2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Stochastic Nash Games for Multimodeling Systems

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Abstract— **In this paper, the linear quadratic infinite horizon Nash games for stochastic multimodeling systems (SMS) are discussed. After establishing the asymptotic structure of the solutions to the cross-coupled stochastic multimodeling algebraic Riccati equation (CSMARE), a parameter independent stochastic Nash strategy set is given on the basis of this structure. The degradation of the cost by means of the proposed strategy is also investigated. As another important feature, a new algorithm for solving the reduced-order CSMARE is derived. The numerical example for a multimachine power system is given to demonstrate the efficiency and feasibility of the proposed algorithm.**

I. INTRODUCTION

The problem of designing a feedback strategy for a multimodeling system has been subject of many papers during the past three decades (see e.g., [6], [7], [8], [9]). Recent advance in theory of the descriptor systems approach has allowed a revisiting of the Nash games for the multiparameter singularly perturbed systems [13]. These literatures, however, do not address the issue of robustness against stochastic uncertainty such as standard Wiener process.

The stochastic control problems governed by Itô's differential equation have become a popular research topic during the past decade [1], [2], [3]. It has attracted much attention and has been widely applied to various control problems. Recently, the stochastic H_2/H_∞ control with state-dependent noise has been addressed [4]. Although these results are very elegant in theory and despite the fact that it is easy to design a feedback controller, a control problem with multiple decision makers is an issue that remains to be considered.

In this paper, the linear quadratic infinite horizon Nash games for stochastic multimodeling systems (SMS) are investigated. It should be noted that the obtained result is different although related to those of [13]. Particularly, although the Nash games for multiparameter singularly perturbed systems have been investigated and the parameters independent strategy set was also given in [13], there exists a difference for the considered SMS. The above papers mentioned consider for the deterministic SMS. We consider stochastic uncertain Nash games with standard Wiener process. Furthermore, the nature of stochastic uncertainty considered here is different from that of the aforementioned paper [9]. Indeed, although we consider the SMS that is governed by Itô's differential equation, in [9], LQG Nash game problem with white Gaussian noise is investigated. The contributions of this paper are as follows. First, the stochastic model of SMS corresponding to the problem of the Nash games is defined. After establishing the asymptotic structure of the solutions to the cross-coupled stochastic multimodeling algebraic Riccati equation (CSMARE), a parameter independent stochastic Nash strategy set is established. Furthermore, the degradation of the cost by means of the proposed strategy is also investigated. As another important feature, in order to solve the reduced-order CSMARE, a new algorithm on the basis of the fixed point iterations is derived. Finally, in order to demonstrate the efficiency of the proposed algorithm, a numerical example for a multimachine power system is solved.

Notation: The notations used in this paper are fairly standard. $\det L$ denotes the determinant of square matrix L. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. **block diag** denotes the block diagonal matrix. $\mathrm{vec}M$ denotes the column vector of the matrix M [5]. $\otimes$ denotes the Kronecker product. $\oplus$ denotes the Kronecker sum such that $M \oplus N := M \otimes I_n + I_m \otimes N$, $M \in \Re^{m\times m}$, $N \in \Re^{n\times n}$. U_{lm} denotes a permutation matrix in the Kronecker matrix sense [5] such that $U_{lm}\mathrm{vec}M = \mathrm{vec}M^T$, $M \in \Re^{l\times m}$. $E[\cdot]$ denotes the expection operator.

II. PRELIMINARY RESULT

Let us consider the following SMS that consist of N-fast subsystems with specific structure of lower level interconnected through the dynamics of a higher level slow subsystem.

$$\begin{aligned} dx_0(t) &= \left[\sum_{j=0}^{N} A_{0j}x_j(t) + \sum_{j=1}^{N} B_{0j}u_j(t)\right] dt \\ &\quad + \sum_{p=1}^{M}\left[A_{p00}x_0(t) + \mu\sum_{j=1}^{N} A_{p0j}x_j(t)\right] dw_p(t), \\ x_0(0) &= x_0^0, \end{aligned} \tag{1a}$$

$$\begin{aligned} \varepsilon_i dx_i(t) &= [A_{i0}x_0(t) + A_{ii}x_i(t) + B_{ii}u_i(t)]dt \\ &\quad + \varepsilon^{\delta}\sum_{p=1}^{M}[A_{pi0}x_0(t) + A_{pii}x_i(t)]dw_p(t), \\ x_i(0) &= x_i^0,\ i = 1,\ \dots,N, \end{aligned} \tag{1b}$$

where $x_i(t) \in \Re^{n_i},\ i = 0,\ 1,\ \dots,N$ are the state vectors, $u_i(t) \in \Re^{m_i},\ i = 1,\ \dots,N$ are the control inputs. $\varepsilon_i > 0$,

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This research was supported by the Grant-in-Aid for Scientific Research (C)-20500014 from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

$i = 1, \ldots, N$ and $\mu \geq 0$ are small parameters and $\delta > 1/2$ is independent of $\bar{\varepsilon} := \min\{\varepsilon_1, \ldots, \varepsilon_N\}$ [10], [11]. It should be noted that the parameters μ and δ have been introduced in [10], [11] for the first time. Moreover, it may be noted that the introducing of $\delta = 1/2$ was originated in [9]. $w_p(t) \in \Re$, $p = 1, \ldots, M$ is a one-dimensional standard Wiener process defined in the filtered probability space [4], [10], [11].

