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A Discrete-time Robust Extended Kalman Filter

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***Abstract*— A discrete-time robust extended Kalman filter (REKF) formulation for uncertain systems expressed in terms of a *set-valued state estimator* is described in this paper. The robust filter and Riccati equations are derived as an approximate solution to a reverse-time optimal control problem defining this *set-valued state estimator*. As presented, the uncertainties are modeled by a sum quadratic constraint (SQC) that takes into account both modeling uncertainties as well as uncertainties introduced from exogenous noise sources.**

I. INTRODUCTION

The family of Kalman filters have been applied for state as well as parameter estimation for numerous linear as well as nonlinear systems. Though the standard Kalman filter is considered an optimal estimator in case of linear systems with Gaussian noise characteristics, its nonlinear suboptimal counterpart, the extended Kalman filter (EKF) is known to diverge under influences of severe nonlinearities and uncertainties [1], [2]. As a solution to this problem robust forms of the filter have been formulated for a wide class of uncertainties [3]–[9]. A popular approach developed for state estimation of uncertain systems is the set membership state estimation approach. Such approaches, with a deterministic interpretation of the Kalman filter in terms of a *set-valued state estimator*, are described in [10], [11]. The set membership state estimation approach of [10] was extended by Savkin and Petersen in [4], [12] in order to accommodate uncertainties in continuous-time linear systems based on an integral quadratic constraint (IQC). A discrete-time equivalent was presented in [12], [13] where the uncertainties were modeled by a sum quadratic constraint (SQC). As a nonlinear extension to the robust linear filter in [4], James and Petersen proposed a robust extended Kalman filter (REKF) for continuous-time uncertain systems in [6].

In this paper, we present a discrete-time REKF for an uncertain discrete-time nonlinear system. The formulation of such a discrete-time filter is motivated by the fact that most modern-day sensors provide data in the discrete domain.

The set-valued state estimation filtering approach of [6] is derived as an approximate solution to a reverse-time optimal control problem obtained by applying the method of dynamic programming to a set of forward-time system equations. This, however, is not straightforward in the discrete-time case for which we consider the discrete-time system dynamics reverse in time. This defines the contribution in the paper. The set-valued state estimation problem in this case will be solved by obtaining an approximate solution to the optimal control problem in terms of a forward-time dynamic programming equation. The discrete-time uncertain system dynamics is considered in reverse-time with an SQC-based uncertainty description.

The remainder of this paper is organized as follows: Section II describes the formulation of the reverse-time discrete-time nonlinear uncertain system and introduces the concept of set-valued state estimation. This set-valued state estimation problem is then expressed in terms of a corresponding optimal control problem. Section III provides an approximate solution to this optimal control problem which leads to the Riccati and filter equations that define the discrete-time REKF. Finally, Section IV presents an illustrative example and compares results obtained by applying the discrete-time REKF and the discrete-time EKF to a spring-mass-damper system.

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II. PROBLEM FORMULATION

In this section, we consider a reverse-time discrete-time uncertain nonlinear system which is derived from a forward-time continuous-time uncertain nonlinear system. The uncertainties in the discrete-time uncertain system are described by an SQC, which is derived from the corresponding continuous-time uncertainty description in the form of an IQC. Furthermore, the concept of set-valued state estimation is introduced and related to a corresponding optimal control problem.

A. Reverse-time Discrete-time Uncertain Nonlinear System

We begin with a forward-time continuous-time uncertain nonlinear system of the form,

$$\begin{aligned} \dot{x}(t) &= a_c(t,x(t),u(t)) + D_c(t)\,w(t), \\ z(t) &= k_c(t,x(t),u(t)), \\ y(t) &= c_c(t,x(t)) + v(t), \end{aligned} \tag{1}$$

where $x(\cdot) \in \mathbb{R}^n$ is the *state*, $u(\cdot) \in \mathbb{R}^m$ is the known control input, $w(\cdot) \in \mathbb{R}^p$ and $v(\cdot) \in \mathbb{R}^l$ are the process and measurement uncertainty inputs respectively, $z(\cdot) \in \mathbb{R}^q$ is the uncertainty output and $y(\cdot) \in \mathbb{R}^l$ is the measured output. $a_c(\cdot)$, $k_c(\cdot)$ and $c_c(\cdot)$ are given nonlinear functions and $D_c(\cdot)$ is a given matrix function of time.

