation parameters of the same order of magnitude are present in the dynamic model of a physical system, the control problem is usually approached as multimodeling systems. The control problem of the multimodeling systems has been widely studied during the past few decades (see e.g., [8], [9], [10]).

Recently, stochastic H_∞ control problem with state- and control dependent noise were considered in [1], [2], [15], [16]. It has attracted much attention and has been widely applied to various fields. For example, the stochastic H_2/H_∞ control with state-dependent noise has been addressed [5]. However, to the best of our knowledge, no results have been obtained for the H_∞ control problem of the multimodeling systems with stochastic uncertainty such as standard Wiener process even though the asymptotic properties of input-output operations norm for singularly stochastic perturbed systems have been treated in [6].

In order to design the stochastic H_∞ controller, the stochastic algebraic Riccati equation (SARE) needs to be solved. The reliable approach for solving the SARE has been well documented in [4]. This algorithm is based on the revised Kleinman algorithm and it is easy to show that this algorithm is equivalent to the Newton's method. Hence, the quadratic convergence is attained if the initial guess is close to the required solution. However, the considered SARE has the positive definite quadratic term. Therefore, we cannot apply the existing result to the SARE that has the sign indefinite quadratic term directly, where it arise in stochastic H_∞ control. More recently, in [11], [12], the Newton's method for solving the sign-indefinite multimodeling algebraic Riccati equation (MARE) has been developed. Although this result is very elegant in theory, the stochastic multimodeling systems (SMS) situation is an issue that remains to be considered.

In this paper, the numerical solution to the stochastic multimodeling algebraic Riccati equation (SMARE) with sign-indefinite quadratic term related to the stochastic H_∞ control problem with state-dependent noise is investigated. The main objective of this paper is to obtain numerically controllers and to prove that the high-order approximate controller can be used reliably on the original SMS. The difficulty in extending the results is that since the stochastic uncertainty exists, the derivation of asymptotic structure of SMARE cannot be determined by using the existing assumption. Therefore, in order to avoid the complication for the derivation, the appropriate assumption is made for the coefficient matrix related to the stochastic uncertainty. The contributions of this paper are as follows. In order to obtain the initial guess of the algorithm, the asymptotic structure of the sign indefinite SMARE is established for the first time. Then, our new concept is to set the initial condition to the solutions of the reduced-order SARE. Because of such a choice, it can be proved that the proposed algorithm converges to a required solution by using Newton-Kantorovich theorem [13]. As another important feature, a high-order state feedback H_∞ controller on the basis of the obtained iterative solution is given. Moreover, the degradation of the H_∞ performance is investigated for the stochastic systems for the first time. Finally, in order to demonstrate the efficiency of the proposed algorithm, a numerical example for a two-area electric energy system is solved.

Notation: The notations used in this paper are fairly standard. I_n denotes the $n \times n$ identity matrix. Superscript T denotes the matrix transpose. $\| \cdot \|$ denotes its Euclidean norm for a matrix. The space of the $\Re^k$-valued functions that are quadratically integrable on $(0,\ \infty)$ are denoted by $L_2^k(0,\ \infty)$. $\|\omega\|_2^2 := E\int_s^t \|\omega(t)\|^2 dt$, $\omega(t) \in L_2^k(s,\ t)$ denotes L_2 norm in a Hilbert space. $\textbf{block diag}$ denotes the block diagonal matrix. $E[\cdot]$ denotes the expection operator.

II. PRELIMINARY RESULT

We consider the following SMS that consist of N-fast subsystems with specific structure of lower level interconnected through the dynamics of a higher level slow subsystem.

$$dx(t) = [A_e x(t) + B_e u(t) + D_e v(t)]dt + \sum_{p=1}^{M} A_{pe} x(t) dw_p(t), \tag{1a}$$

$$z(t) = Cx(t) + Hu(t), \tag{1b}$$

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where $\bar{n} := \sum_{j=0}^{N} n_j$, $\bar{m} := \sum_{j=1}^{N} m_j$, $\bar{l} := \sum_{j=1}^{N} l_j$,

$$
\begin{aligned}
x(t) &:= \begin{bmatrix} x_0^T(t) & x_1^T(t) & \cdots & x_N^T(t) \end{bmatrix}^T \in \Re^{\bar{n}}, \\
u(t) &:= \begin{bmatrix} u_1^T(t) & \cdots & u_N^T(t) \end{bmatrix}^T \in \Re^{\bar{m}}, \\
v(t) &:= \begin{bmatrix} v_1^T(t) & \cdots & v_N^T(t) \end{bmatrix}^T \in \Re^{\bar{l}}, \\
\Pi_e &:= \textbf{block diag}\begin{pmatrix} \varepsilon_1 I_{n_1} & \cdots & \varepsilon_N I_{n_N} \end{pmatrix}, \\
A_e &:= \begin{bmatrix} A_{00} & A_{0f} \\ \Pi_e^{-1} A_{f0} & \Pi_e^{-1} A_f \end{bmatrix}, \\
A_{0f} &:= \begin{bmatrix} A_{01} & \cdots & A_{0N} \end{bmatrix}, \\
A_{f0} &:= \begin{bmatrix} A_{10}^T & \cdots & A_{N0}^T \end{bmatrix}^T, \\
A_f &:= \textbf{block diag}\begin{pmatrix} A_{11} & \cdots & A_{NN} \end{pmatrix}, \\
A_{pe} &:= \begin{bmatrix} A_{p00} & \mu A_{p0f} \\ \Pi_e^{-1} \bar{\varepsilon}^{\delta} A_{pf0} & \Pi_e^{-1} \bar{\varepsilon}^{\delta} A_{pf} \end{bmatrix}, \\
A_{p0f} &:= \begin{bmatrix} A_{p01} & \cdots & A_{p0N} \end{bmatrix}, \\
A_{pf0} &:= \begin{bmatrix} A_{p10}^T & \cdots & A_{pN0}^T \end{bmatrix}^T, \\
A_{pf} &:= \textbf{block diag}\begin{pmatrix} A_{p11} & \cdots & A_{pNN} \end{pmatrix}, \\
B_e &:= \begin{bmatrix} B_0 \\ \Pi_e^{-1} B_f \end{bmatrix}, \ B_0 := \begin{bmatrix} B_{01} & \cdots & B_{0N} \end{bmatrix}, \\
B_f &:= \textbf{block diag}\begin{pmatrix} B_{11} & \cdots & B_{NN} \end{pmatrix}, \\
D_e &:= \begin{bmatrix} D_0 \\ \Pi_e^{-1} D_f \end{bmatrix}, \ D_0 := \begin{bmatrix} D_{01} & \cdots & D_{0N} \end{bmatrix}, \\
D_f &:= \textbf{block diag}\begin{pmatrix} D_{11} & \cdots & D_{NN} \end{pmatrix}, \\
C &:= \begin{bmatrix} C_0 & C_f \end{bmatrix}, \\
C_0 &:= \begin{bmatrix} C_{00} \\ C_{10} \\ \vdots \\ C_{N0} \end{bmatrix}, \ C_f := \begin{bmatrix} 0 & \cdots & 0 \\ C_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_{NN} \end{bmatrix}, \\
H &:= \textbf{block diag}\begin{pmatrix} H_{11} & \cdots & H_{NN} \end{pmatrix}.
\end{aligned}
$$

