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Consensus in Networked Multi-Agent Systems via Sampled Control: Fixed Topology Case

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Abstract— **In this paper, consensus problems of continuous-time networked multi-agent systems via sampled control are investigated. The sampled control protocols are induced from continuous-time linear consensus protocol by using periodic sampling technology and zero-order hold circuit. Two cases are considered: 1) networks without sampling delay; 2) networks with sampling delay. For each case, an algebraic-type necessary and sufficient condition is obtained under which consensus is achieved. Some numerical simulations are presented, which are consistent with our theoretical results.**

I. INTRODUCTION

Consensus problem in networked multi-agent systems has been attracted increasing attention in recent years. It is a comprehensive interdisciplinary subject including control theory, mathematics, biology, physics, computer science, robotics, artificial intelligence and so on. The applications of multi-agent systems are extensive, ranging from multiple space-craft alignment, heading direction in flocking behavior, average in distributed computation and rendezvous of multiple vehicles. Based on certain quantities of interest, consensus problems of multi-agent systems have been studied by many researchers(see [1],[2] and the references therein).

In the field of system and control, the development of consensus theory was mainly impelled by Vicsek's particles swarm model mentioned in [3]. In [4], Jadbabaie et al. provided a theoretical explanation of the consensus behavior of the Vicsek's model and derived convergence results for several similarly inspired models. A systematical framework of consensus problem in networks of dynamic agents with fixed/switching topology and communication time-delays was established in [5] by Olfati-Saber and Murray. In [6], Ren and Beard investigated more comprehensive discrete-time and continuous-time consensus scheme which included Jadbabaie's result as a special case. In [7], by using set-valued Lyapunov theory, a simple but compelling model of network of agents interacting via time-dependent communication links was studied.

In the past few years, consensus problems of multi-agent systems have been developed vary fast and several research topics have been addressed, such as networks with switching topologies and time-varying delays [8], [9], [10], higher-order consensus [11], [12], [13], asynchronous consensus [14], [15], agreement with random networks [16], [17], finite-time consensus [18], [19] and so on.

With the development of digital sensors and controllers, in many cases, though the system itself is a continuous process, the synthesis of control law can only use the data sampled at the discrete sampling instants. Compare to continuous-time system with continuous-time controller or direct discrete-time system, continuous-time system via sampled control has many advantages. On the one hand, the digital controller which is designed based on the sampled controller has obvious advantages in control accuracy , control speed, performance and price, and has better generality. On the other hand, in engineering applications, continuous signals will require broad bandwidth of networks, and in most cases, will not be available in practice. Therefore, sampled control for continuous-time system is more coincident with applications in our real life. Sampled control is applied extensively nowadays. Robots, vehicles, airplanes, satellites, and almost all of modern artificial products are controlled by digital controller where continuous signals are transferred into discrete ones.

For consensus problems of continuous-time multi-agent systems via sampled control, there are only a few relevant results. In [20] and [21], formation control of multi-agent systems with intermittent information exchange between the agents was considered. They both derived stability conditions under a predetermined sampling period. In [22], sampled-data based average-consensus control for networks consisting of continuous-time first-order integrator agents under a noisy distributed communication environment was considered. They proved that when the sampling size was sufficiently small, the static mean square error between the individual state and the average initial states of all nodes was arbitrarily small. However, in the real applications, we always want to know how large the sampling period would be chosen to guarantee the system run well. This requires us to find an upper bound of sampling period. Moreover, sampling delay can not be ignored and sometimes may play a key role in the stability analysis of the whole network. Therefore, we will also consider the case when sampling delay exists and is less than a sampling period.

The main contribution of this work is that a framework for studying consensus problem of multi-agent systems via sampled control is introduced. Consensus problems with sampled data and sampling delay are considered. We analyze two cases: 1) networks without sampling delay; 2) networks

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with sampling delays. Two consensus protocols for networks without and with sampling delay are introduced. We will establish conditions to guarantee consensus achieving. Finally, numerical examples are given to illustrate the utility of our results.

An outline of the rest of this paper is as follows. In Section II, we review graph theory and the consensus problems on networks. Section III introduces the sampled-data based control protocols for networks. Section IV presents the main results. The simulation results are presented in Section V. Finally, Section VI concludes the whole paper.