The uncertainty associated with system (1) can be described in terms of an IQC as in [6], [12],

$$(x(0)-x_0)^T N(x(0)-x_0) + \int_0^s [w(t)^T Q_c(t)\, w(t) + v(t)^T R_c(t)\, v(t)] \leq d + \int_0^s \| z(t) \|^2 \, dt, \tag{2}$$

where $||\cdot||$ denotes the Euclidean norm. Also, $w(\cdot)$ and $v(\cdot)$ represent admissible uncertainties described by,

$$\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} = \phi(t, x(\cdot)), \tag{3}$$

where $\phi(\cdot)$ is a nonlinear time-varying dynamic uncertainty function. Also, $N = N^T > 0$ is a given matrix, $x_0 \in \mathbb{R}^n$ is a given state vector, $d > 0$ is a given constant and $Q_c(\cdot)$, $R_c(\cdot)$ are given positive-definite, symmetric matrix functions of time.

In order to derive a discrete-time robust set-valued state estimator, it is necessary to discretize this continuous-time uncertain system. However, as mentioned in the introduction, the discrete-time set-valued state estimator is most straightforward to derive if this system is discretized in reverse-time rather than in forward-time. In order to discretize the nonlinear uncertain system (1), standard techniques such as the Euler or Runge-Kutta methods can be applied. This will lead to a nonlinear reverse-time discrete-time uncertain system described by the state equations,

$$\begin{aligned} x(k) &= A(k, x(k+1), u(k)) - B(k)\, w(k), \\ z(k+1) &= K(k, x(k+1), u(k)), \\ y(k+1) &= C(k, x(k+1)) + v(k+1), \end{aligned} \tag{4}$$

where $A(\cdot)$, $K(\cdot)$ and $C(\cdot)$ represent discrete-time nonlinear functions.

The uncertainty associated with the reverse-time discrete-time system (4) can be obtained by discretizing the IQC (2) to obtain an SQC,

$$(x(0)-x_0)^T N(x(0)-x_0) + \sum_{k=0}^{T-1} (w(k)^T Q(k)\, w(k) + v(k+1)^T R(k+1)\, v(k+1)) \leq d + \sum_{k=0}^{T-1} \| z(k+1) \|^2, \tag{5}$$

where $v(k+1) = [y(k+1) - C(k, x(k+1))]$, $z(k+1) = K(k, x(k+1), u(k))$, and the admissible uncertainties $w(\cdot)$ and $v(\cdot)$ are described as,

$$\begin{bmatrix} w(k) \\ v(k+1) \end{bmatrix} = \psi(k, x(\cdot)) \tag{6}$$

and $\psi(\cdot)$ is a nonlinear time-varying dynamic uncertainty function.

The uncertain system (4), with the corresponding SQC uncertainty description (5) will be used to derive the robust filter and Riccati equations, which define the discrete-time REKF.

B. Set-Valued State Estimation and the Corresponding Optimal Control Problem

Consider $y_0(k) = y(k)$ to be a fixed measured output and $u_0(k) = u(k)$ a known control input for the uncertain system (4), (5), for $k = 1, 2, ..., T$. The set-valued state estimation problem involves finding the set $Z_T[x_0, u_0(\cdot)|_1^T, y_0(\cdot)|_1^T, d]$ of all possible states $x(T)$ at time step T for the system in (4) with initial conditions and uncertainty constraints defined in (5), consistent with the measured output sequence $y_0(\cdot)$ and input sequence $u_0(\cdot)$. Given an output sequence $y_0(\cdot)$, it follows from the definition of $Z_T[x_0, u_0(\cdot)|_1^T, y_0(\cdot)|_1^T, d]$, that

$$x_T \in Z_T[x_0, u_0(\cdot)|_1^T, y_0(.)|_1^T, d] \tag{7}$$

if and only if there exists an uncertain input sequence $w(\cdot)$ such that, $V_T[x_T, w(\cdot)] \leq d$ where the cost functional $V_T[x_T, w(\cdot)]$ is derived from the SQC (5) as,