$x_j(t) \in \Re^{n_j}$, $j = 0, 1, \ldots, N$ are the state vectors, $u_j(t) \in \Re^{m_j}$, $j = 1, \ldots, N$ are the control inputs, $v_j(t) \in L_2^{l_j}(0, \infty)$, $j = 1, \ldots, N$ is considered to be an unknown finite-energy deterministic disturbance [1], [5]. $z(t) \in \Re^p$ is the controlled output. $\varepsilon_j > 0$, $j = 1, \ldots, N$ and $\mu \geq 0$ are small parameters and $\delta > 1/2$ is independent of $\bar{\varepsilon} := \min\{\varepsilon_1, \ldots, \varepsilon_N\}$. It should be noted that the parameters μ and δ have been introduced in [6], [7] for the first time. Moreover, the considered SMS consists of N-fast subsystems as compared to [6]. $w_p(t) \in \Re$, $p = 1, \ldots, M$ is a one-dimensional standard Wiener process defined in the filtered probability space [1], [2], [3], [5].

We assume that the ratios of the small positive parameters ε_j, $j = 1, \ldots, N$ and μ are bounded by some positive constants $\underline{k}_{ij}$, $\bar{k}_{ij}$, $\underline{l}$ and $\bar{l}$ and only these bounds are assumed to be known [8], [9]. In other words, they have the same order of magnitude.

$$0 < \underline{k}_{ij} \leq \alpha_{ij} \equiv \frac{\varepsilon_j}{\varepsilon_i} \leq \bar{k}_{ij} < \infty, \ 0 \leq \underline{l} \leq \frac{\mu}{\bar{\varepsilon}} \leq \bar{l} < \infty. \quad (2)$$

Note that one of the fast state matrices A_{jj}, $j = 1, \ldots, N$ may be singular.

Without loss of generality, the stochastic H_∞ control problem for the SMS is investigated under the following basic assumption [1], [5].

Assumption 1: $C^T H = 0$ and $H^T H = I_{\bar{m}}$.

It should be noted that the matrix pair (E, G) is deemed stable, if $dx(t) = Ex(t)dt + Gx(t)dw$ is asymptotically mean square stable [5].

The stochastic H_∞ control problem for SMS is given below [1], [5].

Given a constant $\gamma > 0$, find a matrix K satisfying the following conditions:

i) *The system*

$$dx(t) = [A_e + B_e K]x(t)dt + \sum_{p=1}^{M} A_{pe} x(t) dw_p(t) \quad (3)$$

is exponentially mean-square stable internally, i.e. it satisfies the following equation.

$$E\|x(t)\|^2 \leq \rho e^{-\psi(t-s)} E\|x(s)\|^2, \ \exists \rho, \ \psi > 0. \quad (4)$$

ii) *The closed-loop system*

$$
\begin{aligned}
dx(t) &= [(A_e + B_e K)x(t) + D_e v(t)]dt \\
&\quad + \sum_{p=1}^{M} A_{pe} x(t) dw_p(t), && \text{(5a)} \\
z(t) &= (C + HK)x(t), && \text{(5b)}
\end{aligned}
$$

corresponding to the system in equation (1) *with feedback control $u(t) = Kx(t)$, satisfies following condition.*

$$
\begin{aligned}
&\sup_{\substack{v \in L_2^{\bar{l}}(0, \infty), \\ v \neq 0, \ x(0) = 0}} \frac{\|z\|_2^2}{\|v\|_2^2} \\
&:= \sup_{\substack{v \in L_2^{\bar{l}}(0, \infty), \\ v \neq 0, \ x(0) = 0}} \frac{E \int_0^{+\infty} [x^T(t) C^T C x(t) + u^T(t) u(t)] dt}{E \int_0^{+\infty} v^T(t) v(t) dt} \\
&< \gamma^2. \quad (6)
\end{aligned}
$$

The following result is well known [1], [5].