It is assumed that the ratios of the small positive parameters ε_i, $i = 1, \ldots, N$ and μ are bounded by some positive constants $\underline{k}_{ij}$, $\bar{k}_{ij}$, $\underline{l}$ and $\bar{l}$ and only these bounds are assumed to be known [6], [7]. In other words, they have the same order of magnitude.

$$0 < \underline{k}_{ij} \leq \alpha_{ij} \equiv \frac{\varepsilon_j}{\varepsilon_i} \leq \bar{k}_{ij} < \infty, \ 0 \leq \underline{l} \leq \frac{\mu}{\bar{\varepsilon}} \leq \bar{l} < \infty. \quad (2)$$

Note that one of the fast state matrices A_{jj}, $j = 1, \ldots, N$ may be singular. The performance criterion is given by

$$J_i(u_1, \ldots, u_N) = E\int_0^{\infty} [x^T(t) Q_i x(t) + u_i^T(t) R_i u_i(t)] dt, \ i = 1, \ldots, N \quad (3)$$

where $\bar{n} := \sum_{j=0}^{N} n_j$,

$$\begin{aligned}
x(t) &:= \begin{bmatrix} x_0^T(t) & x_1^T(t) & \cdots & x_N^T(t) \end{bmatrix}^T \in \Re^{\bar{n}}, \\
Q_i &:= C_i^T C_i = \begin{bmatrix} Q_{i00} & Q_{i0f} \\ Q_{i0f}^T & Q_{if} \end{bmatrix}, \ Q_{i00} := C_{i0}^T C_{i0}, \\
Q_{i0f} &:= \begin{bmatrix} 0 & \cdots & 0 & Q_{i0i} & 0 & \cdots & 0 \end{bmatrix}, \\
Q_{if} &:= \textbf{block diag} \begin{pmatrix} 0 & \cdots & 0 & Q_{iii} & 0 & \cdots & 0 \end{pmatrix}, \\
C_1 &:= \begin{bmatrix} C_{10} & C_{11} & 0 & \cdots & 0 \end{bmatrix}, \\
&\vdots \\
C_i &:= \begin{bmatrix} C_{i0} & 0 & \cdots & 0 & C_{ii} & 0 & \cdots & 0 \end{bmatrix}, \\
&\vdots \\
C_N &:= \begin{bmatrix} C_{0N} & 0 & \cdots & 0 & C_{NN} \end{bmatrix}.
\end{aligned}$$

Without loss of generality, the following basic assumption (see e.g., [7]) is made.

Assumption 1: The triples (A_{ii}, B_{ii}, C_{ii}), $i = 1, \ldots, N$ are stabilizable and detectable.

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes. Our purpose is to find a linear feedback strategy set $(u_1^*, \ldots, u_N^*)$ such that

$$J_i(u_1^*, \ldots, u_N^*) \leq J_i(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_N^*), \ i = 1, \ldots, N. \quad (4)$$

The decision makers are required to select the closed loop strategy $u_i^*(t)$, if they exist, such that (4) holds. Moreover, each player uses the strategy $u_i^*(t)$ such that the closed-loop system is asymptotically mean square stable for sufficiently small ε_i [1]. The following lemma is already known [12].

Lemma 1: There exists an admissible strategy such that the inequality (4) holds iff the cross-coupled stochastic multimodeling algebraic Riccati equation (CSMARE),

$$P_{ie}\left(A_e - \sum_{j=1}^{N} S_{je} P_{je}\right) + \left(A_e - \sum_{j=1}^{N} S_{je} P_{je}\right)^T P_{ie} + \sum_{p=1}^{M} A_{pe}^T P_{ie} A_{pe} + P_{ie} S_{ie} P_{ie} + Q_i = 0, \quad (5)$$

have solutions $P_{ie} := \Phi_e P_i \geq 0$, where

$$\begin{aligned}
\Phi_e &:= \textbf{block diag} \begin{pmatrix} I_{n_0} & \Pi_e \end{pmatrix}, \\
\Pi_e &:= \textbf{block diag} \begin{pmatrix} \varepsilon_1 I_{n_1} & \cdots & \varepsilon_N I_{n_N} \end{pmatrix}, \\
A_e &:= \Phi_e^{-1} A, \ A_{pe} := \Phi_e^{-1} A_p, \\
B_{ie} &:= \Phi_e^{-1} B_i, \ S_{ie} := B_{ie} R_i^{-1} B_{ie}^T, \\
P_i &:= \begin{bmatrix} P_{i00} & P_{if0}^T \Pi_e \\ P_{if0} & P_{if} \end{bmatrix}, \ P_{i00} := P_{i00}^T, \\
P_{if0} &:= \begin{bmatrix} P_{i10}^T & \cdots & P_{iN0}^T \end{bmatrix}^T, \ \Pi_e P_{if} := P_{if}^T \Pi_e, \\
P_{if} &:= \begin{bmatrix} P_{i11} & \alpha_{12} P_{i21}^T & \alpha_{13} P_{i31}^T & \cdots & \alpha_{1N} P_{iN1}^T \\ P_{i21} & P_{i22} & \alpha_{23} P_{i32}^T & \cdots & \alpha_{2N} P_{iN2}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{i(N-1)1} & P_{i(N-1)2} & P_{i(N-1)3} & \cdots & \alpha_{(N-1)N} P_{iN(N-1)}^T \\ P_{iN1} & P_{iN2} & P_{iN3} & \cdots & P_{iNN} \end{bmatrix}, \\
A &:= \begin{bmatrix} A_{00} & A_{0f} \\ A_{f0} & A_f \end{bmatrix}, \ A_p := \begin{bmatrix} A_{p00} & \mu A_{p0f} \\ \bar{\varepsilon}^{\delta} A_{pf0} & \bar{\varepsilon}^{\delta} A_{pf} \end{bmatrix}, \\
A_{0f} &:= \begin{bmatrix} A_{01} & \cdots & A_{0N} \end{bmatrix}, \\
A_{f0} &:= \begin{bmatrix} A_{10}^T & \cdots & A_{N0}^T \end{bmatrix}^T, \\
A_f &:= \textbf{block diag} \begin{pmatrix} A_{11} & \cdots & A_{NN} \end{pmatrix}, \\
A_{p0f} &:= \begin{bmatrix} A_{p01} & \cdots & A_{p0N} \end{bmatrix}, \\
A_{pf0} &:= \begin{bmatrix} A_{p10}^T & \cdots & A_{pN0}^T \end{bmatrix}^T, \\
A_{pf} &:= \textbf{block diag} \begin{pmatrix} A_{p11} & \cdots & A_{pNN} \end{pmatrix}, \\
B_1 &:= \begin{bmatrix} B_{10}^T & B_{11}^T & 0 & \cdots & 0 \end{bmatrix}^T, \\
&\vdots \\
B_i &:= \begin{bmatrix} B_{i0}^T & 0 & \cdots & 0 & B_{ii}^T & 0 & \cdots & 0 \end{bmatrix}^T, \\
&\vdots \\
B_N &:= \begin{bmatrix} B_{0N}^T & 0 & \cdots & 0 & B_{NN}^T \end{bmatrix}^T.
\end{aligned}$$

