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Guaranteed Cost for Stochastic Systems with Unknown Transitions Jump Rates

El-Kébir Boukas

Abstract— **This paper deals with the class of continuous-time stochastic systems with totally known and totally unknown, but bounded with some known bounds, transition jump rates. The guaranteed cost control problem of this class of systems is tackled. New sufficient conditions for guaranteed cost are developed. A design procedure for the guaranteed cost controller which guarantees that the closed-loop dynamics will be stable is proposed. It is shown that the addressed problems can be solved if the corresponding developed convex optimization problems are feasible. A numerical example is employed to show the usefulness of the proposed results.**

Keywords: Stochastic systems, unknown and known transition jump rates, Wiener process, stochastic stability, guaranteed cost, state feedback controller.

I. Introduction

Stochastic systems represent an important class of systems that has attracted a lot of researchers from mathematics and control communities. For more details of what it has been done on this topics we refer the reader [8] and the references therein. Many problems for this class of systems either in the continuous-time and discrete-time have been tackled and interesting results have been reported in the literature. Among these contributions we quote those of [4] where the authors have considered the optimal stabilizing controller for a discrete-time version of the stochastic systems we are treating in this paper. In [2], the authors consider a discrete-time version with constant time-delay and apply the fuzzy theory to deal with some problems in the networked control systems with dropout. The continuous-time version has been considered in [1], [3], [5]. In [1] the authors have dealt with the output feedback stabilization of the continuous-time stochastic systems. In [3], the H_∞ control for the stochastic time-delay systems with Markovian switching is tackled. The authors in [5] considered the continuous-time class of stochastic systems and proposed a way to compute the controller that guarantees the optimal decaying rate. The stabilizing controllers in all these studies are computed using convex optimization problems.

On the other hand the class of Markovian jump systems has been found more appropriate to describe practical systems with random abrupt changes in their structures such as components failures or repairs, sudden environment disturbance, interconnections changing and operating in different point of a nonlinear plant. This fact was the cause of the tremendous interest to the Markovian jump systems. For more details of this class of systems or on what has been done of the different problems, we refer the reader to [8], [10], [9] and the references therein.

Guaranteed cost control is one of the approaches that have been proposed in the literature to robustly stabilize dynamical uncertain systems. For deterministic systems and Markov jump systems, this control problem has been studied by many authors see [9] and the references . This paper deals with the class of continuous-time stochastic systems with random abrupt changes and which combines the two previous classes of systems. Two cases will be considered. The first one treats the case where the transition jump rates are totally known while the second one tackles the case where the transition jump rates are totally unknown but bounded with some known bounds in each mode. For these two cases, we will consider the guaranteed cost control problem and design the stabilizing controller that provides the minimum cost. To the best of the author's knowledge this class of systems has not been fully investigated so far. Our goal in this paper is to establish conditions that permit the design of a state feedback controller that makes the closed-loop system stochastically stable and at the same time assures the guaranteed cost. The conditions we will develop will be tractable if a corresponding convex optimization is feasible.

The rest of this paper is organized as follows. In Section 2, the problem is stated and the goal of the paper is given. In Section 3, the main results are developed and they include results on the design of an optimal guaranteed cost controller that makes the closed-loop system stochastically stable for the totally known and totally unknown, but bounded with some known bounds, transition jump rates. Section 4 provides a numerical example to show the usefulness of the proposed results for the two cases.

Notation: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. The superscript "$^\top$" denotes matrix transposition and the notation $X \geq Y$(respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $\mathbb{I}$ is the identity matrices with compatible dimensions.

II. Problem statement

Consider a linear stochastic system with random abrupt changes that has N modes, i.e., $\mathscr{S} = \{1, 2, \ldots, N\}$. The mode switching is assumed to be governed by a continuous-time Markov process $\{r_t, t \geq 0\}$ taking values in the state space $\mathscr{S}$ and having the following infinitesimal generator

$$\Lambda = [\lambda_{ij}], i, j \in \mathscr{S},$$

where $\lambda_{ij} \geq 0, \forall j \neq i, \lambda_{ii} = -\sum_{j\neq i} \lambda_{ij}$.