Lemma 1: Suppose that Assumption 1 is satisfied. The stochastic H_∞ state-feedback control problem has a solution if and only if there exists a symmetric non-negative definite solution P_e to the following SMARE

$$
\begin{aligned}
\mathcal{G}(P_e) &:= A_e^T P_e + P_e A_e + \sum_{p=1}^{M} A_{pe}^T P_e A_{pe} \\
&\quad - P_e (B_e B_e^T - \gamma^{-2} D_e D_e^T) P_e + C^T C = 0 \quad (7)
\end{aligned}
$$

such that the stochastic system

$$
\begin{aligned}
dx(t) &= [A_e - B_e B_e^T P_e + \gamma^{-2} D_e D_e^T P_e] x(t) dt \\
&\quad + \sum_{p=1}^{M} A_{pe} x(t) dw_p(t) \quad (8)
\end{aligned}
$$

is exponentially mean-square stable.

The controller solving this H_∞ problem is given by equation (9).

$$u(t) = Kx(t) = -B_e^T P_e x(t). \quad (9)$$

III. ASYMPTOTIC STRUCTURE OF SMARE

In this section, we need to first analyze the asymptotic structure of SMARE (7) to obtain the controller. In order to simplify the presentation, the following matrices are defined.

$$\begin{aligned}
S_e &:= B_e B_e^T - \gamma^{-2} D_e D_e^T = \begin{bmatrix} S_{00} & S_{0f}\Pi_e^{-1} \\ \Pi_e^{-1} S_{0f}^T & \Pi_e^{-1} S_f \Pi_e^{-1} \end{bmatrix}, \\
S_{0f} &:= \begin{bmatrix} S_{01} & \cdots & S_{0N} \end{bmatrix}, \\
S_f &:= \textbf{block diag} \begin{pmatrix} S_{11} & \cdots & S_{NN} \end{pmatrix}, \\
Q &:= C^T C = \begin{bmatrix} Q_{00} & Q_{0f} \\ Q_{0f}^T & Q_f \end{bmatrix}, \\
Q_{0f} &:= \begin{bmatrix} Q_{01} & \cdots & Q_{0N} \end{bmatrix}, \\
Q_f &:= \textbf{block diag} \begin{pmatrix} Q_{11} & \cdots & Q_{NN} \end{pmatrix}.
\end{aligned}$$

Since the matrices A_e, A_{pe}, B_e and D_e contain the term of ε_j^{-1}, a solution P_e of the SMARE (7), if it exists, must contain terms of ε_j. Taking this fact into consideration, we look for a solution P_e of the SMARE (7) with the structure

$$P_e := \begin{bmatrix} P_{00} & P_{f0}^T \Pi_e \\ \Pi_e P_{f0} & \Pi_e P_f \end{bmatrix}, \ P_{00} = P_{00}^T,$$

$$P_{f0} := \begin{bmatrix} P_{10}^T & \cdots & P_{N0}^T \end{bmatrix}^T,$$

$$P_f := \begin{bmatrix} P_{11} & \alpha_{12}P_{21}^T & \alpha_{13}P_{31}^T & \cdots & \alpha_{1N}P_{N1}^T \\ P_{21} & P_{22} & \alpha_{23}P_{32}^T & \cdots & \alpha_{2N}P_{N2}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{(N-1)1} & P_{(N-1)2} & P_{(N-1)3} & \cdots & \alpha_{(N-1)N}P_{N(N-1)}^T \\ P_{N1} & P_{N2} & P_{N3} & \cdots & P_{NN} \end{bmatrix},$$

$$\Pi_e P_f = P_f^T \Pi_e.$$

Before investigating the optimal control problem, we investigate the asymptotic structure of the SMARE (7). Substituting the matrices A_e, A_{pe}, S_e, Q and P_e into SMARE (7) results in the following partitioned equation (10).

$$\begin{aligned}
f_1 &= P_{00}A_{00} + A_{00}^T P_{00} + P_{f0}^T A_{f0} + A_{f0}^T P_{f0} \\
&\quad + \sum_{p=1}^{M} A_{p00}^T P_{00} A_{p00} + \bar{\varepsilon}^{\delta} \sum_{p=1}^{M} A_{pf0}^T P_{f0} A_{p00} \\
&\quad + \bar{\varepsilon}^{\delta} \sum_{p=1}^{M} A_{p00}^T P_{f0}^T A_{pf0} + \bar{\varepsilon}^{2\delta} \sum_{p=1}^{M} A_{pf0}^T P_f \Pi_e^{-1} A_{pf0} \\
&\quad - P_{00} S_{00} P_{00} - P_{f0}^T S_f P_{f0} \\
&\quad - P_{00} S_{0f} P_{f0} - P_{f0}^T S_{0f}^T P_{00} + Q_{00} = 0, \qquad (10a)\\
f_2 &= A_{00}^T P_{f0}^T \Pi_e + A_{f0}^T P_f + P_{00} A_{0f} + P_{f0}^T A_f \\
&\quad + \mu \sum_{p=1}^{M} A_{p00}^T P_{00} A_{p0f} + \bar{\varepsilon}^{\delta} \mu \sum_{p=1}^{M} A_{pf0}^T P_{f0} A_{p0f} \\
&\quad + \bar{\varepsilon}^{\delta} \sum_{p=1}^{M} A_{p00}^T P_{f0}^T A_{pf} + \bar{\varepsilon}^{2\delta} \sum_{p=1}^{M} A_{pf0}^T P_f \Pi_e^{-1} A_{pf} \\
&\quad - P_{00} S_{00} P_{f0}^T \Pi_e - P_{f0}^T S_{0f}^T P_{f0}^T \Pi_e \\
&\quad - P_{00} S_{0f} P_f - P_{f0}^T S_f P_f + Q_{0f} = 0, \qquad (10b)\\
f_3 &= P_f^T A_f + A_f^T P_f + \Pi_e P_{f0} A_{0f} + A_{0f}^T P_{f0}^T \Pi_e \\
&\quad + \mu^2 \sum_{p=1}^{M} A_{p0f}^T P_{00} A_{p0f} + \bar{\varepsilon}^{\delta} \mu \sum_{p=1}^{M} A_{pf}^T P_{f0} A_{p0f} \\
&\quad + \bar{\varepsilon}^{\delta} \mu \sum_{p=1}^{M} A_{p0f}^T P_{f0}^T A_{pf} + \bar{\varepsilon}^{2\delta} \sum_{p=1}^{M} A_{pf}^T P_f \Pi_e^{-1} A_{pf} \\
&\quad - P_f^T S_f P_f - P_f^T S_{0f}^T P_{f0}^T \Pi_e - \Pi_e P_{f0} S_{0f} P_f \\
&\quad - \Pi_e P_{f0} S_{00} P_{f0}^T \Pi_e + Q_f = 0. \qquad (10c)
\end{aligned}$$