In this paper, $\mathbf{i}$ is the imaginary unit. Given a complex number $\lambda \in \mathbb{C}$, $Re(\lambda)$, $Im(\lambda)$, $\arg(\lambda)$ and $|\lambda|$ are the real part, the imaginary part, the argument principal value and the modulus of λ, respectively, then we have $\lambda = Re(\lambda) + \mathbf{i}Im(\lambda) = |\lambda|\cos(\arg(\lambda)) + \mathbf{i}|\lambda|\sin(\arg(\lambda))$. Notation $\mathbf{1}$ is the column vector $[1,\cdots,1]^T$ with appropriate dimension. Notation $triag\{a_1,\cdots,a_M\}$ represents the upper triangular matrix $\begin{bmatrix} a_1 & * & * \\ & \ddots & * \\ 0 & & a_M \end{bmatrix}$, where $*$ are the elements to be not concerned.

II. BRIEF REVIEW OF GRAPH AND CONSENSUS PROBLEM IN NETWORKS

In this section, we introduce networks and consensus problems. Let $\mathscr{G} = (\mathscr{V},\mathscr{E},\mathscr{A})$ be a digraph with the set of vertices $\mathscr{V} = \{1,2,\cdots,M\}$ and the set of edges $\mathscr{E} \subseteq \mathscr{V}\times\mathscr{V}$, and a weighted adjacency matrix $\mathscr{A} = [a_{ij}]$ with nonnegative adjacency elements a_{ij}. An edge of $\mathscr{G}$ is denoted by $e_{ij} = (j,i)$, where j is called the parent vertex of i and i the child vertex of j. The adjacency elements associated with the edges are positive, i.e., $e_{ij} \in \mathscr{E} \Leftrightarrow a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathscr{V}$. The set of *neighbors* of node i is denoted by $\mathscr{N}_i = \{j \in \mathscr{V} : (j,i) \in \mathscr{E}\}$.

A *directed path* between each distinct vertices i and j is meant a finite ordered sequence of distinct edges of $\mathscr{G}$ of the form $(i,k_1),(k_1,k_2),\cdots,(k_l,j)$ in a digraph. A directed tree is a directed graph, where every vertex has exactly one parent vertex except for one vertex, called a root vertex, which has no parent and has a directed path to every other vertex. A subgraph $\mathscr{G}_s$ of $\mathscr{G}$ is a graph such that $\mathscr{V}(\mathscr{G}_s) \subset \mathscr{V}(\mathscr{G})$ and $\mathscr{E}(\mathscr{G}_s) \subset \mathscr{E}(\mathscr{G})$. $\mathscr{G}_s$ is called a spanning subgraph if $\mathscr{V}(\mathscr{G}_s) = \mathscr{V}(\mathscr{G})$. For any $i,j \in \mathscr{V}(\mathscr{G}_s)$, if $(i,j) \in \mathscr{E}(\mathscr{G}_s) \Leftrightarrow (i,j) \in \mathscr{E}(\mathscr{G})$, then $\mathscr{E}(\mathscr{G})$ is called an induced subgraph of $\mathscr{G}$, and we also say $\mathscr{G}_s$ is induced by $\mathscr{V}(\mathscr{G}_s)$. *A directed spanning tree* of $\mathscr{G}$ is a directed tree which is a spanning subgraph of $\mathscr{G}$. $\mathscr{G}$ is said to have a spanning tree if some edges of $\mathscr{G}$ can form a spanning tree of $\mathscr{G}$.

A weighted directed graph $\mathscr{G}(\mathscr{A})$ is a directed graph $\mathscr{G}$ plus a nonnegative matrix $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{M\times M}$, where a_{ij} is called the weight of edge e_{ij}, and $a_{ij} > 0 \Leftrightarrow e_{ij} \in \mathscr{E}(\mathscr{G})$.

A digraph is called *strongly connected* if and only if any two distinct nodes of the graph can be connected via a directed path. A digraph is strongly connected if and only if its adjacency matrix is irreducible.

A digraph is said to be *balanced* if

$$\sum_j a_{ij} = \sum_j a_{ji}, \forall i = 1,2,\cdots,M.$$

For a graph, if $(i,j) \in \mathscr{E}$, then $(j,i) \in \mathscr{E}$, then it is an *undirected graph*. A strongly connected undirected graph is only said *connected*.

The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{M\times M}$ of $\mathscr{G}(\mathscr{A})$ is defined as

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{k=1,k\neq i}^{M} a_{ik}, & i = j \end{cases}$$

Lemma 1: [6] Given a digraph $\mathscr{G}$, then the Laplacian L associated with the graph has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open right half plane. Furthermore, it has exactly one zero eigenvalue if and only if the graph has a spanning tree.

Given a graph $\mathscr{G}$, denote $\Lambda^+(L)$ as the set of nonzero eigenvalues of the Laplacian L of $\mathscr{G}$.