Then the closed-loop linear Nash equilibrium solutions to the full-order problem are given by

$$u_i^*(t) = -R_i^{-1} B_i^T P_i x(t). \quad (6)$$

III. SYMPTOTIC STRUCTURE AND LOCAL UNIQUENESS

In order to obtain the approximate Nash strategies for the CSMARE (5), asymptotic structure and local uniqueness are investigated.

$$\det\begin{bmatrix} \tilde{A}_s^T \oplus \tilde{A}_s^T + \sum_{p=1}^{M} A_{p00}^T \otimes A_{p00}^T & -(S_{s_2}\bar{P}_{100}) \oplus (S_{s_2}\bar{P}_{100}) & \cdots & -(S_{s_N}\bar{P}_{100}) \oplus (S_{s_N}\bar{P}_{100}) \\ -(S_{s_1}\bar{P}_{200}) \oplus (S_{s_1}\bar{P}_{200}) & \tilde{A}_s^T \oplus \tilde{A}_s^T + \sum_{p=1}^{M} A_{p00}^T \otimes A_{p00}^T & \cdots & -(S_{s_N}\bar{P}_{200}) \oplus (S_{s_N}\bar{P}_{200}) \\ \vdots & \vdots & \ddots & \vdots \\ -(S_{s_1}\bar{P}_{N00}) \oplus (S_{s_1}\bar{P}_{N00}) & -(S_{s_2}\bar{P}_{N00}) \oplus (S_{s_2}\bar{P}_{N00}) & \cdots & \tilde{A}_s^T \oplus \tilde{A}_s^T + \sum_{p=1}^{M} A_{p00}^T \otimes A_{p00}^T \end{bmatrix} \neq 0, \quad (9)$$

where $\tilde{A}_s := A_s - \sum_{j=1}^{N} S_{s_j}\bar{P}_{j00}$ and $\tilde{A}_s$ are stable matrix.

Under Assumption 1, the following zeroth-order equations of the CSMARE (5) are given as $\|\nu\| := \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_N^2 + \mu^2} \to 0^+$.

$$\bar{P}_{i00}\tilde{A}_s + \tilde{A}_s^T\bar{P}_{i00} + \sum_{p=1}^{M} A_{p00}^T\bar{P}_{i00}A_{p00} + \bar{P}_{i00}S_{s_i}\bar{P}_{i00} + Q_{s_i} = 0, \quad (7a)$$

$$A_{ii}^T\bar{P}_{iii} + \bar{P}_{iii}A_{ii} - \bar{P}_{iii}S_{iii}\bar{P}_{iii} + Q_{iii} = 0, \quad (7b)$$

$$\bar{P}_{ikl} = 0,\ k > l,\ \bar{P}_{ijj} = 0,\ i \neq j \quad (7c)$$

$$\begin{aligned} &\begin{bmatrix} \bar{P}_{110} & \bar{P}_{210} & \cdots & \bar{P}_{N10} \end{bmatrix} \\ &= \begin{bmatrix} \bar{P}_{111} & -I_{n_1} \end{bmatrix} T_{111}^{-1}T_{110} \begin{bmatrix} I_{n_0} & 0 & \cdots & 0 \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix}, \\ &\begin{bmatrix} \bar{P}_{120} & \bar{P}_{220} & \cdots & \bar{P}_{N20} \end{bmatrix} \\ &= \begin{bmatrix} \bar{P}_{222} & -I_{n_2} \end{bmatrix} T_{222}^{-1}T_{220} \begin{bmatrix} 0 & I_{n_0} & \cdots & 0 \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix}, \\ &\qquad \vdots \\ &\begin{bmatrix} \bar{P}_{1N0} & \bar{P}_{2N0} & \cdots & \bar{P}_{NN0} \end{bmatrix} \\ &= \begin{bmatrix} \bar{P}_{NNN} & -I_{n_N} \end{bmatrix} T_{NNN}^{-1}T_{NN0} \\ &\quad \times \begin{bmatrix} 0 & 0 & \cdots & I_{n_0} \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix}, \end{aligned} \quad (7d)$$