The mode transition probabilities are described as follows:

$$Pr[r_{t+h}=j|r_t=i]=\begin{cases}\lambda_{ij}h+o(h), & j\neq i\\ 1+\lambda_{ii}h+o(h), & j=i\end{cases} \quad (1)$$

where $\lim_{h\to 0} o(h)/h=0$.

Let the state equation of this class of systems be defined in a fundamental probability space $(\Omega,\mathcal{F},\mathbb{P})$ and assume that its behavior is described by the following stochastic differential equations:

$$\begin{cases} dx(t)=[A(r_t,t)x(t)+B(r_t,t)u(t)]\,dt\\ \qquad +W(r_t)x(t)d\omega(t),\\ x(s)=x_0\end{cases} \quad (2)$$

where $x(t)\in\mathbb{R}^n$ is the system state at time t, $u(t)\in\mathbb{R}^m$ is the control input of the system at time t, $\omega(t)\in\mathbb{R}$ is a standard Wiener process that is assumed to be independent of the Markov process $\{r_t, t\geq 0\}$, $A(r_t,t)\in\mathbb{R}^{n\times n}$ and $B(r_t,t)\in\mathbb{R}^{n\times m}$ are assumed to have uncertainties, i,e.: $A(r_t,t)=A(r_t)+D_A(r_t)F_A(r_t)E_A(r_t)$ and $B(r_t,t)=B(r_t)+D_B(r_t)F_B(r_t)E_B(r_t)$ with $A(i)\in\mathbb{R}^{n\times n}$, $D_A(i)\in\mathbb{R}^{n\times n_D}$, $E_A(i)\in\mathbb{R}^{n_E\times n}$, $B(i)\in\mathbb{R}^{n\times m}$, $D_B(i)\in\mathbb{R}^{n\times m_D}$ and $E_B(i)\in\mathbb{R}^{m_E\times n}$ are known real matrices with appropriate dimensions for each $i\in\mathscr{S}$ and $F_A(r_t)\in\mathbb{R}^{n_D\times n_E}$ and $F_B(r_t)\in\mathbb{R}^{m_D\times m_E}$ satisfy $F_A^\top(i)F_A(i)\leq\mathbb{I}$ and $F_B^\top(i)F_B(i)\leq\mathbb{I}$ for each $i\in\mathscr{S}$, and $W(i)$ is a known matrix with appropriate dimension.

Remark 2.1: When the uncertainties are equal to zero the system will be referred to as nominal system. The ones that satisfies the previous conditions are referred to as admissible. The uncertainties we are considering in this paper are known in the literature as norm bounded uncertainties.

More often the transition jump rates can not be easily obtained and an alternate to overcome this case is required. The following assumption will be used in this paper to develop new results for the case of totally known and totally unknown, but bounded with some known bounds, transition jump rates.

Assumption 2.1: The jump rates are assumed to satisfy the following:

$$0<\underline{\lambda}_i\leq\lambda_{ij}\leq\bar{\lambda}_i, \forall i,j\in\mathscr{S}, j\neq i \quad (3)$$

where $\underline{\lambda}_i$ and $\bar{\lambda}_i$ are known parameters for each mode or may represent the lower and upper bounds when all the jump rates are known, i.e.: $0<\underline{\lambda}_i=\min_{j\in\mathscr{S}}\{\lambda_{ij}\neq 0, i\neq j\}$, $0<\bar{\lambda}_i=\max_{j\in\mathscr{S}}\{\lambda_{ij}, i\neq j\}$, with $\underline{\lambda}_i\leq\bar{\lambda}_i$.