Given a digraph $\mathscr{G}$, let $x_i \in \mathbb{R}$ denote the state of node i. We refer to $(\mathscr{G},x)$ with $x = [x_1,x_2,\cdots,x_M]^T \in \mathbb{R}^M$ as a *network* with state x and communication topology $\mathscr{G}$. Suppose each node of a graph is a dynamic agent with dynamics

$$\dot{x}_i(t) = u_i(t) \tag{1}$$

where x_i is aforementioned state of node i and u_i is the control input that will be used for consensus problem.

Let $\chi : \mathbb{R}^M \to \mathbb{R}$ be a function of the state of the network $x(t)$. The *χ-consensus problem* in a network $(\mathscr{G},x)$ is a distributed way to calculated $\chi(x_0)$ by applying inputs u_i that only depend on the states of itself and its neighbors. We say a feedback

$$u_i(t) = k_i(x_{j_1}(t),x_{j_2}(t),\cdots,x_{j_{L_i}}(t)) \tag{2}$$

is a *control protocol* with topology $\mathscr{G}$ if the cluster $\{j_1,\cdots,j_{L_i}\} = \{i\}\bigcup\mathscr{N}_i$, $i = 1,\cdots,M$.

We say protocol (2) asymptotically solves the χ-consensus problem if and only if there exists an asymptotically stable equilibrium x^* of the network satisfying $x_i^* = \chi(x(0)), i = 1,\cdots,M$. Whenever the nodes of a network are all in consensus, the common value of all nodes is called the *network decision value*. A special case with $\chi(x) = \mathrm{Ave}(x) = \frac{1}{M}(\sum_{i=1}^{M} x_i)$ is called *average-consensus* problem.

III. SAMPLED-DATA CONTROL PROTOCOLS AND INDUCED NETWORK DYNAMICS

In this section, we investigate distributed solutions of the consensus problem via sampled-data linear control. In [5] the following continuous-time linear consensus protocol was introduced:

$$u_i(t) = \sum_{j\in\mathscr{N}_i} a_{ij}(x_j(t) - x_i(t)), i = 1,\cdots,M. \tag{3}$$

Here a sampled-data control protocol is induced from (3) by using period sampling technology and zero-order hold circuit. Let $h > 0$ be the sampling period, the obtained protocol is given as:

$$\begin{aligned} &u_i(t) = \sum_{j\in\mathscr{N}_i} a_{ij}(x_j(kh) - x_i(kh)), \\ &\text{if } t \in [kh,kh+h), k = 0,1,2,\cdots; i = 1,\cdots,M. \end{aligned} \tag{4}$$

By using the protocol (4), the network dynamics is summarized as follows:

$$x(kh+h) = \Phi x(kh), k = 0,1,2,\cdots. \tag{5}$$

where

$$\Phi = I - hL \tag{6}$$

with L the aforementioned Laplacian associate with the graph $\mathscr{G}$.

If sampling induced time delay is concerned, the situation becomes complicated. We assume that the sampling delay τ is fixed and less than the sampling period, i.e., $0 < \tau < h$. In this situation, the protocol becomes

$$u_i(t) = \begin{cases} \sum_{j\in\mathscr{N}_i} a_{ij}(x_j(kh-h) - x_i(kh-h)), \text{if } t \in [kh, kh+\tau) \\ \sum_{j\in\mathscr{N}_i} a_{ij}(x_j(kh) - x_i(kh)), \text{if } t \in [kh+\tau, kh+h) \end{cases}$$
$$k = 0,1,2,\cdots; i = 1,\cdots,M. \tag{7}$$

Then the network dynamics is given as follows:

$$\begin{bmatrix} x(kh+h) \\ x(kh) \end{bmatrix} = \Psi \begin{bmatrix} x(kh) \\ x(kh-h) \end{bmatrix}, k = 0,1,2,\cdots. \tag{8}$$

where

$$\Psi = \begin{bmatrix} I-(h-\tau)L, & -\tau L \\ I, & 0 \end{bmatrix}. \tag{9}$$

IV. CONSENSUS ANALYSIS

Before giving our main results, firstly, we analyze the solution for general discrete-time linear time-invariant system. Consider the system

$$x(t+1) = Ax(t) \tag{10}$$

where $x \in \mathbb{R}^M$ is the state, $A \in \mathbb{R}^{M\times M}$ is constant matrix.

Lemma 2: For system (10), the limit of the solution $x(t)$ is $c\mathbf{1}$ ($c \in \mathbb{R}$ is a constant) if and only if

i) $A\mathbf{1} = \mathbf{1}$;

ii) 1 is algebraically simple eigenvalue of A, and is the unique eigenvalue of maximum modulus.