It is assumed that the limit of α_{ij} exists as ε_i and ε_j tend to zero [8], [9], that is

$$\bar{\alpha}_{ij} = \lim_{\substack{\varepsilon_j \to +0 \\ \varepsilon_i \to +0}} \alpha_{ij}. \qquad (11)$$

Let $\bar{P}_{00}$, $\bar{P}_{f0}$ and $\bar{P}_f$ be the limiting solutions of the above equation (10) as $\mu \to +0$, $\varepsilon_j \to +0$, $j = 1, \ \dots \ , N$, then we obtain the following reduced-order equations (12).

$$\begin{aligned}
&\bar{P}_{00}A_{00} + A_{00}^T \bar{P}_{00} + \bar{P}_{f0}^T A_{f0} + A_{f0}^T \bar{P}_{f0} + \sum_{p=1}^{M} A_{p00}^T \bar{P}_{00} A_{p00} \\
&\quad - \bar{P}_{00} S_{00} \bar{P}_{00} - \bar{P}_{f0}^T S_f \bar{P}_{f0} \\
&\quad - \bar{P}_{00} S_{0f} \bar{P}_{f0} - \bar{P}_{f0}^T S_{0f}^T \bar{P}_{00} + Q_{00} = 0, \qquad (12a)\\
&A_{f0}^T \bar{P}_f + \bar{P}_{00} A_{0f} + \bar{P}_{f0}^T A_f \\
&\quad - \bar{P}_{00} S_{0f} \bar{P}_f - \bar{P}_{f0}^T S_f \bar{P}_f + Q_{0f} = 0, \qquad (12b)\\
&\bar{P}_f^T A_f + A_f^T \bar{P}_f - \bar{P}_f^T S_f \bar{P}_f + Q_f = 0, \qquad (12c)
\end{aligned}$$

where

$$\bar{P}_f := \begin{bmatrix} \bar{P}_{11} & \bar{\alpha}_{12}\bar{P}_{21}^T & \bar{\alpha}_{13}\bar{P}_{31}^T & \cdots & \bar{\alpha}_{1N}\bar{P}_{N1}^T \\ \bar{P}_{21} & \bar{P}_{22} & \bar{\alpha}_{23}\bar{P}_{32}^T & \cdots & \bar{\alpha}_{2N}\bar{P}_{N2}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{(N-1)1} & \bar{P}_{(N-1)2} & \bar{P}_{(N-1)3} & \cdots & \bar{\alpha}_{(N-1)N}\bar{P}_{N(N-1)}^T \\ \bar{P}_{N1} & \bar{P}_{N2} & \bar{P}_{N3} & \cdots & \bar{P}_{NN} \end{bmatrix},$$

$$\bar{P}_{jj} = \bar{P}_{jj}^T, \ j = 0, \ 1, \ \dots \ , N.$$

It should be noted that the algebraic Riccati equation (ARE) (12c) admits an asymmetric solution. However, it can be verified that there exists at least a symmetric positive semidefinite stabilizing solution of ARE (12c) because of the following reasons [11], [12].

First, the following AREs are introduced.

$$\bar{P}_{jj}^* A_{jj} + A_{jj}^T \bar{P}_{jj}^* - \bar{P}_{jj}^* S_{jj} \bar{P}_{jj}^* + Q_{jj} = 0. \qquad (13)$$

Moreover, let us define the following sets.

$\Gamma_{f_j} = \{\gamma > 0 |$ the ARE (13) with $S_{jj} = B_{jj}B_{jj}^T - \gamma^{-2} D_{jj} D_{jj}^T$ has a positive semidefinite and stabilizing solution $\bar{P}_{jj}^* \}$, $j = 1, \ \dots \ , N$.

Assumption 2: The sets Γ_{f_j} are not empty.

Lemma 2: Under Assumption 2, the ARE (12c) admits a unique symmetric positive semidefinite stabilizing solution $\bar{P}_f$ which can be written as

$$\bar{P}_f^* := \textbf{block diag} \begin{pmatrix} \bar{P}_{11}^* & \cdots & \bar{P}_{NN}^* \end{pmatrix}. \qquad (14)$$

Assumption 2 ensures that $A_{jj} - S_{jj}\bar{P}_{jj}^*$, $j = 1, \ \dots \ , N$ are nonsingular. Substituting the solution of (12c) into (12b) and substituting $\bar{P}_{f0}^*$ into (12a) and making some lengthy calculations (the detail is omitted for brevity), we obtain the following 0-order equations (15).