$$V_T[x_T, w(\cdot)] \triangleq (x(0)-x_0)^T N(x(0)-x_0) + \sum_{k=0}^{T-1} (w(k)^T Q(k)\, w(k) + v(k+1)^T R(k+1)\, v(k+1)) \leq d + \sum_{k=0}^{T-1} \| z(k+1) \|^2 \tag{8}$$

with $v(k+1) = [y_0(k+1) - C(k, x(k+1))]$, $z(k+1) = K(k, x(k+1), u_0(k))$. Here, the vector $x(\cdot)$ is the solution to the reverse-time discrete-time system (4), with input uncertainty $w(\cdot)$ and terminal condition $x(T) = x_T$. Hence,

$$Z_T[x_0, u_0(\cdot)|_1^T, y_0(\cdot)|_1^T, d] = \left\{ x_T \in \mathbb{R}^n : \inf_{w(\cdot)} V_T[x_T, w(\cdot)] \leq d \right\}. \tag{9}$$

The optimization problem

$$\inf_{w(\cdot)} V_T[x_T, w(\cdot)] \tag{10}$$

for the system (4), defines a *nonlinear optimal control problem* with a sign indefinite quadratic cost function. The discrete-time REKF will be derived by finding an approximate solution to this optimal control problem.

III. Discrete-time Robust Extended Kalman Filter

The optimal control problem (10) for the reverse-time discrete-time system (4) can be solved via dynamic programming. Indeed, the corresponding discrete-time (forward-time) Hamilton-Jacobi-Bellman (HJB) equation for this optimal control problem is given by,

$$\begin{aligned} & V_{k+1}^*(x(k+1)) \\ & = \min_{w(k)} \{ V_k^*[A(k, x(k+1), u_0(k)) - B(k)\, w(k)] \\ & \quad + w(k)^T Q(k)\, w(k) + v(k+1)^T R(k+1)\, v(k+1) \\ & \quad - z(k+1)^T z(k+1) \} \end{aligned} \tag{11}$$

with initial condition

$$V_0^*(x(0)) = (x(0) - x_0)^T N (x(0) - x_0). \tag{12}$$

As a first step towards obtaining an approximate solution to the nonlinear optimal control problem (10), (4) we approximate $V_k^*(\cdot)$ as:

$$V_k^*(x(k)) \approx (x(k)-\hat{x}(k))^T X(k)(x(k)-\hat{x}(k))+\Phi(k), \quad (13)$$

where $\hat{x}(k)$ is a vector representing the robust state estimate and $X(k)$ is a symmetric matrix.

Applying the approximate solution (13) to the HJB equation (11), we obtain,

$$\begin{aligned}
&(x(k+1)-\hat{x}(k+1))^T X(k+1)(x(k+1)-\hat{x}(k+1)) \\
&\quad + \Phi(k+1) \\
&= \min_{w(k)} \Big\{ [A(k,x(k+1),u_0(k))-B(k)\,w(k)-\hat{x}(k)]^T X(k) \\
&\quad [A(k,x(k+1),u_0(k))-B(k)\,w(k)-\hat{x}(k)] \\
&\quad + v(k+1)^T R(k+1)\, v(k+1) \\
&\quad + \Phi(k)+w(k)^T Q(k)\, w(k)-z(k+1)^T z(k+1) \Big\}. \quad (14)
\end{aligned}$$

Furthermore, comparing (12) and (13), the initial conditions become $\hat{x}(0)=x_0$, $X(0)=N$ and $\Phi(0)=0$.