Proof: The proof of Lemma 2 is obvious, we delete it for space saving. ■

Remark 1: Different from the system proposed in [23] where the system matrix A was assumed to be a non-negative matrix. Here, we don't request all the elements of A are non-negative.

In the case when the sampling delay is zero, we have the following proposition.

Proposition 1: Consider a network $(\mathscr{G}, x)$, the protocol (4) globally asymptotically solves the χ-consensus problem if and only if the digraph $\mathscr{G}$ has a directed spanning tree and the sampling period h satisfies

$$0 < h < \min_{\lambda\in\Lambda^+(L)} \tfrac{2Re(\lambda)}{|\lambda|^2}. \tag{11}$$

Moreover, denote γ such that $\gamma^T L = 0$ and $\gamma^T\mathbf{1} = 1$, then $\chi(x(0)) = \gamma^T x(0)$.

Proof: From Lemma 2, we know that the protocol (4) globally asymptotically solves the χ-consensus problem if and only if $\Phi\mathbf{1} = \mathbf{1}$ and 1 is algebraically simple eigenvalue of Φ, and is the unique eigenvalue of maximum modulus. It is easy to verify that $\Phi\mathbf{1} = \mathbf{1}$. Since the eigenvalues of Φ are 1 and $1-h\lambda, \lambda \in \Lambda^+(L)$, from Lemma 1, we know that 1 is an algebraically simple eigenvalue. To guarantee that 1 is the unique eigenvalue of maximum modulus, we should have:

$$|1-h\lambda| < 1, \lambda \in \Lambda^+(L) \tag{12}$$

Noticing

$$|1-h\lambda| < 1 \quad \Leftrightarrow \quad 0 < h < \tfrac{2Re(\lambda)}{|\lambda|^2}$$

we get that the protocol (4) globally asymptotically solves the χ-consensus problem if and only if a digraph has a spanning tree and (11) holds. Moreover, $\chi(x(0)) = \gamma^T x(0)$ is obvious. ■

For the nonzero sampling delay situation, the following theorem is obtained.

Theorem 1: Consider a network $(\mathscr{G}, x)$, the protocol (7) globally asymptotically solves the χ-consensus problem if and only if the digraph $\mathscr{G}$ has a directed spanning tree and the sampling delay τ and the sampling period h satisfy

$$0 \le \tau < 2\min_{\lambda\in\Lambda^+(L)} \frac{\cos(\frac{\arg(\lambda)+\pi}{3})}{|\lambda|} \tag{13}$$

and

$$\tau < h < 2\min_{\lambda\in\Lambda^+(L)} \frac{(Re(\lambda)-\tau|\lambda|^2)(1-\tau^2|\lambda|^2)}{|\lambda|^2(1-2Re(\lambda)\tau+\tau^2|\lambda|^2)} \tag{14}$$

Moreover, if the condition holds, denote γ such that $\gamma^T L = 0$ and $\gamma^T\mathbf{1} = 1$, then $\chi(x(0)) = \gamma^T x(0)$.

Before giving the proof, we present some necessary preliminaries. In the analysis of discrete-time systems, by using a bilinear transformation [25], problem of determining Schur stability of a discrete-time system can be transformed into the problem of determining Hurwitz stability of a continuous-time system. Given a polynomial with complex coefficients:

$$a(s) = \gamma_n s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1 s + \gamma_0 \tag{15}$$

where $\gamma_i \in \mathbb{C}, i = 1,\ldots,n$. Applying the bilinear transformation

$$\sigma = \varphi(s) = \tfrac{s+1}{s-1} \tag{16}$$

to $a(s)$, we get a new polynomial

$$r(\sigma) = (\sigma-1)^n a(\tfrac{\sigma+1}{\sigma-1}) = \rho_0 + \rho_1\sigma + \cdots + \rho_n\sigma^n \tag{17}$$

where $\rho_i = \alpha_i + \mathbf{i}\beta_i, \alpha_i, \beta_i \in \mathbb{R}, i = 0,1,\cdots,n$. Then we know that the Schur stability of $a(s)$ is equivalent to the Hurwitz stability of $r(\sigma)$. Substituting $\sigma = \mathbf{i}\omega$ into the above complex polynomial $r(\sigma)$, we have

$$r(\mathbf{i}\omega) = m(\omega) + \mathbf{i}n(\omega) \tag{18}$$

where $m(\omega), n(\omega) \in \mathbb{R}[\omega]$, and

$$m(\omega) = \alpha_0 - \beta_1\omega - \alpha_2\omega^2 + \beta_3\omega^3 + \alpha_4\omega^4 - \cdots \tag{19}$$

$$n(\omega) = \beta_0 + \alpha_1\omega - \beta_2\omega^2 - \alpha_3\omega^3 + \beta_4\omega^4 - \cdots \tag{20}$$

In order to determine the Hurwitz stability of $r(\sigma)$, the following famous theorem is introduced.