$$\bar{P}_{00}^*\boldsymbol{A} + \boldsymbol{A}^T\bar{P}_{00}^* + \sum_{p=1}^{M} A_{p00}^T\bar{P}_{00}^*A_{p00} \\ -\bar{P}_{00}^*\boldsymbol{S}\bar{P}_{00}^* + \boldsymbol{Q} = 0, \tag{15a}$$

$$\bar{P}_{j0}^{*T} := \begin{bmatrix} \bar{P}_{jj}^* & -I_{n_j} \end{bmatrix} T_{jj}^{-1}T_{j0}\begin{bmatrix} I_{n_0} \\ \bar{P}_{00}^* \end{bmatrix}, \tag{15b}$$

$$\bar{P}_{jj}^*A_{jj} + A_{jj}^T\bar{P}_{jj}^* - \bar{P}_{jj}^*S_{jj}\bar{P}_{jj}^* + Q_{jj} = 0, \tag{15c}$$

where $\bar{P}_{f0}^* := \begin{bmatrix} \bar{P}_{10}^{*T} & \cdots & \bar{P}_{N0}^{*T} \end{bmatrix}^T$,

$$\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{S} \\ -\boldsymbol{Q} & -\boldsymbol{A}^T \end{bmatrix} := T_{00} - \sum_{j=1}^{N} T_{0j}T_{jj}^{-1}T_{j0},$$

$$T_{00} := \begin{bmatrix} A_{00} & -S_{00} \\ -Q_{00} & -A_{00}^T \end{bmatrix},\ T_{0j} := \begin{bmatrix} A_{0j} & -S_{0j} \\ -Q_{0j} & -A_{j0}^T \end{bmatrix},$$

$$T_{j0} := \begin{bmatrix} A_{j0} & -S_{0j}^T \\ -Q_{0j}^T & -A_{0j}^T \end{bmatrix},\ T_{jj} := \begin{bmatrix} A_{jj} & -S_{jj}^T \\ -Q_{jj}^T & -A_{jj}^T \end{bmatrix},$$

$j = 1,\ \dots,N$.

Remark 1: For each $j \in \{1,\ \dots,N\}$ equation (13) is a Riccati equation arising in connection with the deterministic H_∞ problem. Hence, if Γ_{f_i} is not empty then $\Gamma_{f_i} = (\gamma_{f_i},\ \infty)$. On the other hand, if $\gamma \in \Gamma_{f_i}$ then the matrix $A_{jj} - S_{jj}\bar{P}_{jj}^*$ is a stable matrix. Therefore the hamiltonian T_{jj} is invertible.

The ARE (15c) produces a positive semidefinite solution if γ is sufficiently large. Hence, let us define the set.
$\Gamma_s = \{\gamma > 0|$ the SARE (15a) has a positive semidefinite and stabilizing solution $\bar{P}_{00}^*\}$.

We introduce the assumption:

Assumption 3: The set Γ_s is not empty and it has the form $\Gamma_s = (\gamma_s,\ \infty)$.

Remark 2:
a) In the considered general case it is not clear how the coefficients $\boldsymbol{A}$, $\boldsymbol{S}$, $\boldsymbol{Q}$ are depending upon γ. That is why we have to introduce as an assumption the fact that the set Γ_s takes the form of a right unbounded interval. It is worth mentioning that this happens if all matrices A_{jj} are invertible.
b) The fact that $\bar{P}_{00}^*$ is the stabilizing solution of (15a) means that the trajectory $x = 0$ of the Ito differential equation

$$dx_0(t) = \tilde{\boldsymbol{A}}x_0(t)dt + \sum_{p=1}^{M} A_{p00}x_0(t)dw_p(t) \tag{16}$$

is exponentially stable in mean square (ESMS), where $\tilde{\boldsymbol{A}} := \boldsymbol{A} - \boldsymbol{S}\bar{P}_{00}^*$. This is equivalent to the fact that the Lyapunov operator $X \to \tilde{\boldsymbol{A}}^TX + X\tilde{\boldsymbol{A}} + \sum_{p=1}^{M} A_{p00}^TXA_{p00}$ are located in the half plane $\mathbf{Re}\lambda < 0$. This means that (16) is true.

The limiting behavior of P_e as the parameter $\|\nu\| \to +0$ is described by the following theorem.

Theorem 1: Under Assumptions 1-3, if a parameter $\gamma > \bar{\gamma} := \max\{\gamma_s,\ \gamma_{f_1},\ \dots,\gamma_{f_N}\}$ is selected, there exists a small σ^* such that for all $\|\nu\| \in (0,\ \sigma^*)$, the SMARE (7) admits the unique symmetric positive semidefinite stabilizing solution P_e for stochastic system (5) which can be written as

$$P_e := \Phi_e\bar{P} + O(\|\nu\|), \tag{17}$$

where

$$\bar{P} := \begin{bmatrix} \bar{P}_{00}^* & 0 \\ \bar{P}_{f0}^* & \bar{P}_f^* \end{bmatrix},\ \Phi_e := \textbf{block diag}\begin{pmatrix} I_{n_0} & \Pi_e \end{pmatrix}.$$

Proof: This can be proved by applying the implicit function theorem to (10). Since the proof is similar to that mentioned in [12], it is omitted. ■

It should be noted that there is no solution of the SMARE (7) as long as there is no positive semi-definite solutions $\bar{P}_{jj}$ of the SARE (15c). Conversely, the asymptotic structure of the solution for the SMARE (7) can be established by using the reduced-order solution $\bar{P}_{jj}$ of the SARE (15c) via implicit function theorem. Therefore, the existence of the reduced-order solution $\bar{P}_{jj}$ of the SARE (15c) will play an important role in this study. In this case, it is easy to verify that the magnitude of disturbance attenuation level γ_{f_i} influences to the existence of the reduced-order solution $\bar{P}_{jj}$. In fact, when γ_{f_i} tends to zero, it is hard to obtain the reduced-order solution $\bar{P}_{jj}$ except for the special case. Finally, in this study, the problem considered here is restricted for the disturbance attenuation level γ_{f_i} such that the reduced-order SAREs (15c) have the solutions $\bar{P}_{jj}$.