Solving the minimization problem in (14), the following nonlinear difference equation is obtained,

$$\begin{aligned}
&(x(k+1)-\hat{x}(k+1))^T X(k+1)(x(k+1)-\hat{x}(k+1)) \\
&\quad +\Phi(k+1) \\
&= \Big\{ A(k,x(k+1),u_0(k))^T\, \Xi(k)\, A(k,x(k+1),u_0(k)) \\
&\quad -A(k,x(k+1),u_0(k))^T\, \Xi(k)\, \hat{x}(k) \\
&\quad +C(k,x(k+1))^T\, R(k+1)\, C(k,x(k+1)) \\
&\quad +y_0(k+1)^T\, R(k+1)\, y_0(k+1) \\
&\quad -2\,y_0(k+1)^T\, R(k+1)\, C(k,x(k+1)) \\
&\quad -\hat{x}(k)^T\, \Xi(k)\, A(k,x(k+1),u_0(k))+\hat{x}(k)^T\, \Xi(k)\, \hat{x}(k) \\
&\quad -z(k+1)^T z(k+1)+\Phi(k) \Big\}, \quad (15)
\end{aligned}$$

where

$$\Xi(k)=X(k)-X(k)B(k)[Q(k)+B(k)^T X(k)B(k)]^{\#}B(k)^T X(k). \quad (16)$$

In the above equation, $(\cdot)^{\#}$ denotes the Moore-Penrose pseudo-inverse. If the matrix $(Q(k)+B(k)^T X(k)B(k))$ is positive definite, the pseudo-inverse in (16) can be replaced by a normal matrix inverse. This would hold in all cases for which a suitable solution exists for $X(k)$; see [12].

In order to obtain an approximate solution to the nonlinear difference equation (15), a first order linearization of the nonlinear system (4) is performed about the point $\hat{x}(k)$,

$$\begin{aligned}
&A(k,x(k+1),u_0(k)) \\
&\quad \approx A(k,\hat{x}(k),u_0(k)) + \nabla_x A(k,\hat{x}(k),u_0(k))\,(x(k+1)-\hat{x}(k)), \\
&C(k,x(k+1)) \\
&\quad \approx C(k,\hat{x}(k)) + \nabla_x C(k,\hat{x}(k))\,(x(k+1)-\hat{x}(k)), \\
&K(k,x(k+1),u_0(k)) \\
&\quad \approx K(k,\hat{x}(k),u_0(k)) + \nabla_x K(k,\hat{x}(k),u_0(k))\,(x(k+1)-\hat{x}(k)). \quad (17)
\end{aligned}$$

Substituting the linearized terms from (17) into (15) and comparing coefficients of various time-dependent variables we obtain the following set of recursive equations,

Riccati Difference Equation

$$X(k+1)=\begin{bmatrix} \nabla_x A(k,\hat{x}(k),u_0(k))^T\, \Xi(k)\, \nabla_x A(k,\hat{x}(k),u_0(k)) \\ +\, \nabla_x C(k,\hat{x}(k))^T\, R(k+1)\, \nabla_x C(k,\hat{x}(k)) \\ -\, \nabla_x K(k,\hat{x}(k),u_0(k))^T\, \nabla_x K(k,\hat{x}(k),u_0(k)) \end{bmatrix},$$
$$X(0)=N. \quad (18)$$

Filter State Equation

$$\hat{x}(k+1)=\hat{x}(k)+X(k+1)^{-1}\Lambda,$$
$$\hat{x}(0) = x_0, \quad (19)$$

where,

$$\Lambda=\begin{bmatrix} \nabla_x A(k,\hat{x}(k),u_0(k))^T\, \Xi(k)\, \hat{x}(k) \\ -\, \nabla_x A(k,\hat{x}(k),u_0(k))^T\, \Xi(k)\, A(k,\hat{x}(k),u_0(k)) \\ +\, \nabla_x C(k,\hat{x}(k))^T\, R(k+1)\, y_0(k+1) \\ -\, \nabla_x C(k,\hat{x}(k))^T\, R(k+1)\, C(k,\hat{x}(k)) \\ +\, \nabla_x K(k,\hat{x}(k),u_0(k))^T K(k,\hat{x}(k),u_0(k)) \end{bmatrix}. \quad (20)$$