Lemma 3: [25] (Hermite-Biehler Theorem) The polynomial $r(\sigma)$ is Hurwitz stability if and only if the related pair $m(\omega), n(\omega)$ is interlaced, and $m(0)n'(0) - m'(0)n(0) > 0$.

Now we give the proof of Theorem 1.

Proof: Assume the digraph $\mathscr{G}$ has a spanning tree. Due to the expression of Ψ given by (9), it is easy to verify that $\Psi\mathbf{1} = \mathbf{1}$. Let $0, \lambda_2, \cdots, \lambda_M$ denote the eigenvalues of L, it follows that L is similar to $triag\{0, \lambda_2, \cdots, \lambda_M\}$. Thus, we get that Ψ is similar to

$$\begin{bmatrix} triag\{1, 1-(h-\tau)\lambda_2, \cdots, 1-(h-\tau)\lambda_M\} & triag\{0, -\tau\lambda_2, \cdots, -\tau\lambda_M\} \\ I & 0 \end{bmatrix} \tag{21}$$

Moreover, Ψ is similar to

$$triag\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1-(h-\tau)\lambda_2 & -\tau\lambda_2 \\ 1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 1-(h-\tau)\lambda_M & -\tau\lambda_M \\ 1 & 0 \end{bmatrix}\} \tag{22}$$

It is easy to see that 0 and 1 are two eigenvalues of Ψ. By Lemma 2, we only need to guarantee that all the other eigenvalues of Ψ are in the unit circle. From the expression of (22), we can see that all the other eigenvalues have same properties. Therefore, we can analyze them uniformly. Consider a 2×2 matrix

$$\begin{bmatrix} 1-(h-\tau)\lambda & -\tau\lambda \\ 1 & 0 \end{bmatrix} \tag{23}$$

where $Re(\lambda) > 0$. Its characteristic polynomial is

$$a(s) = s^2 + ((h-\tau)\lambda - 1)s + \tau\lambda \tag{24}$$

By applying the bilinear transformation (16), we get

$$r(\sigma) = h\lambda\sigma^2 + 2(1-\tau\lambda)\sigma + 2 + (2\tau-h)\lambda \tag{25}$$

We rewrite it as

$$r(\sigma) = \sigma^2 + \frac{2(\bar{\lambda}-\tau|\lambda|^2)}{h|\lambda|^2}\sigma + \frac{2\bar{\lambda}+(2\tau-h)|\lambda|^2}{h|\lambda|^2} \tag{26}$$

Then the polynomial (24) is Schur stable if and only if the polynomial (26) is Hurwitz stable.

Let $\lambda = Re(\lambda) + \mathbf{i}Im(\lambda)$, we get

$$r(\sigma) = \sigma^2 + \frac{2(Re(\lambda)-\tau|\lambda|^2)-2Im(\lambda)\mathbf{i}}{h|\lambda|^2}\sigma + \frac{(2Re(\lambda)+(2\tau-h)|\lambda|^2)-2Im(\lambda)\mathbf{i}}{h|\lambda|^2} \tag{27}$$

It follows that

$$m(\omega) = -\omega^2 + \frac{2Im(\lambda)}{h|\lambda|^2}\omega + \frac{(2Re(\lambda)+(2\tau-h)|\lambda|^2)}{h|\lambda|^2} \tag{28}$$

$$n(\omega) = \frac{2(Re(\lambda)-\tau|\lambda|^2)}{h|\lambda|^2}\omega + \frac{-2Im(\lambda)}{h|\lambda|^2} \tag{29}$$

Then, from Lemma 3, the polynomial (26) is Hurwitz stable if and only if the following conditions hold:

a) the polynomial $m(\omega)$ has two distinct roots $m_1 < m_2$;

b) the interlaced condition holds, i.e., $m_1 < n_1 < m_2$, where n_1 is the unique root of the polynomial $n(\omega)$;

c) $m(0)n'(0) - m'(0)n(0) > 0$.