Remark 2.2: We have to mention that some alternatives have been proposed to handle such case by considering uncertainties on the jump rates (see [6], [7], [11], [12], [13]). Our approach is totally different and requires only two bounds in each mode to establish the results we propose in this paper.

For the system (2), we have the following definitions:

Definition 2.1: ([8], [9]) Nominal free system (2) is said to be stochastically stable if there exists a constant $M(x_0,r_0)$ such that

$$\mathbb{E}\left[\int_0^\infty \|x(t)\|^2dt\middle|x_0,r_0\right]\leq M(x_0,r_0); \quad (4)$$

Definition 2.2: ([8], [9]) Uncertain free system (2) is said to be robust stochastically stable if there exists a constant $M(r_0,x_0)$ such that (4) holds for all admissible uncertainties.

let $R_1(i)$ and $R_2(i)$, $i\in\mathscr{S}$ be two given symmetric and positive-definite matrices and consider the following cost function:

$$J(x_0,r_0)=\mathbb{E}\left[\int_0^\infty \tilde{x}^\top(t)M(r_t)\tilde{x}(t)dt\right] \quad (5)$$

where $M(r_t)=\begin{bmatrix} R_1(r_t) & 0\\ 0 & R_2(r_t)\end{bmatrix}$, $\tilde{x}^\top(t)=\begin{bmatrix} x^\top(t) & u^\top(t)\end{bmatrix}$, x_0 and r_0 are respectively the initial state and the initial mode of the system.

Definition 2.3: If there exist a control law, $u(.)$ and a positive scalar ϱ representing the upper bound of the cost (5) such that the closed-loop system is stochastically stable, and the cost (5) is bounded by ϱ, then ϱ is called the guaranteed cost, also referred to as the optimal guaranteed cost, and $u(.)$ is the associated guaranteed cost control law.

The goal of this paper is to design a controller of the following form:

$$u(t)=K(r_t)x(t) \quad (6)$$

where $K(i)$ is a design parameter that has to be determined for every $i\in\mathscr{S}$.

The guaranteed cost problem can be stated as follows: given two symmetric and positive-definite matrices find a state feedback controller of the form (6) such that the closed-loop system is stochastically stable and at the same time the cost (5) is bounded for all admissible uncertainties.

The aim of this paper is to develop LMI conditions that can be used to design a state feedback controller which guarantees that the closed-loop system of the uncertain system is robust stochastically stable and the cost (5) is bounded for all admissible norm bounded uncertainties when the transition jump rates are either totally known or totally unknown but bounded with some known bounds for each mode. The conditions we will develop here will be in terms of the solutions of convex optimization problems that can be easily solved using existing tools.

Lemma 2.1: ([8]) The free uncertain stochastic system (2) ($u(t)=0$ for all $t\geq 0$) is stochastically stable if there exist a set of symmetric and positive-definite matrices $P=(P(1),\cdots,P(N))$, with $P(i)\in\mathbb{R}^{n\times n}$ and a set of positive scalars $\varepsilon_A=(\varepsilon_A(1),\cdots,\varepsilon_A(N))$ such that the following set of coupled LMIs holds for each $i\in\mathscr{S}$ and for all admissible uncertainties:

$$\begin{bmatrix} J_1(i) & P(i)D_A(i) & W^\top(i)P(i)\\ D_A^\top(i)P(i) & -\varepsilon_A(i)\mathbb{I} & 0\\ P(i)W(i) & 0 & -P(i)\end{bmatrix}<0, \quad (7)$$

where

$$J_1(i)=P(i)A(i)+A^\top(i)P(i)+\varepsilon_A(i)E_A^\top(i)E_A(i) + \sum_{j=1}^N \lambda_{ij}P(j).$$

Lemma 2.2: Let Z, E, Δ, F be matrices with appropriate dimensions. Suppose Z is symmetric and $\Delta^\top \Delta \leq \mathbb{I}$, then

$$Z + E\Delta F + F^\top \Delta^\top E^\top < 0$$

if and only if there exists scalar $\varepsilon > 0$ satisfying

$$Z + \varepsilon E E^\top + \frac{1}{\varepsilon} F^\top F < 0.$$

In the rest of this paper we will assume that we have complete access to the system modes and states at each time t for feedback purpose.