Filter Constant Equation

$$\begin{aligned}
&\Phi(k+1) \\
&= 2\hat{x}(k)^T \begin{bmatrix} -\, \nabla_x A(k,\hat{x}(k),u_0(k))^T\, \Xi(k)\, A(k,\hat{x}(k),u_0(k)) \\ -\, \Xi(k)\, A(k,\hat{x}(k),u_0(k)) \\ +\, \nabla_x C(k,\hat{x}(k))^T\, R(k+1)\, y_0(k+1) \\ -\, \nabla_x C(k,\hat{x}(k))^T\, R(k+1)\, C(k,\hat{x}(k)) \\ +\, \nabla_x K(k,\hat{x}(k),u_0(k))^T\, K(k,\hat{x}(k),u_0(k)) \end{bmatrix} \\
&+\, \hat{x}(k)^T \begin{bmatrix} \nabla_x A(k,\hat{x}(k),u_0(k))^T\, \Xi(k)\, \nabla_x A(k,\hat{x}(k),u_0(k)) \\ +\, 2\,\nabla_x A(k,\hat{x}(k),u_0(k))^T\, \Xi(k) \,+\, \Xi(k) \\ +\, \nabla_x C(k,\hat{x}(k))^T\, R(k+1)\, \nabla_x C(k,\hat{x}(k)) \\ -\, \nabla_x K(k,\hat{x}(k),u_0(k))^T\, \nabla_x K(k,\hat{x}(k),u_0(k)) \end{bmatrix} \hat{x}(k) \\
&+ \begin{bmatrix} A(k,\hat{x}(k),u_0(k))^T\, \Xi(k)\, A(k,\hat{x}(k),u_0(k)) \\ +\, C(k,\hat{x}(k))^T\, R(k+1)\, C(k,\hat{x}(k)) \\ -\, K(k,\hat{x}(k),u_0(k))^T\, K(k,\hat{x}(k),u_0(k)) \\ +\, y_0(k+1)^T\, R(k+1)\, y_0(k+1) \\ -\, 2\,C(k,\hat{x}(k))^T\, R(k+1)\, y_0(k+1) \end{bmatrix} \\
&+\, \left[\Phi(k) \,-\, \hat{x}(k+1)^T\, X(k+1)\, \hat{x}(k+1)\right], \\
&\Phi(0) \,=\, 0, \quad (21)
\end{aligned}$$

where $\Xi(k)$ is defined in (16).

Assuming that the recursions (18), (19), (21) have solutions $\hat{x}(k)$, $X(k)$, $\Phi(k)$ such that $X(k)>0$ and $(Q(k)+B(k)^T X(k)B(k))>0$ for $k=1,2,...,T$, the corresponding approximate set-valued state estimate is given by,

$$\begin{aligned}
&Z_T[x_0,\, u_0(\cdot)|_1^T,\, y_0(\cdot)|_1^T,\, d] \\
&\cong \left\{ \begin{matrix} x_T \in \mathbb{R}^n \,: \\ (x(T)\,-\,\hat{x}(T))^T\, X(T)\, (x(T)\,-\,\hat{x}(T)) \leq\, d \,-\, \Phi(T) \end{matrix} \right\}. \quad (22)
\end{aligned}$$

It should be noted that, if the reverse-time discrete-time uncertain system (4), (5) is in fact linear as in [12], [13], then the nonlinear Riccati difference equation (18), the filter state equation (19) and the filter constant equation (21) yield a discrete-time linear robust Kalman filter that agrees with the solution presented in [12], [13].

IV. Illustrative Example

The effectiveness of the discrete-time REKF over the discrete-time EKF is illustrated by applying both methods to a spring-mass-damper system in which the total energy in the system is measured. The total energy can be indirectly measured using available inertial or motion sensors, and applied to structural health monitoring and vibration analysis of aircrafts, spacecrafts and missiles. Computing the energy also aids in understanding the total energy profile during various phases of a missile trajectory with the aim of improving range and accuracy.