where $\tilde{A}_s := A_s - \sum_{j=1}^{N} S_{s_j}\bar{P}_{j00}$

$$\begin{bmatrix} A_s & \star \\ \star & -A_s^T \end{bmatrix} = \begin{bmatrix} A_{00} & \star \\ \star & -A_{00}^T \end{bmatrix} - \sum_{i=1}^{N} T_{i0i}T_{iii}^{-1}T_{ii0},$$

$$\begin{bmatrix} \star & -S_{s_i} \\ -Q_{s_i} & \star \end{bmatrix} = T_{i00} - T_{i0i}T_{iii}^{-1}T_{ii0},$$

$$T_{i00} = \begin{bmatrix} A_{00} & -S_{i00} \\ -Q_{i00} & -A_{00}^T \end{bmatrix},\ T_{i0i} = \begin{bmatrix} A_{0i} & -S_{i0i} \\ -Q_{i0i} & -A_{i0}^T \end{bmatrix},$$

$$T_{ii0} = \begin{bmatrix} A_{i0} & -S_{i0i}^T \\ -Q_{i0i}^T & -A_{0i}^T \end{bmatrix},\ T_{iii} := \begin{bmatrix} A_{ii} & -S_{iii} \\ -Q_{iii} & -A_{ii}^T \end{bmatrix},$$

$i = 1,\ \ldots,N$.

Before establishing the asymptotic structure of the reduced-order solution, we introduce the following assumption.

Assumption 2: The cross-coupled stochastic algebraic Riccati equation (CSARE) (7a) has stabilizing solution $\bar{P}_{i00}$, $i = 1,\ \ldots\ N$. This means that the solution $x_0(t) = 0$ of the closed-loop stochastic system

$$dx_0(t) = \tilde{A}_s x_0(t)dt + \sum_{p=1}^{M} A_{p00}x_0(t)dw_p(t) \quad (8)$$

is exponentially stable in mean square.

It may be noted that the stochastic stabilizability is necessary condition for the existence of the stabilizing solution of CSARE.

The following theorem shows the relation between the solutions P_i and the zeroth-order solutions $\bar{P}_{ikl}$, $i = 1,\ \ldots,N,\ k \geq l,\ 0 \leq k,\ l \leq N$.

Theorem 1: Under Assumptions 1 and 2, suppose that the condition (9) that is given at the top of this page holds. There is a neighborhood $\mathcal{V}(0)$ of $\|\nu\| = 0$ such that for all $\|\nu\| \in \mathcal{V}(0)$ there exists a solution $P_i = P_i(\varepsilon_1,\ \ldots,\varepsilon_N)$. These solutions are unique in a neighborhood of $\bar{P}_i = P_i(0,\ \ldots,0)$. Then, the CSMARE (5) possess the power series expansion at $\|\nu\| = 0$. That is, the following form is satisfied.

$$P_i = \bar{P}_i + O(\|\nu\|) = \begin{bmatrix} \bar{P}_{i00} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \bar{P}_{i10} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{ii0} & 0 & \cdots & 0 & \bar{P}_{iii} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{iN0} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + O(\|\nu\|). \quad (10)$$

Proof: First, zeroth-order solutions for the asymptotic structure of CSMARE (5) are established. Under Assumption 1, the following equality holds.

$$T_{iii} = \begin{bmatrix} I_{n_i} & 0 \\ \bar{P}_{iii} & I_{n_i} \end{bmatrix} \begin{bmatrix} \hat{A}_{ii} & -S_{ii} \\ 0 & -\hat{A}_{ii}^T \end{bmatrix} \begin{bmatrix} I_{n_i} & 0 \\ -\bar{P}_{iii} & I_{n_i} \end{bmatrix} \quad (11)$$

where $\hat{A}_{ii} := A_{ii} - S_{iii}\bar{P}_{iii}$. Since T_{iii} is nonsingular, $\hat{A}_{ii}$ is also nonsingular. This means that T_{iii}^{-1} can be expressed explicitly in terms of $\hat{A}_{ii}^{-1}$. Therefore, using the above result, the formulations (7) are obtained. These transformations can be done by the lengthy, but direct algebraic manipulations [13], which are omitted here.

For the local uniqueness of the solutions $P_i = P_i(\varepsilon_1,\ \ldots,\varepsilon_N)$, it is enough to verify that the corresponding Jacobian is nonsingular at $\|\nu\| = 0$. Formally calculating the derivative of the CSMARE (5) and after some tedious algebra, the left-hand side of (9) is obtained. Setting $\|\nu\| = 0$ and using (7), the condition (9) is obtained. Finally, the implicit function theorem implies that there is a unique solutions map $P_i = P_i(\varepsilon_1,\ \ldots,\varepsilon_N)$ and a neighborhood $\mathcal{V}(0)$ of $\|\nu\| = 0$ because the condition (9) is equivalent to the corresponding Jacobian at $\|\nu\| = 0$. ■

It is noteworthy that the local uniqueness is newly shown compared with the existing results [6], [7], [13]. Moreover, it may be noted that the formulas under the equation (7) have been used to simplify the expressions for the first time to the stochastic case.

IV. FIXED POINT ITERATIONS

In order to obtain the strategy set, we have to solve CSARE (7a). Now, we give a new numerical computation method on the basis of the fixed point iteration.