Consider condition a), the quadratic polynomial $m(\omega)$ has two distinct roots if and only if

$$\Delta_m = \left(\frac{2Im(\lambda)}{h|\lambda|^2}\right)^2 + 4\frac{(2Re(\lambda)+(2\tau-h)|\lambda|^2)}{h|\lambda|^2} > 0$$

Then we have

$$\Delta_m > 0 \Leftrightarrow \quad h^2 - 2\frac{Re(\lambda)+\tau|\lambda|^2}{|\lambda|^2}h - \frac{Im^2(\lambda)}{|\lambda|^4} < 0$$

$$\Leftrightarrow \frac{Re(\lambda)+\tau|\lambda|^2}{|\lambda|^2} - \frac{\sqrt{1+2Re(\lambda)\tau+\tau^2|\lambda|^2}}{|\lambda|} < h < \frac{Re(\lambda)+\tau|\lambda|^2}{|\lambda|^2} + \frac{\sqrt{1+2Re(\lambda)\tau+\tau^2|\lambda|^2}}{|\lambda|}$$

Noticing that

$$\frac{Re(\lambda)+\tau|\lambda|^2}{|\lambda|^2} - \frac{\sqrt{1+2Re(\lambda)\tau+\tau^2|\lambda|^2}}{|\lambda|} \le 0 \quad \Leftrightarrow Re^2(\lambda) \le |\lambda|^2$$

Thus, we have $\Delta_m > 0$ if and only if

$$h < \frac{Re(\lambda)+\tau|\lambda|^2}{|\lambda|^2} + \frac{\sqrt{1+2Re(\lambda)\tau+\tau^2|\lambda|^2}}{|\lambda|} \tag{30}$$

This inequality gives an upper bound of h.

By simple calculation, the roots of $m(\omega)$ are given by

$$m_1 = \frac{Im(\lambda) - \sqrt{Im^2(\lambda)+(2Re(\lambda)+(2\tau-h)|\lambda|^2)|\lambda|^2 h}}{h|\lambda|^2}$$

$$m_2 = \frac{Im(\lambda) + \sqrt{Im^2(\lambda)+(2Re(\lambda)+(2\tau-h)|\lambda|^2)|\lambda|^2 h}}{h|\lambda|^2}$$

The root of $n(\omega)$ is given by

$$n_1 = \frac{Im(\lambda)}{Re(\lambda)-\tau|\lambda|^2}$$

According to the condition b), we have

$$\frac{Im(\lambda) - \sqrt{Im^2(\lambda)+(2Re(\lambda)+(2\tau-h)|\lambda|^2)|\lambda|^2 h}}{h|\lambda|^2} < \frac{Im(\lambda)}{Re(\lambda)-\tau|\lambda|^2} < \frac{Im(\lambda) + \sqrt{Im^2(\lambda)+(2Re(\lambda)+(2\tau-h)|\lambda|^2)|\lambda|^2 h}}{h|\lambda|^2}$$

$$\Leftrightarrow h|\lambda|^2(1-2Re(\lambda)\tau+\tau^2|\lambda|^2) < 2(Re(\lambda)-\tau|\lambda|^2)(1-\tau^2|\lambda|^2)$$

Then the condition b) holds if and only if

$$h < 2\frac{(Re(\lambda)-\tau|\lambda|^2)(1-\tau^2|\lambda|^2)}{|\lambda|^2(1-2Re(\lambda)\tau+\tau^2|\lambda|^2)} \tag{31}$$

This inequality gives another upper bound of h.

Noticing

$$\frac{(Re(\lambda)-\tau|\lambda|^2)(1-\tau^2|\lambda|^2)}{|\lambda|^2(1-2Re(\lambda)\tau+\tau^2|\lambda|^2)} \le \frac{Re(\lambda)+\tau|\lambda|^2}{|\lambda|^2} \quad \Leftrightarrow Re^2(\lambda) \le |\lambda|^2$$

and

$$\frac{Re(\lambda)+\tau|\lambda|^2}{|\lambda|^2} \le \frac{\sqrt{1+2Re(\lambda)\tau+\tau^2|\lambda|^2}}{|\lambda|} \quad \Leftrightarrow Re(\lambda)^2 \le |\lambda|^2$$

we get that

$$2\frac{(Re(\lambda)-\tau|\lambda|^2)(1-\tau^2|\lambda|^2)}{|\lambda|^2(1-2Re(\lambda)\tau+\tau^2|\lambda|^2)} \le \frac{Re(\lambda)+\tau|\lambda|^2}{|\lambda|^2} + \frac{\sqrt{1+2Re(\lambda)\tau+\tau^2|\lambda|^2}}{|\lambda|} \tag{32}$$

This means that (31) implies (30).