III. MAIN RESULTS

Let us consider that the control is equal to zero for all $t \geq 0$ and see under which conditions the systems (2) will have a bounded cost (5). When the control is equal to zero for all $t \geq 0$, the cost (5) becomes:

$$J(x_0, r_0) = \mathbb{E}\left[\int_0^\infty \left[x^\top(t) R_1(r_t) x(t)\right] dt\right].$$

It is trivial that we should satisfy some conditions to guarantee the existence of the solution and the stochastic stability to assure that the cost function is bounded. The following theorem gives the desired results:

Theorem 3.1: If there exist a set of symmetric and positive-definite matrices $P = (P(1), \cdots, P(N))$, $P(i) \in \mathbb{R}^{n \times n}$ and a set of positive scalars $\varepsilon_A = (\varepsilon_A(1), \cdots, \varepsilon_A(N))$ such that the following set of coupled matrix inequalities holds for each $i \in \mathscr{S}$:

$$\begin{bmatrix} J(i) & P(i)D_A(i) & W^\top(i)P(i) \\ D_A^\top(i)P(i) & -\varepsilon_A(i)\mathbb{I} & 0 \\ P(i)W(i) & 0 & -P(i) \end{bmatrix} < 0 \qquad (8)$$

where

$$J(i) = P(i)A(i) + A^\top(i)P(i) + \varepsilon_A(i)E_A^\top(i)E_A(i) + R_1(i) + \sum_{j=1}^N \lambda_{ij} P(j),$$

then the system (2) is stochastically stable and the cost (5) satisfies the following for all admissible uncertainties:

$$J(x_0, r_0) \leq tr\left[P(r_0)x_0 x_0^\top\right] \qquad (9)$$

Proof: Since $R_1(i) > 0$, for all $i \in \mathscr{S}$, from the inequality (8) of the theorem, we get:

$$\begin{bmatrix} J_1(i) & P(i)D_A(i) & W^\top(i)P(i) \\ D^\top(i)P(i) & -\varepsilon_A(i)\mathbb{I} & 0 \\ P(i)W(i) & 0 & -P(i) \end{bmatrix} < 0 \qquad (10)$$

with

$$J_1(i) = P(i)A(i) + A^\top(i)P(i) + \varepsilon_A(i)E_A^\top(i)E_A(i) + \sum_{j=1}^N \lambda_{ij} P(j).$$

Using Lemma 2.1 we conclude that the free uncertain stochastic system with random abrupt changes (2) is stochastically stable.

Let (x, i) denote respectively the state of the vector state, $x(t)$, and the mode, r_t, at time t and consider the following Lyapunov function:

$$V(x(t), r_t) = x^\top(t) P(r_t) x(t)$$

The weak infinitesimal operator, $\mathscr{L}V(.)$ emanating from (x, i) at time t is given by (see [8]):

$$\mathscr{L}V(x, i) = x^\top(t)\Big[P(i)A(i) + A^\top(i)P(i) + P(i)D_A(i)F_A(i)E_A(i) + E_A^\top(i)F_A^\top(i)D_A^\top(i)P(i) + W^\top(i)P(i)W(i) + \sum_{j=1}^N \lambda_{ij}P(j)\Big]x(t)$$