IV. NEWTON'S METHOD

Let us consider Newton's method (18).

$$P_e^{(n+1)}(A_e - S_eP_e^{(n)}) + (A_e - S_eP_e^{(n)})^TP_e^{(n+1)} \\ + \sum_{p=1}^{M} A_{pe}^TP_e^{(n+1)}A_{pe} + P_e^{(n)}S_eP_e^{(n)} + Q = 0, \tag{18}$$

where $n = 0,\ 1,\ \dots$, and the initial conditions are chosen as follows.

$$P_e^{(0)} := \begin{bmatrix} \bar{P}_{00}^* & \bar{P}_{f0}^{*T}\Pi_e \\ \bar{P}_{f0}^* & \bar{P}_f^* \end{bmatrix}. \tag{19}$$

The algorithm represented by equation (18) has the feature given in the following theorem for the SMS.

Theorem 2: Suppose that Assumptions 1-3 are satisfied. If the parameter-independent reduced-order SARE (15c) has a positive semidefinite solution, there exists a small $\tilde{\sigma}$ such that for all $\|\nu\| \in (0,\ \tilde{\sigma})$, $0 < \tilde{\sigma} \le \bar{\sigma}$, the iterative algorithm represented by equation (18) converges to the exact solution of P_e with a rate equal to that of quadratic convergence; here, $P_e^{(n)}$ is positive semidefinite. Moreover, the convergence solutions equal those of P_e in the SMARE (7) in the neighborhood of the initial condition $P_e^{(0)} = \bar{P}$. In other words, the following condition is satisfied.

$$\|P_e^{(n)} - P_e\| = \frac{(2\theta)^{2^n}}{\phi\lambda 2^n} = \frac{O(\|\nu\|^{2^n})}{\phi\lambda 2^n},\ n = 0,\ 1,\ \dots, \tag{20}$$

where $\lambda = 2\|S_e\| < \infty$, $\phi = \|[\nabla\mathcal{G}(P_e^{(0)})]^{-1}\|$, $\eta = \phi\cdot\|\mathcal{G}(P_e^{(0)})\|$, $\theta = \phi\eta\lambda < 2^{-1}$.

Proof: The proof follows directly by applying the Newton-Kantorovich theorem [13]. Since the proof is similar to that mentioned in [11], [12], it is omitted. ■

V. A HIGH-ORDER STATE H_∞ CONTROLLER

In this section, we apply the controller $u^{(n)}(t) = -B_e^T P_e^{(n)} x(t)$ to the SMS (1) and compare it with the exact optimal control (9).

Theorem 3: Under the conditions given in Theorem 1, if the controller gain matrix $K^{(n)} := -B_e^T P_e^{(n)}$ is designed for a prescribed disturbance attenuation level $\gamma > \bar{\gamma}$ and the resulted controller $u^{(n)}(t) = -B_e^T P_e^{(n)} x(t)$ is applied to the SMS (1), then the following inequality will be satisfied:

$$\begin{aligned} &\|(C+HK^{(n)}) \cdot (sI_{\bar{n}} - A_e - B_e K^{(n)})^{-1} D_e\|_\infty \\ &= \|(C+HK)(sI_{\bar{n}} - A_e - B_e K)^{-1} D_e\|_\infty + O(\|\nu\|^{n+1}) \\ &< \gamma + O(\|\nu\|^{n+1}) \end{aligned} \tag{21}$$

where $K = -B_e^T P_e$ and $K^{(n)} = -B_e^T P_e^{(n)}$.

Proof: Applying the optimal controller $u(t) = Kx(t) = -B_e^T P_e x(t)$ to (1) yields

$$\begin{aligned} dx(t) &= [(\bar{A}_e + \|\nu\| L_e) x(t) + D_e v(t)] dt \\ &\quad + \sum_{p=1}^{M} A_{pe} x(t) dw_p(t), \ x^0 = 0, \end{aligned} \tag{22a}$$

$$J = E \int_0^\infty x^T(t) \bar{Q} x(t) dt, \tag{22b}$$

where

$$\bar{A}_e = \begin{bmatrix} \bar{A}_{00} & \bar{A}_{0f} \\ \Pi_e^{-1} \bar{A}_{f0} & \Pi_e^{-1} \bar{A}_f \end{bmatrix},$$

$\bar{A}_{00} := A_{00} - S_{00} \bar{P}_{00} - S_{0f} \bar{P}_{f0}$, $\bar{A}_{0f} := A_{0f} - S_{0f} \bar{P}_f$,
$\bar{A}_{f0} := A_{f0} - S_{0f}^T \bar{P}_{00} - S_f \bar{P}_{f0}$, $\bar{A}_f := A_f - S_f \bar{P}_f$,
$L_e = -\|\nu\|^{-1} (B_e B_e^T P_e - A_e + \bar{A}_e)$, $\bar{Q} = Q + P_e B_e B_e^T P_e$.

Since $\bar{A}_f$ is stable, there is transformation $y(t) = T^{-1} x(t)$ such that $T^{-1} \bar{A}_e T = \mathbf{block\ diag} \begin{pmatrix} \hat{A}_s & \Pi_e^{-1} \hat{A}_f \end{pmatrix}$, where $\hat{A}_s := \bar{A}_{00} - \bar{A}_{0f} \bar{A}_f^{-1} \bar{A}_{f0} + O(\|\nu\|)$ and $\hat{A}_f := \bar{A}_f + O(\|\nu\|)$ [10].