Let us consider the following fixed point algorithm for solving the CSARE (7a).

$$\begin{aligned}&\bar{P}_{i00}^{(n+1)}\boldsymbol{A}(n)+\boldsymbol{A}^T(n)\bar{P}_{i00}^{(n+1)}\\&\quad+\sum_{p=1}^{M}A_{p00}^T\bar{P}_{i00}^{(n+1)}A_{p00}+\bar{P}_{i00}^{(n)}S_{s_i}\bar{P}_{i00}^{(n)}+Q_{s_i}=0,\\&n=0,\ 1,\ \dots,\end{aligned}\tag{12}$$

where $\boldsymbol{A}(n):=A_s-\sum_{j=1}^{N}S_{s_j}\bar{P}_{j00}^{(n)}$ and $\bar{P}_{i00}^{(0)}$ is the solutions of the following stochastic algebraic Riccati equation (SARE).

$$\begin{aligned}&\bar{P}_{100}^{(0)}A_s+A_s^T\bar{P}_{100}^{(0)}-\bar{P}_{100}^{(0)}S_{s_1}\bar{P}_{100}^{(0)}\\&\quad+\sum_{p=1}^{M}A_{p00}^T\bar{P}_{100}^{(0)}A_{p00}+Q_{s_1}=0,\\&\bar{P}_{200}^{(0)}(A_s-S_{s_1}\bar{P}_{100}^{(0)})+(A_s-S_{s_1}\bar{P}_{100}^{(0)})^T\bar{P}_{200}^{(0)}\\&\quad-\bar{P}_{200}^{(0)}S_{s_2}\bar{P}_{200}^{(0)}+\sum_{p=1}^{M}A_{p00}^T\bar{P}_{200}^{(0)}A_{p00}+Q_{s_2}=0,\\&\qquad\vdots\\&\bar{P}_{N00}^{(0)}\left(A_s-\sum_{j=1}^{N-1}S_{s_j}\bar{P}_{j00}^{(0)}\right)+\left(A_s-\sum_{j=1}^{N-1}S_{s_j}\bar{P}_{j00}^{(0)}\right)^T\bar{P}_{N00}^{(0)}\\&\quad-\bar{P}_{N00}^{(0)}S_{s_N}\bar{P}_{N00}^{(0)}+\sum_{p=1}^{M}A_{p00}^T\bar{P}_{N00}^{(0)}A_{p00}+Q_{s_N}=0.\end{aligned}$$

Theorem 2: Suppose the positive semidefinite solutions of the CSARE (7a) exist. Under Assumptions 1 and 2, there exists a small $\hat{\sigma}$ such that for all $\|\nu\|\in(0,\ \hat{\sigma}),\ \hat{\sigma}\le\sigma^*$ the fixed point algorithm (12) converges to the exact solution $\bar{P}_{i00}^*$. Moreover, $\bar{P}_{i00}^{(n)}$ is positive semidefinite and $\boldsymbol{A}(n)$ is stable.

Proof: We give the proof by using the successive approximation technique [3]. Firstly, we take any stabilizable linear strategy $\bar{u}_i^{(0)}(t,\ \bar{x})=-R_i^{-1}B_{i0}^T\bar{P}_{i00}^{(0)}\bar{x}(t)$. Then, the following minimization problems need to be considered.

$$\begin{aligned}d\bar{x}(t)&=\left[\left(A-\sum_{j=1,\ j\neq i}^{N}S_{s_j}\bar{P}_{j00}^{(0)}\right)\bar{x}(t)+B_i\bar{u}_i(t)\right]dt\\&\quad+\sum_{p=1}^{M}A_{p00}\bar{x}(t)dw_p(t),\end{aligned}\tag{13a}$$

$$\begin{aligned}&V_i(t,\ \bar{x})\\&=\min_{\bar{u}_i}E\int_t^{\infty}[\bar{x}^T(\tau)Q_{s_i}\bar{x}(\tau)+\bar{u}_i^T(\tau)R_i\bar{u}_i(\tau)]d\tau.\end{aligned}\tag{13b}$$

Corresponding Hamiltonians to the stochastic Nash differential games for each control agent are given below.

$$\begin{aligned}&H_i\left(t,\ \bar{x},\ \bar{u}_1^{(0)},\ \dots,\bar{u}_{i-1}^{(0)},\ \bar{u}_i,\ \bar{u}_{i+1}^{(0)},\ \dots,\bar{u}_N^{(0)},\ p_i^{(0)}\right)\\&=\frac{1}{2}\mathbf{Tr}\left[\sum_{p=1}^{M}\bar{x}^T(t)A_{p00}^T\frac{\partial^2V_i^{(0)}}{\partial\bar{x}^2}A_{p00}\bar{x}(t)\right]\\&\quad+\bar{x}(t)^TQ_{s_i}\bar{x}(t)+\bar{u}_i(t)^TR_i\bar{u}_i(t)\\&\quad+\left(\frac{\partial V_i^{(0)}}{\partial\bar{x}}\right)^T\left[\left(A-\sum_{j=1,\ j\neq i}^{N}S_{s_j}\bar{P}_{j00}^{(0)}\right)\bar{x}(t)+B_i\bar{u}_i(t)\right],\end{aligned}\tag{14}$$

where

$$\begin{aligned}&d\bar{x}(t)=\boldsymbol{A}(n)\bar{x}(t)dt+\sum_{p=1}^{M}A_{p00}\bar{x}(t)dw_p(t),\\&V_i^{(0)}:=V_i^{(0)}(t,\ \bar{x})\\&\qquad=E\int_t^{\infty}\bar{x}^T(\tau)\left(Q_{s_i}+\bar{P}_{i00}^{(0)}S_{s_i}\bar{P}_{i00}^{(0)}\right)\bar{x}(\tau)d\tau.\end{aligned}$$

The equilibrium controls must satisfy the following equation

$$\frac{\partial H_i}{\partial\bar{u}_i}=0\ \Rightarrow\ \bar{u}_i^{(1)}(t,\ \bar{x})=-\frac{1}{2}R_i^{-1}B_i^T\left(\frac{\partial V_i^{(0)}}{\partial\bar{x}}\right).\tag{15}$$