Based on the Lemma 2.2 we get:

$$\mathscr{L}V(x, i) \leq x^\top(t)\Big[P(i)A(i) + A^\top(i)P(i) + \varepsilon_A(i)E_A^\top(i)E_A(i) + \varepsilon_A^{-1}(i)P(i)D_A(i)D_A^\top(i)P(i) + W^\top(i)P(i)W(i) + \sum_{j=1}^N \lambda_{ij}P(j)\Big]x(t)$$

which gives

$$\mathscr{L}V(x, i) + x^\top(t)R_1(i)x(t) \leq x^\top(t)\Big[P(i)A(i) + A^\top(i)P(i) + \varepsilon_A(i)E_A^\top(i)E_A(i) + \varepsilon_A^{-1}(i)P(i)D_A(i)D_A^\top(i)P(i) + R_1(i) + W^\top(i)P(i)W(i) + \sum_{j=1}^N \lambda_{ij}P(j)\Big]x(t)$$

Combining this with the theorem conditions and Schur complement, the following holds for any $T > 0$:

$$\int_0^T \mathscr{L}V(x(t), r_t)dt + \int_0^T x^\top(t)R_1(r_t)x(t)dt < 0$$

Using now the Dynkin formula and the fact that the system is stochastically stable, we get:

$$\mathbb{E}\left[\int_0^\infty x^\top(t)R_1(r_t)x(t)dt\right] \leq \mathbb{E}\left[x_0^\top P(r_0)x_0\right]$$

which implies that the cost function is bounded and this ends the proof of the theorem. □

For the totally unknown, but bounded with some known bounds, transition jump rates, using the Assumption 2.1 we get the following relations:

$$\sum_{j=1, j\neq i}^N \lambda_{ij}P(j) \leq \bar{\lambda}_i \sum_{j=1, j\neq i}^N P(j)$$

$$\lambda_{ii}P(i) = -\sum_{j=1, j\neq i}^N \lambda_{ij}P(i) \leq -(N_i - 1)\underline{\lambda}_i P(i)$$

with N_i is the number of modes from the mode i including itself.

Based on these realtions we have:

$$\sum_{j=1}^N \lambda_{ij}P(j) \leq -(N_i - 1)\underline{\lambda}_i P(i) + \bar{\lambda}_i \sum_{j=1, j\neq i}^N P(j)$$

Combining this with the previous conditions we get the following results.

Corollary 3.1: If there exist a set of symmetric and positive-definite matrices $P = (P(1), \cdots, P(N))$, $P(i) \in \mathbb{R}^{n\times n}$ and a set of positive scalars $\varepsilon_A = (\varepsilon_A(1), \cdots, \varepsilon_A(N))$ such that the following set of coupled matrix inequalities holds for each $i \in \mathscr{S}$:

$$\begin{bmatrix} J(i) & P(i)D_A(i) & W^\top(i)P(i) \\ D^\top(i)P(i) & -\varepsilon_A(i)\mathbb{I} & 0 \\ P(i)W(i) & 0 & -P(i) \end{bmatrix} < 0 \tag{11}$$

where

$$J(i) = P(i)A(i) + A^\top(i)P(i) + \varepsilon_A(i)E_A^\top(i)E_A(i) + R_1(i) - (N_i - 1)\,\underline{\lambda}_i P(i) + \sum_{j=1,j\neq i}^{N} \bar{\lambda}_i P(j),$$

then the uncertain stochastic system with random abrupt changes (2) is stochastically stable and the cost (5) satisfies the following for all admissible uncertainties:

$$J(x_0, r_0) \le tr\left[P(r_0)x_0x_0^\top\right] \tag{12}$$

In the rest of this section we will deal with the design of the state feedback controller that makes the closed-loop system stochastically stable and at same time assures that the cost function is bounded. For this purpose, notice that the cost function with the control law (6) becomes:

$$J(x_0, r_0) = \mathbb{E}\left[\int_0^\infty \left[x^\top(t)\widetilde{R}_1(r_t)x(t)\right] dt\right]$$

with $\widetilde{R}_1(i) = R_1(i) + K^\top(i)R_2(i)K(i)$ for all $i \in \mathscr{S}$.

The closed-loop state equation is given by:

$$dx(t) = A_{cl}(r_t, t)x(t) + W(r_t)x(t)d\omega(t)$$

with $A_{cl}(r_t, t) = A(r_t, t) + B(r_t, t)K(r_t)$.