Using the transformation T, we obtain

$$\begin{aligned} \begin{bmatrix} dy_0 \\ \Pi_e dy_f \end{bmatrix} &= \left(\begin{bmatrix} \hat{A}_s & 0 \\ 0 & \hat{A}_f \end{bmatrix} \begin{bmatrix} y_0 \\ y_f \end{bmatrix} + \|\nu\| \begin{bmatrix} L_s \\ L_f \end{bmatrix} y \right. \\ &\quad \left. + \begin{bmatrix} D_{0s} \\ D_{ff} \end{bmatrix} v \right) dt + \sum_{p=1}^{M} T^{-1} A_{pe} x(t) dw_p(t), \end{aligned} \tag{23}$$

where $J = E \int_0^\infty y^T(t) T^T \bar{Q} T y(t) dt$, $y^0 = y(0)$, $y(t) = [y_0^T(t) \quad y_f^T(t)]^T$ and $[L_s^T \quad \Pi_e^{-1} L_f^T]^T = T^{-1} L_e$, $[D_{0s}^T \quad \Pi_e^{-1} D_{ff}^T]^T = T^{-1} D_e$. From (23), if $\|\nu\|$ is small enough, then we have $\|y\|_2^2 \le c_1 \|v\|_2^2, c_1 > 0$. Similarly, substituting $u(t) = -K^{(n)} \hat{x}(t)$ and $f(t) = T^{-1} \hat{x}(t)$ into system (1), we get

$$\begin{aligned} \begin{bmatrix} df_0 \\ \Pi_e df_f \end{bmatrix} &= \left(\begin{bmatrix} \hat{A}_s & 0 \\ 0 & \hat{A}_f \end{bmatrix} \begin{bmatrix} f_0 \\ f_f \end{bmatrix} + \|\nu\| \begin{bmatrix} \hat{L}_s \\ \hat{L}_f \end{bmatrix} y \right. \\ &\quad \left. + \begin{bmatrix} D_{0s} \\ D_{ff} \end{bmatrix} v \right) dt + \sum_{p=1}^{M} T^{-1} A_{pe} \hat{x}(t) dw_p(t), \end{aligned} \tag{24}$$

where $\hat{J} = E \int_0^\infty f^T(t) T^T \hat{Q} T f(t) dt$, $\hat{Q} = Q + P_e^{(n)} B_e B_e^T P_e^{(n)}$, $f^0 = f(0)$, $[\hat{L}_s^T \quad \hat{L}_f^T]^T = -\|\nu\|^{-1} T^{-1} (B_e B_e^T P_e^{(n)} - A_e + \bar{A}_e)$. Hence, from (24), one can derive $\|f\|_2^2 \le c_2 \|v\|_2^2$, $c_2 > 0$. Subtracting (24) from (23) we get the following equation (25).

$$\begin{aligned} \begin{bmatrix} de_0 \\ \Pi_e de_f \end{bmatrix} &= \left(\begin{bmatrix} \hat{A}_s & 0 \\ 0 & \hat{A}_f \end{bmatrix} \begin{bmatrix} e_0 \\ e_f \end{bmatrix} + \|\nu\| \begin{bmatrix} \hat{L}_s \\ \hat{L}_f \end{bmatrix} e \right. \\ &\quad \left. + \begin{bmatrix} O(\|\nu\|^{n+1}) \\ O(\|\nu\|^{n+1}) \end{bmatrix} y \right) dt \\ &\quad + \sum_{p=1}^{M} T^{-1} A_{pe} e(t) dw_p(t), \end{aligned} \tag{25}$$

where $e(t) = y(t) - f(t)$. From (25), we obtain $\|e\|_2^2 \le c_3 \|\nu\|^{2n+2} \|y\|_2^2 \le c_4 \|\nu\|^{2n+2} \|v\|_2^2$, $c_3, c_4 > 0$.

Then, we note that $T^{-1} \hat{L}_e T - T^{-1} L_e T = O(\|\nu\|^n)$, $\|\hat{Q} - \bar{Q}\| = m_0 \|\nu\|^{n+1}$, $m_0 > 0$.

Applying the Schwartz inequality yields

$$\begin{aligned} |J - \hat{J}| &\le E \int_0^\infty [m_1 \|e(t)\| \cdot \|y(t)\| + m_1 \|e(t)\| \cdot \|f(t)\| \\ &\quad + m_2 \|\nu\|^{n+1} \|f(t)\|^2] dt \\ &\le \bar{m} [\|e\|_2 (\|y\|_2 + \|f\|_2) + \|\nu\|^{n+1} \|f\|_2^2], \end{aligned} \tag{26}$$

where $\bar{m} = \max\{m_1, \ m_2\}$, $m_1 = \|T^T \bar{Q} T\|$, $m_2 = \|T^T B B^T T\|$. Moreover, substituting $\|y\|_2^2 \le c_1 \|v\|_2^2$, $\|f\|_2^2 \le c_2 \|v\|_2^2$ and $\|e\|_2^2 \le c_4 \|\nu\|^{2n+2} \|v\|_2^2$ into (26) yields

$$\begin{aligned} &|J - \hat{J}| \\ &\le \bar{m} [\sqrt{c_4} (\sqrt{c_1} + \sqrt{c_2}) + c_2] \|\nu\|^{n+1} \|v\|_2^2 \le \bar{m}_0 \|\nu\|^{n+1} \|v\|_2^2 \end{aligned} \tag{27}$$

Finally, by using condition $J \le \gamma^2 \|v\|_2^2$, we have

$$\hat{J} \le [\gamma^2 + O(\|\nu\|^{n+1})] \|v\|_2^2 = [\gamma + O(\|\nu\|^{n+1})]^2 \|v\|_2^2, \tag{28}$$

that is, an $O(\|\nu\|^n)$ accuracy controller $u^{(n)}(t) = -K^{(n)} x(t)$ achieves the performance level $\gamma + O(\|\nu\|^{n+1})$. ■