Since $h > 0$, it follows that

$$(Re(\lambda) - \tau|\lambda|^2)(1-\tau^2|\lambda|^2) > 0$$

Since $\tau|\lambda|<1$ due to the Schur stability of matrix (23), we have

$$\tau < \frac{Re(\lambda)}{|\lambda|^2} \tag{33}$$

This gives an upper bound of τ.

By simple calculation, we get

$$m(0)=\frac{2Re(\lambda)+(2\tau-h)|\lambda|^2}{h|\lambda|^2},\quad n(0)=\frac{-2Im(\lambda)}{h|\lambda|^2}$$

and

$$m'(0)=\frac{2Im(\lambda)}{h|\lambda|^2},\quad n'(0)=\frac{2(Re(\lambda)-\tau|\lambda|^2)}{h|\lambda|^2}$$

According to condition c), we have

$$\frac{2Re(\lambda)+(2\tau-h)|\lambda|^2}{h|\lambda|^2}\frac{2(Re(\lambda)-\tau|\lambda|^2)}{h|\lambda|^2}-\frac{2Im(\lambda)}{h|\lambda|^2}\frac{(-2Im(\lambda))}{h|\lambda|^2}>0$$

$$\Leftrightarrow (Re(\lambda)-\tau|\lambda|^2)h<2-2\tau^2|\lambda|^2$$

Then condition c) holds if and only if

$$h<\frac{2-2\tau^2|\lambda|^2}{Re(\lambda)-\tau|\lambda|^2} \tag{34}$$

We get another upper bound of h.

Noticing

$$2\frac{(Re(\lambda)-\tau|\lambda|^2)(1-\tau^2|\lambda|^2)}{|\lambda|^2(1-2Re(\lambda)\tau+\tau^2|\lambda|^2)}\le\frac{2-2\tau^2|\lambda|^2}{Re(\lambda)-\tau|\lambda|^2}\quad\Leftrightarrow Re^2(\lambda)\le|\lambda|^2$$

Thus, we know that (31) implies (34), as well.

From $\tau<h$, we get

$$\tau<2\frac{(Re(\lambda)-\tau|\lambda|^2)(1-\tau^2|\lambda|^2)}{|\lambda|^2(1-2Re(\lambda)\tau+\tau^2|\lambda|^2)}\Leftrightarrow \tau^3|\lambda|^4-3\tau|\lambda|^2+2Re(\lambda)>0$$

Let

$$f(z)=z^3|\lambda|^4-3z|\lambda|^2+2Re(\lambda) \tag{35}$$

The trajectory of $f(z)$ is illustrated in Fig.1. We have

$$f(0)>0, f\left(\frac{Re(\lambda)}{|\lambda|^2}\right)<0,\quad \text{and}\quad f'\left(\frac{1}{|\lambda|}\right)=f'\left(-\frac{1}{|\lambda|}\right)=0.$$

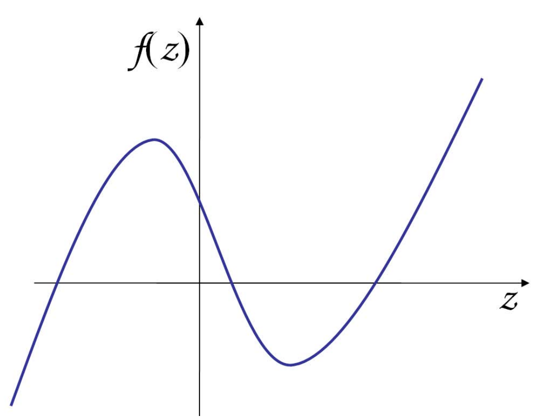

Fig. 1: The trajectory of $f(z)$.

By solving the univariate cubic equation $f(z)=0$, we get three real roots given by

$$z_1=-\frac{2}{|\lambda|}\cos\left(\frac{\arg(\lambda)}{3}\right), z_2=\frac{2}{|\lambda|}\cos\left(\frac{\arg(\lambda)+\pi}{3}\right),$$
$$z_3=\frac{2}{|\lambda|}\cos\left(\frac{\pi-\arg(\lambda)}{3}\right),$$

which satisfy $z_1<0<z_2<z_3$.

Since $\tau\ge 0$, we know that $f(\tau)>0$ if and only if

$$\tau<\frac{2}{|\lambda|}\cos\left(\frac{\arg(\lambda)+\pi}{3}\right) \tag{36}$$

it is another upper bound of τ.