We are interested in estimating the velocity $x_1(t)$ and position $x_2(t)$ of such a system in the presence of modeling uncertainties as well as exogenous disturbances. Consider a continuous-time uncertain spring-mass-damper system described by the state equations,

$$\begin{aligned} \dot{x}(t) &= A_c\, x(t) + B_c\, w(t), \\ y(t) &= C_c(t, x(t)) + v(t), \\ z(t) &= K_c(t, x(t)), \end{aligned} \tag{23}$$

where,

$$\begin{aligned} x(t) &= [x_1(t)\ \ x_2(t)]^T, \\ A_c &= \begin{bmatrix} -2\zeta\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix}, \\ B_c &= [1\ \ 0]^T, \\ K_c(t, x(t)) &= [1\ \ 1]\, x(t), \\ w(t) &= \left(\omega_n^2 \tilde{w}(t) + \Delta_1 z(t)\right), \\ C_c(t, x(t)) &= \frac{1}{2}\left(m x_1(t)^2 + k x_2(t)^2\right), \\ v(t) &= \left(\tilde{v}(t) + \Delta_2 z(t)\right). \end{aligned} \tag{24}$$

Here, $\tilde{w}(t)$, $\tilde{v}(t)$ represent exogenous disturbances, and the terms $\Delta_1 z(t)$ and $\Delta_2 z(t)$ represent uncertainties in the model dynamics. In the discretized uncertain system model presented below, these quantities will be bounded by an SQC. The description and values of various constants in the system are detailed in Table I.

TABLE I

Spring-mass-damper System Constants

Constant	Value	Units	Description
ζ	0.2	-	Damping ratio
ω_n	1	Hz	Natural frequency
m	5	kg	Mass
k	$\omega_n^2 m$	N m^{-1}	Spring constant

The measurement equation in (24) represents measurement of the total energy in the system, which can be measured using standard sensors in a physical application. In order to simulate the spring-mass-damper system, the continuous-time system in (23)-(24) is descretized as,

$$\begin{aligned} x(k+1) &= A_d\, x(k) + B_d\, w(k), \\ y(k+1) &= C_d(k, x(k+1)) + v(k+1), \\ z(k+1) &= K_d(k, x(k+1)), \end{aligned} \tag{25}$$

where,

$$\begin{aligned} x(k) &= [x_1(k)\ \ x_2(k)]^T, \\ A_d &= e^{A_c h}, \\ B_d &= \int_0^{kh} e^{A_c \tau} d\tau\ \ B_c, \\ K_d(k, x(k+1)) &= [1\ \ 1]\, x(k+1), \\ w(k) &= \left(\omega_n^2 \tilde{w}(k) + \Delta_1 z(k+1)\right), \\ C_d(k, x(k+1)) &= \frac{1}{2}\left(m x_1(k+1)^2 + k x_2(k+1)^2\right), \\ v(k) &= \left(\tilde{v}(k+1) + \Delta_2 z(k+1)\right). \end{aligned} \tag{26}$$

Here, $k = t/h$ and the sample time is chosen as $h = 10\,ms$. The system of equations (25)-(26) was used to simulate the discrete-time spring-mass-damper system with initial conditions $x(0) = [0\ \ 0]^T$. The uncertainty in the system is defined by the SQC,

$$\sum_{k=0}^{10/h} \left(0.05\, w(k)^2 + 0.5\, v(k+1)^2\right) \le d + |z(k+1)|^2. \tag{27}$$

As described in Section III, the dynamics of the discrete-time uncertain system under consideration needs to be modeled in reverse-time in order to construct the corresponding Riccati and filter equations. This can be calculated using the forward-time continuous-time dynamics as,

$$\begin{aligned} x(k) &= A_k\, x(k+1) + B_k\, w(k), \\ y(k+1) &= C_k(k, x(k+1)) + v(k+1), \\ z(k+1) &= K_k(k, x(k+1)), \end{aligned} \tag{28}$$

where,

$$\begin{aligned} x(k) &= [x_1(k)\ \ x_2(k)]^T, \\ A_k &= e^{-A_c h}, \\ B_k &= \int_0^{kh} e^{-A_c \tau} d\tau\ \ B_c, \\ K_k(k, x(k+1)) &= K_d(k, x(k+1)), \\ C_k(k, x(k+1)) &= C_d(k, x(k+1)). \end{aligned} \tag{29}$$

The uncertainty in both process as well as measurement equations remains the same as in (26) and (27).

The REKF Riccati equation (18) and filter equation (19) were applied to the simulated system (25)-(26) using the system dynamic model described by (27)-(29). The discrete-time EKF was also applied to the simulated system using the Riccati equation (18) and filter equation (19) for the system (27)-(29) obtained by setting $K_k(k, x(k+1)) = [0\ \ 0]$. A random initial state of $x_{init} = [0.86\ \ 0.09]^T$ was used for the estimator process.