Note that $\partial V_i^{(0)}/\partial\bar{x}$ along the system trajectory can be calculated from (16).

$$\begin{aligned}dV_i^{(0)}(t,\ \bar{x})&=\frac{\partial V_i^{(0)}}{\partial\bar{x}}\left(A_s-\sum_{j=1}^{N}S_{s_j}\bar{P}_{j00}^{(0)}\right)\bar{x}(t)dt\\&\quad+\frac{1}{2}\mathbf{Tr}\left[\sum_{p=1}^{M}\bar{x}^T(t)A_{p00}^T\frac{\partial^2V_i^{(0)}}{\partial\bar{x}^2}A_{p00}\bar{x}(t)\right]dt\\&\quad+2\sum_{p=1}^{M}\bar{x}^T(t)A_{p00}^T\bar{P}_{i00}dw_p(t)\\&=\left[\frac{d}{dt}V_i^{(0)}(t,\ \bar{x})\right]dt,\ i=1,\ \dots,N.\end{aligned}\tag{16}$$

Assume that these simple partial differential equations (16) have solutions of the following form

$$V_i^{(0)}(t,\ \bar{x})=\bar{x}^T(t)\bar{P}_{i00}^{(1)}\bar{x}(t).\tag{17}$$

A partial differentiation to (17) gives

$$\frac{\partial}{\partial\bar{x}}V_i^{(0)}(t,\ \bar{x})=2\bar{P}_{i00}^{(1)}\bar{x}(t),\ \frac{\partial^2}{\partial\bar{x}^2}V_i^{(0)}(t,\ \bar{x})=2\bar{P}_{i00}^{(1)}.\tag{18}$$

Moreover, we have

$$\begin{aligned}dV_i^{(0)}(t,\ \bar{x})&=\left[\frac{d}{dt}V_i^{(0)}(t,\ \bar{x})\right]dt\\&=\bar{x}^T(t)\left(Q_{s_i}+\bar{P}_{i00}^{(0)}S_{s_i}\bar{P}_{i00}^{(0)}\right)\bar{x}(t)dt\\&\quad+2\sum_{p=1}^{M}\bar{x}^T(t)A_{p00}^T\bar{P}_{i00}dw_p(t).\end{aligned}\tag{19}$$

Therefore, using (16) and (18), for any $\bar{x}(t)$ we have

$$\bar{P}_{i00}^{(1)}A_s(0)+A_s(0)^T\bar{P}_{i00}^{(1)} + \sum_{p=1}^{M}A_{p00}^T\bar{P}_{i00}^{(1)}A_{p00}+\bar{P}_{i00}^{(0)T}S_{s_i}\bar{P}_{i00}^{(0)}+Q_{s_i}=0. \quad (20)$$

Thus, from (15) and (18), we get

$$\bar{u}_i^{(1)}(t,\ \bar{x})=-R_i^{-1}B_{i0}^T\bar{P}_{i00}^{(1)}\bar{x}(t),\ \bar{P}_{i00}^{(1)}\geq 0. \quad (21)$$

Repeating the above steps, we get $\bar{u}_i^{(2)}(t,\ \bar{x}) = -R_i^{-1}B_i^T\bar{P}_{i00}^{(2)}\bar{x}(t),\ \bar{P}_{i00}^{(2)}\geq 0$. Continuing the same procedure, we get the sequences of the solution matrices. Finally, by using the monotonicity result of the successive approximations and the minimization technique in the negative gradient direction [3], we get a monotone decreasing sequence

$$V_i^{(n+1)}(t,\ \bar{x})\leq V_i^{(n)}(t,\ \bar{x}), \quad (22)$$

where $V_i^{(n)}(t,\ \bar{x})\geq 0$. Thus, these sequences are convergent. Note that the sequence $\bar{u}_i^{(n)}(t,\ \bar{x})$ is also convergent, since $\bar{u}_i^{(n)}(t,\ \bar{x}) = -R_i^{-1}B_{i0}^T\bar{P}_{i00}^{(n)}\bar{x}(t)$. Consequently, from the method of successive approximations [3], the convergence proof is completed.

Second, we prove that $\bar{P}_{i00}^{(n)}$ is positive semidefinite and $\boldsymbol{A}(n) := A_s-\sum_{j=1}^{N}S_{s_j}\bar{P}_{j00}^{(n)}$ is stable. The first stage is to prove that $\boldsymbol{A}(n)$ is stable. The proof is done by using mathematical induction. When $n=0$, $\boldsymbol{A}(0)$ is stable because $\bar{P}_{i00}^{(0)}$ is the stabilizing solution of the SARE. Next $n=q$, we assume that $\boldsymbol{A}(q)$ is stable. Substituting $n=q$ into (14) instead of $n=0$, the minimization problem (14) produce a stabilizing control given by

$$\bar{u}_i^{(q+1)}(t,\ \bar{x})=-R_i^{-1}B_{i0}^T\bar{P}_{i00}^{(q+1)}\bar{x}(t). \quad (23)$$