VI. NUMERICAL EXAMPLE FRO MEGAWAT-FREQUENCY STOCHASTIC H_∞ CONTROL

In order to demonstrate the efficiency of the stochastic H_∞ control for SMS, we present results for the practical multiarea electric energy systems. The state variable model of the megawatt-frequency control problem was developed in [14]. The system matrices are given by the top of the next page. It is assumed that time constant of the governers represent the small singular perturbations. Hence, small parameters are $T_{gv1} := \varepsilon_1 = 0.030$ and $T_{gv2} := \varepsilon_2 = 0.029$. Moreover, it should be noted that $\mu = 0$.

It should be noted that the deterministic disturbance distribution $v(t) := [\Delta P_{d1} \ \Delta P_{d2}]^T = [0.1 \ 0.1]^T$ and the state dependent noise related to the load frequency constant [14] are both considered compared with the existing results [11], [12]. We suppose that the error of the load frequency constant is within 5% of the nominal value. Therefore, the proposed design method is very useful because the resulting strategy can be implemented to more practical SMS.

$$A_{00} = \begin{bmatrix} 0 & 0.315 & 0 & -0.315 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1.888 & -0.0498 & 1.888 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1.888 & 0 & -1.888 & -0.0498 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.333 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3.333 \end{bmatrix},\ A_{01} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3.333 \\ 0 \end{bmatrix},\ A_{02} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ 0 \\ 3.333 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 0 & 0 & 0.41666 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.41666 & 0 & 0 \end{bmatrix},\ A_{20} = \begin{bmatrix} 0 & 0 & 0.41666 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.41666 & 0 & 0 \end{bmatrix},\ A_{11} = A_{22} = -1,$$

$$A_{100} = \textbf{block diag} \begin{pmatrix} 0 & 0 & 0.00249 & 0 & 0.00249 & 0 & 0 \end{pmatrix},\ A_{101} = A_{102} = 0 \in \Re^{7 \times 1},$$

$$A_{110} = A_{120} = 0 \in \Re^{1 \times 7},\ A_{111} = A_{112} = A_{122} = 0,\ B_{01} = B_{02} = 0 \in \Re^{7 \times 1},\ B_{11} = 1,\ B_{22} = 1,$$

$$D_{01} = \begin{bmatrix} 0 & 0 & -0.6 & 0 & 0 & 0 & 0 \end{bmatrix}^T,\ D_{02} = \begin{bmatrix} 0 & 0 & 0 & 0 & -0.6 & 0 & 0 \end{bmatrix}^T,\ D_{11} = D_{22} = 0,\ Q = \textbf{block diag} \begin{pmatrix} I_7 & 0.25 I_2 \end{pmatrix}.$$

$$u^{(5)}(t) = \begin{bmatrix} 1.5893 & 9.4531e-1 & 4.1393 & 1.6120 & 1.8547e-1 & 4.2214 & -2.8374e-2 & 4.6816e-1 & 2.1536e-2 \\ -7.8321e-1 & 1.7522e-3 & 2.3204e-1 & 1.1581 & 9.5872e-1 & 2.6205e-1 & 9.3331e-2 & 2.2279e-2 & 2.6668e-1 \end{bmatrix} x(t).$$

For every boundary value $\gamma > \bar{\gamma} := \max\{\gamma_s,\ \gamma_{f_1},\ \gamma_{f_2}\} = 2.2608e-1$, the SMARE (7) has the positive definite stabilizing solution because the AREs (15c) and the SARE (15a) have the positive definite solution, where $\gamma_s = 2.2608e-1$, $\gamma_{f_1} = \gamma_{f_2} = \infty$.

Now, we choose $\gamma = 0.3\ (> \bar{\gamma})$ to solve the MSARE (7). The efficiency of the Newton's method (18) is demonstrated. It is easy to verify that algorithm (18) converges to the exact solution with an accuracy of $\|\mathcal{G}(P_e^{(n)})\| < 1.0e-11$ after five iterations.

Table 1. Errors per iterations.

n	$\|\mathcal{G}(P_e^{(n)})\|$
0	1.5667
1	$4.2489e-01$
2	$3.3631e-03$
3	$2.0470e-05$
4	$1.5710e-11$
5	$9.1508e-12$

In order to verify the accuracy of the solution, the remainder per iteration is substituted by $P_e^{(n)}$ into SMARE (7). In Table 1, the results of the error $\|\mathcal{G}(P_e^{(n)})\|$ per iteration are given. It can be seen that algorithm (18) yields quadratic convergence. Using the obtained iterative solution, the high-order approximate stochastic H_∞ controller is given by the top of the this page.

VII. CONCLUSION

In this paper, stochastic H_∞ control problem for the SMS has been discussed. Particularly, a new iterative algorithm for solving the SMARE that has sign-indefinite quadratic form has been proposed. The proposed algorithm consist of the Newton's method. As a result, it has been proven that the solution of the SMARE converges to a positive semi-definite stabilizing solution with the rate of convergence of $O(\|\nu\|^{2^n})$. As another important feature, the degradation of the H_∞ performance via a high-order state feedback controller by means of the obtained iterative solution was given for the stochastic case for the first time. Finally, for the practical megawatt-frequency control problem, the numerical examples have shown excellent results that the proposed algorithm has succeeded in reducing the computational workspace and the quadratic convergence has been attained. It is worth pointing out that although the stochastic and the deterministic uncertainty are both included in the SMS, we can construct the H_∞ controller with high-accuracy.

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