The results obtained by applying the discrete-time REKF as well as the discrete-time EKF to the discrete-time uncertain spring-mass-damper system ((25)-(27)) are now presented. In these simulations, the disturbance sequences $\tilde{w}(k)$ and $\tilde{v}(k+1)$ were taken as uncorrelated white noise sequences with covariances $E(\tilde{w}(k)^2) = 0.62$ and $E(\tilde{v}(k+$

$1)^2) = 6.23e-5$. Also, the uncertain parameters were taken as $\Delta_1 = 2.68$ and $\Delta_2 = 0.85$. These values are consistent with the SQC (27). Fig. 1 compares the absolute velocity errors whereas Fig. 2 compares the absolute position errors obtained by applying the discrete-time REKF and EKF methods to the uncertain spring-mass-damper system. It is evident from these plots that the discrete-time REKF estimates states of the uncertain spring-mass-damper system accurately with a steady decrease in absolute error, whereas the EKF shows clear signs of divergence.

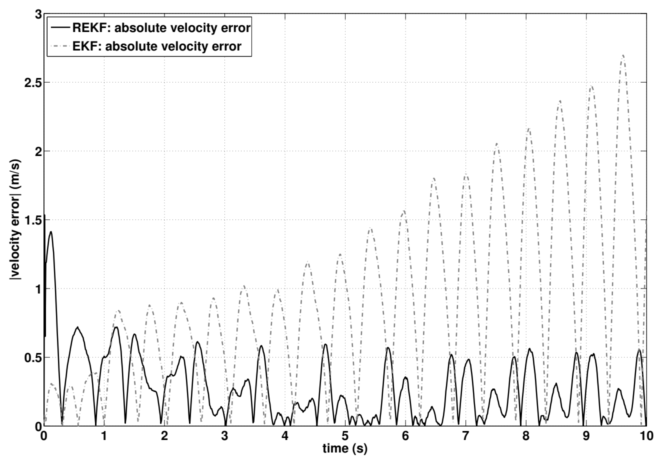

Fig. 1. Comparison of the absolute velocity errors obtained by applying discrete-time REKF and EKF to the uncertain spring-mass-damper system.

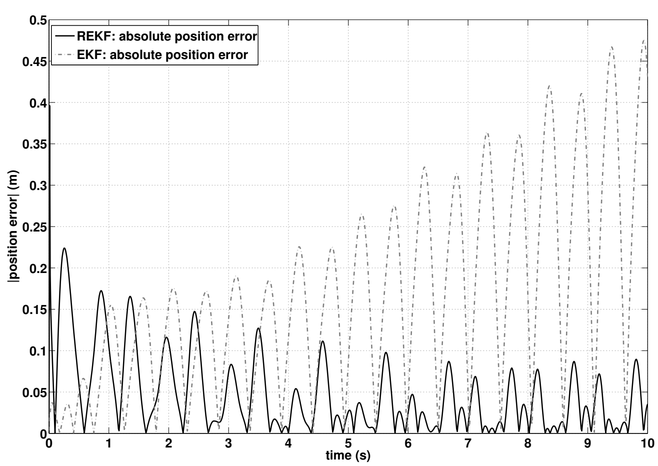

Fig. 2. Comparison of the absolute position errors obtained by applying discrete-time REKF and EKF to the uncertain spring-mass-damper system.

V. CONCLUSIONS AND FUTURE WORKS

In this paper, we have presented a novel discrete-time robust extended Kalman filter (REKF) based on the reverse-time discretization of a continuous-time nonlinear uncertain system. Filter and Riccati difference equations were derived as an approximate set-valued state estimator obtained from the solution to a corresponding optimal control problem. The performance of the new filter was illustrated by comparing its absolute estimation error results with that of a standard EKF for an uncertain spring-mass-damper system. The new filter can be applied to nonlinear discrete-time systems with uncertainties modeled by a sum quadratic constraint (SQC) uncertainty description.

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