It is obvious from the method of successive approximations that $\boldsymbol{A}(q+1)$ is stable since it is the stable matrix of the closed-loop stochastic system. Thus, $\boldsymbol{A}(n)$ is stable for all n. The next stage is to prove that $\bar{P}_{i00}^{(n)}$ is positive semidefinite matrix. This proof is also done by using mathematical induction. When $n=0$, it is obvious that $\bar{P}_{i00}^{(n)}$ is positive semidefinite matrix because $\bar{P}_{i00}^{(0)}$ is the positive semidefinite solution of the SARE. Next $n=q$, we assume that $\bar{P}_{i00}^{(q)}$ is positive semidefinite matrix. Using the theory of [2] and fact that $\boldsymbol{A}(q)$ is stable, $\bar{P}_{i00}^{(q+1)}$ is positive semidefinite matrix. Thus, $\bar{P}_{i00}^{(n)}$ is positive semidefinite matrix for all n. Consequently, the proof of Theorem 2 is completed. ■

V. APPROXIMATE NASH STRATEGY

Since there exists a case that the parameters represent small unknown perturbations whose values are not known exactly, it is desirable to have the parameter independent strategy set. Therefore, a parameter independent stochastic Nash strategy set is considered. Using the result (10), the N-order approximate stochastic Nash strategy is given.

$$\tilde{u}_i(t):=-R_{ii}^{-1}B_i^T\bar{P}_ix(t),\ i=1,\ \dots,N, \quad (24)$$

Before introducing the theorem, the following assumption is imposed [4].

Assumption 3: Define $\bar{S}_e := B_eR^{-1}B$. There exists a small $\bar{\sigma}$ such that for all $\|\nu\|\in(0,\ \bar{\sigma})$ the following facts hold.

i) $\left[A_e-\sum_{j=1}^{N}S_{je}\bar{P}_{je},\ A_{1e},\ \dots,A_{Me}\ \middle|\ C_i^TC_i\right]$ is exactly observable.

ii) $\left(A_e-\sum_{j=1}^{N}S_{je}\bar{P}_{je},\ A_{1e},\ \dots,A_{Me}\right)$ is stable.

Theorem 3: Under Assumptions 1-3, the use of the approximate stochastic Nash strategy (24) results in $J_i(\tilde{u}_1,\ \dots,\ \tilde{u}_N)$ satisfying

$$J_i(\tilde{u}_1,\ \dots,\ \tilde{u}_N)=J_i(u_1^*,\ \dots,\ u_N^*)+O(\|\nu\|), \quad (25)$$

where $J_i(u_1^*,\ \dots,\ u_N^*)$ are the optimal equilibrium values of the cost functions (3).

Proof: When $\tilde{u}_i(t)$ is used, the equilibrium value of the cost performances are

$$J_i(\tilde{u}_1,\ \dots,\ \tilde{u}_N)=x^T(0)X_{ie}x(0), \quad (26)$$

where X_{ie} is the positive semidefinite solution of the following multimodeling stochastic algebraic Lyapunov equation (MSALE)

$$X_{ie}\tilde{A}_e+\tilde{A}_e^TX_{ie}+\sum_{p=1}^{M}A_{pe}^TX_{ie}A_{pe}+Q_i+\bar{P}_{ie}S_{ie}\bar{P}_{ie}=0, \quad (27)$$

where $\tilde{A}_e:=A_e-\sum_{j=1}^{N}S_{je}\bar{P}_{je}$ and $\bar{P}_{ie}:=\Phi_e\bar{P}_i$.

Subtracting (5) from (27) and using the relation $\bar{P}_{ie}-P_{ie}=O(\|\nu\|)$, it is easy to verify that $V_{ie}=X_{ie}-P_{ie}$ satisfies the following MSALE.

$$V_{ie}\tilde{A}_e+\tilde{A}_e^TV_{ie}+\sum_{p=1}^{M}A_{pe}^TV_{ie}A_{pe}=O(\|\nu\|). \quad (28)$$

Thus, under Assumption 3, it is easy to verify that $V_{ie}=O(\|\nu\|)$ because $A_e-\sum_{j=1}^{N}S_{je}\bar{P}_{je}$ is stable by using the standard stochastic Lyapunov theorem [4] for sufficiently small $\|\nu\|$. Consequently, the equality (25) holds. ■

Although ε_i is unknown, it is possible to design the approximate stochastic Nash strategy which achieves the $O(\|\nu\|)$ approximation for the equilibrium value of the cost functional.

VI. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the stochastic Nash games for SMS, we present results for the control problem of multimachine power systems. The model is based on the multi-stage decomposition of two interconnected areas [6]. The matrices of the systems model are given as [6]. The other ones related to the stochastic uncertainty are defined as follows.

$$M=1,\ A_{100}=hA_{00},\ A_{10f}=A_{0f},\ \mu=h,$$
$$A_{1f0}=hA_{f0},\ A_{1f}=hA_f,\ h=0.01.$$

The initial state x_0 is assumed to be a random variable with a covariance matrix $E[x(0)x^T(0)]=I_n$.

$$\tilde{u}_1(t) = \left[-3.1623e-01 \;\; -6.9834e-06 \;\; -4.2335 \;\; 1.7058 \;\; -4.3742e-02 \;\; -3.1653e-02 \;\; -6.3792e-02 \;\; 0 \;\; 0 \right] x(t),$$

$$\tilde{u}_2(t) = \left[-6.9834e-06 \;\; -3.1623e-01 \;\; 1.7058 \;\; -4.2335 \;\; 4.3742e-02 \;\; 0 \;\; 0 \;\; -3.1653e-02 \;\; -6.3792e-02 \right] x(t),$$

$$u_1^*(t) = \left[-3.1623e-01 \;\; -2.0187e-06 \;\; -4.2370 \;\; 1.7078 \;\; -4.2875e-02 \;\; -3.2507e-02 \;\; -6.4193e-02 \;\; 3.3142e-04 \;\; 1.5571e-04 \right] x(t),$$

$$u_2^*(t) = \left[-2.0187e-06 \;\; -3.1623e-01 \;\; 1.7078 \;\; -4.2370 \;\; 4.2875e-02 \;\; 3.3142e-04 \;\; 1.5571e-04 \;\; -3.2507e-02 \;\; -6.4193e-02 \right] x(t).$$

Table 2.