Compare (33) and (36), by using $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ and $\cos(3\alpha)=4\cos^3\alpha-3\cos\alpha, \forall\alpha,\beta$, we have

$$\frac{2}{|\lambda|}\cos\left(\frac{\arg(\lambda)+\pi}{3}\right)<\frac{Re(\lambda)}{|\lambda|^2}\Leftrightarrow\sin\left(2\frac{\arg(\lambda)}{3}\right)<\frac{\sqrt{3}}{2}$$

Since $Re(\lambda)>0$, we have

$$-\frac{\sqrt{3}}{2}<\sin\left(2\frac{\arg(\lambda)}{3}\right)<\frac{\sqrt{3}}{2} \tag{37}$$

This means that (36) implies (33).

Thus, the inequalities (36) and (31) are the necessary and sufficient condition under which the matrix (23) is Schur stable.

As a result, for a digraph, a spanning tree and the inequalities (13) and (14) are the necessary and sufficient condition under which the protocol (7) globally asymptotically solves the χ-consensus problem.

■

V. SIMULATIONS

In this section, numerical simulations will be given to illustrate the theoretical results obtained in the previous sections. The graph in our simulations have $0-1$ weights.

We give the simulations of consensus problem with sampling delay under strongly connected graph.

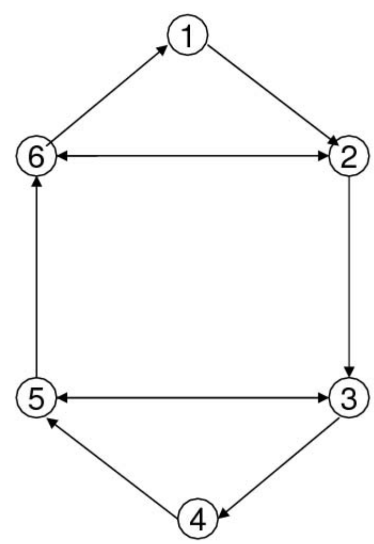

Fig. 2: A strongly connected graph

Example 1 (Network with sampling delay): Consider the network with a digraph given in Fig.2. By (13) and (14), we obtain that the upper bound of sampling delay τ is 1/3 and the upper bound of sampling period h is 4/3. Fig.3 shows the simulation result with $\tau=0.1$ and $h=0.3$. The agreement is achieved. However, The simulation result in Fig.4 shows that agreement does not achieve where $\tau = 0.4$ and $h = 1$. This is coincident with our theoretic results.

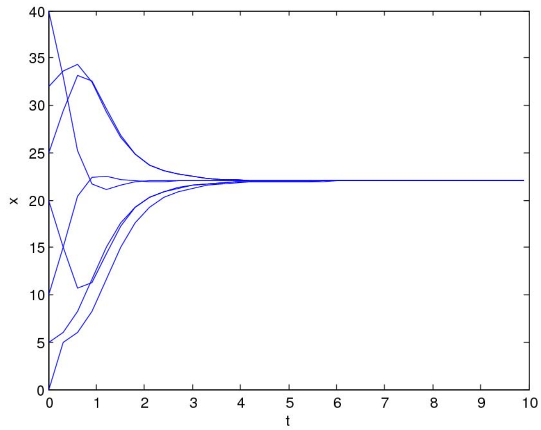

Fig. 3: Simulation result when $\tau = 0.1, h = 0.3$ in Example 2.

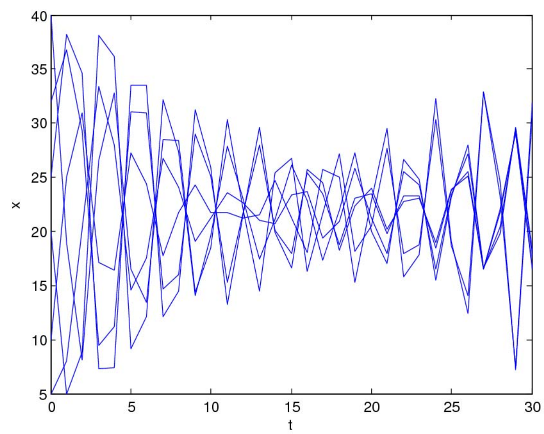

Fig. 4: Simulation result when $\tau = 0.4, h = 1$ in Example 2.

VI. CONCLUSION

In this paper, convergence analysis of consensus control for networks of multi-agent systems via sampled control has been investigated. Our analysis relies on several tools from algebraic graph theory, matrix theory and stability theory. For directed networks with fixed topology, we have established sufficient and necessary conditions for reaching χ-consensus problem without or with sampling delays. The switching topology case is discussed in [26]. The future work include large sampling delay case and time-varying sampling delay case.

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