Based on the results of Theorem 3.1, the closed-loop system will be stochastically stable and the cost function is bounded if there exist a set of symmetric and positive-definite matrices $P = (P(1), \cdots, P(N))$, $P(i) \in \mathbb{R}^{n\times n}$ such that the following hold:

$$A_{cl}^\top(i,t)P(i) + P(i)A_{cl}(i,t) + \widetilde{R}_1(i) + W^\top(i)P(i)W(i) + \sum_{j=1}^{N} \lambda_{ij}P(j) < 0.$$

Using the expression of $A_{cl}(i,t)$ and Lemma 2.2, we get:

$$\begin{aligned} &A^\top(i)P(i) + P(i)A(i) + K^\top(i)B^\top(i)P(i) + P(i)B(i)K(i) \\ &+\varepsilon_A(i)P(i)D_A(i)D_A^\top(i)P(i) + \varepsilon_B(i)P(i)D_B(i)D_B^\top(i)P(i) \\ &+\varepsilon_A^{-1}(i)E_A^\top(i)E_A(i) + W^\top(i)P(i)W(i) + \widetilde{R}_1(i) \\ &+\varepsilon_B^{-1}(i)K^\top(i)E_B^\top(i)E_B(i)K(i) + \sum_{j=1}^{N}\lambda_{ij}P(j) < 0 \end{aligned}$$

where $\varepsilon_A = (\varepsilon_A(1), \cdots, \varepsilon_A(N))$ and $\varepsilon_B = (\varepsilon_B(1), \cdots, \varepsilon_B(N))$ are sets of positive scalars.

This inequality matrix is nonlinear in the decision variables $P(i)$ and $K(i)$. To put it in the LMI setting, let $X(i) = P^{-1}(i)$ and pre- and post-multiplying this inequality by $X(i)$, we get:

$$\begin{aligned} &X(i)A^\top(i) + A(i)X(i) + X(i)K^\top(i)B^\top(i) + B(i)K(i)X(i) \\ &+\varepsilon_A(i)D_A(i)D_A^\top(i) + \varepsilon_B(i)D_B(i)D_B^\top(i) + X(i)\widetilde{R}_1(i)X(i) \\ &+\varepsilon_A^{-1}(i)X(i)E_A^\top(i)E_A(i)X(i) + X(i)W^\top(i)X^{-1}(i)W(i)X(i) \\ &+\varepsilon_B^{-1}(i)X(i)K^\top(i)E_B^\top(i)E_B(i)K(i)X(i) + \lambda_{ii}X(i) \\ &+ \sum_{j=1,j\neq i}^{N}\lambda_{ij}X(i)X^{-1}(j)X(i) < 0 \end{aligned}$$

Defining $\mathcal{X}_i(X)$ and $\mathcal{S}_i(X)$ as follows:

$$\begin{aligned} \mathcal{X}_i(X) &= \operatorname{diag}\left[X(1), \cdots, X(i-1), X(i+1), \cdots, X(N)\right] \\ \mathcal{S}_i(X) &= \Big[\sqrt{\lambda_{i1}}X(i), \cdots, \sqrt{\lambda_{ii-1}}X(i), \sqrt{\lambda_{ii+1}}X(i), \cdots, \sqrt{\lambda_{iN}}X(i)\Big] \end{aligned}$$

and letting $Y(i) = K(i)X(i)$, $S_1(i) = R_1^{\frac{1}{2}}$ and $S_2(i) = R_2^{\frac{1}{2}}$ we get the conditions (13).

The following theorem summarizes the results of this development.