ε_1	ε_2	$J_1(u_1^*(t),\ u_2^*(t))$	$J_2(u_1^*(t),\ u_2^*(t))$	$J_1(\tilde{u}_1(t),\ \tilde{u}_2(t))$	$J_2(\tilde{u}_1(t),\ \tilde{u}_2(t))$	ψ_1	ψ_2
$1.0000e-03$	$1.0000e-04$	$8.5712e+02$	$8.5883e+02$	$8.5652e+02$	$8.5327e+02$	$4.4614e+02$	$3.9267e+03$
$1.0000e-04$	$1.0000e-05$	$8.4741e+02$	$8.4756e+02$	$8.4732e+02$	$8.4699e+02$	$6.7390e+02$	$4.0184e+03$
$1.0000e-05$	$1.0000e-06$	$8.4648e+02$	$8.4650e+02$	$8.4647e+02$	$8.4644e+02$	$6.9548e+02$	$4.0259e+03$
$1.0000e-06$	$1.0000e-07$	$8.4639e+02$	$8.4639e+02$	$8.4639e+02$	$8.4638e+02$	$6.9762e+02$	$4.0267e+03$
$1.0000e-07$	$1.0000e-08$	$8.4638e+02$	$8.4638e+02$	$8.4638e+02$	$8.4638e+02$	$6.9784e+02$	$4.0267e+03$

First, we demonstrate the efficiency of the fixed point algorithm (12). In order to verify the exactitude of the solution, the remainder per iteration is computed by substituting $\bar{P}_{i00}^{(n)}$ into CSARE (7a). Table 1 shows the errors $\|\boldsymbol{F}^{(n)}\|$ per iteration, where $\|\boldsymbol{F}^{(n)}\| := \sum_{i=1}^{2} \|\boldsymbol{F}_i(\bar{P}_{100}^{(n)},\ \bar{P}_{200}^{(n)})\|$, $\boldsymbol{F}_i(\bar{P}_{100},\ \bar{P}_{200} := \bar{P}_{i00}\left(A_s - \sum_{j=1}^{N} S_{s_j}\bar{P}_{j00}\right) + \left(A_s - \sum_{j=1}^{N} S_{s_j}\bar{P}_{j00}\right)^T \bar{P}_{i00} + A_{100}^T \bar{P}_{i00} A_{100} + \bar{P}_{i00} S_{s_i} \bar{P}_{i00} + Q_{s_i}$, $i = 1,\ 2$.

Table 1. Error per iterations.

n	$\|\boldsymbol{F}^{(n)}\|$	n	$\|\boldsymbol{F}^{(n)}\|$
1	$2.6548e+02$	14	$2.1084e-05$
2	$9.9802e+04$	15	$5.2630e-06$
3	$8.1525e+03$	16	$1.3084e-06$
4	$1.4301e+03$	17	$3.2413e-07$
5	$3.1976e+02$	18	$8.0056e-08$
6	$1.0375e+02$	19	$1.9722e-08$
7	$2.5413e+01$	20	$4.8463e-09$
8	1.2030	21	$1.1895e-09$
9	$2.2853e-02$	22	$2.9114e-10$
10	$5.1671e-03$	23	$7.0087e-11$
11	$1.3077e-03$	24	$1.5983e-11$
12	$3.3332e-04$	25	$2.4951e-12$
13	$8.4055e-05$		

It should be noted that algorithm (12) converges to the exact solution with an accuracy of $\|\boldsymbol{F}^{(n)}\| < 1.0e-11$ after 25 iterations. Hence, it can be observed from Table 1 that algorithm (12) attains linear convergence.

Using the proposed design procedure, the parameter independent approximate stochastic Nash strategies (24) and the optimal strategies (6) are given at the top of this page. In this example, $\varepsilon_1 = \varepsilon_2 = 0.0001$ are chosen.

Now, let us evaluate the costs using the parameter independent approximate stochastic Nash strategies (24). The values of the optimal cost performance and the proposed strategies (24) for various ε_i, $i = 1,\ 2$ are given in Table 2, where $\phi_i := |J_i(\tilde{u}_1^*,\ \tilde{u}_2^*) - J_i(u_1^*,\ u_2^*)|/\sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \mu^2}$. It is easy to verify that for each parameters ε_i, $|J_i(\tilde{u}_1^*,\ \tilde{u}_2^*) - J_i(u_1^*,\ u_2^*)| = O(\|\nu\|)$ because of $\phi_i < \infty$. Therefore, the new result for the loss of performance which is indicated by (25) is correct.

VII. CONCLUSION

In this paper, stochastic Nash games for the SMS have been studied. The main contribution of this paper is to propose the new strategy set that are independent of the small parameters. It should be noted that the proposed design method is quite different from the existing method such as the deterministic case [9], [13]. As a result, even if the system is governed by Itô's differential equation, we can obtain the Nash strategy. Moreover, the positive semidefiniteness and the local uniqueness in the neighborhood of the parameters $\varepsilon_i = 0$ have been established. Moreover, it has been newly shown that the resulting strategies achieve $O(\|\nu\|)$ approximation of the optimal cost performance. As another important feature, a new algorithm for solving the CSARE has been established. In fact, the convergence result has been proved. Finally, the numerical example for a multimachine power system was demonstrated for the usefulness of the proposed methodology.

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