Theorem 3.2: There exists a state feedback controller of the form (6) such that the closed-loop state equation of the stochastic system (2) is stochastically stable and moreover the cost function (5) is bounded if there exist a set of symmetric and positive-definite matrices $X = (X(1), \cdots, X(N))$, $X(i) \in \mathbb{R}^{n\times n}$ a set of matrices $Y = (Y(1), \cdots, Y(N))$, $Y(i) \in \mathbb{R}^{m\times n}$ and sets of positive scalars $\varepsilon_A = (\varepsilon_A(1), \cdots, \varepsilon_A(N))$ and $\varepsilon_B = (\varepsilon_B(1), \cdots, \varepsilon_B(N))$ such that the following set of coupled LMIs holds for each $i \in \mathscr{S}$:

$$\begin{bmatrix} \widehat{J}(i) & X(i)E_A^\top(i) & Y^\top(i)E_B^\top(i) & X(i)S_1^\top(i) & Y^\top(i)S_2^\top(i) & X(i)W^\top(i) & \mathcal{S}_i(X) \\ E_A(i)X(i) & -\varepsilon_A(i)\mathbb{I} & 0 & 0 & 0 & 0 & 0 \\ E_B(i)Y(i) & 0 & -\varepsilon_B(i)\mathbb{I} & 0 & 0 & 0 & 0 \\ S_1(i)X(i) & 0 & 0 & -\mathbb{I} & 0 & 0 & 0 \\ S_2(i)Y(i) & 0 & 0 & 0 & -\mathbb{I} & 0 & 0 \\ W(i)X(i) & 0 & 0 & 0 & 0 & -X(i) & 0 \\ \mathcal{S}_i^\top(X) & 0 & 0 & 0 & 0 & 0 & -\mathcal{X}_i(X) \end{bmatrix} < 0, \tag{13}$$

where

$$\begin{aligned} \widehat{J}(i) &= A(i)X(i) + X(i)A^\top(i) + B(i)Y(i) + B^\top(i)Y^\top(i) \\ &\quad +\lambda_{ii}X(i) + \varepsilon_A(i)D_A(i)D_A^\top(i) + \varepsilon_B(i)D_B(i)D_B^\top(i) \\ \mathcal{X}_i(X) &= \operatorname{diag}\left[X(1), \cdots, X(i-1), X(i+1), \cdots, X(N)\right] \\ \mathcal{S}_i(X) &= \Big[\sqrt{\lambda_{i1}}X(i), \cdots, \sqrt{\lambda_{ii-1}}X(i), \sqrt{\lambda_{ii+1}}X(i), \cdots, \sqrt{\lambda_{iN}}X(i)\Big] \end{aligned}$$

The stabilizing controller gain is given by $K(i) = Y(i)X^{-1}(i)$, $i \in \mathscr{S}$. Moreover the cost (5) satisfies the following for all admissible uncertainties:

$$J(x_0, r_0) \leq tr\left[X^{-1}(r_0)x_0x_0^\top\right] \tag{14}$$

Remark 3.1: For the optimal cost, there exists a positive scalar ϱ such that:

$$x_0^\top X^{-1}(r_0)x_0 \leq \varrho$$

can be rewritten as follows:

$$\begin{bmatrix} -\varrho & x_0^\top) \\ x_0 & -X(r_0) \end{bmatrix} \leq 0 \tag{15}$$

The following optimization problem can determine the controller that assures the minimum cost:

$$\mathbb{P}: \begin{cases} \min_{\substack{\varrho>0, \\ X=(X(1),\cdots,X(N)), \\ Y=(Y(1),\cdots,Y(N)), \\ \varepsilon_A=(\varepsilon_A(1),\cdots,\varepsilon_A(N)), \\ \varepsilon_B=(\varepsilon_B(1),\cdots,\varepsilon_B(N))}} \varrho \\ s.t: (13),(15) \end{cases}$$

For the totally unknown, but bounded with some known bounds, transition jump rates case we can follow the same steps as it was done before and we get the following result.

Corollary 3.2: There exists a state feedback controller of the form (6) such that the closed-loop state equation of the nominal system (2) is stochastically stable and moreover the cost function (5) is bounded if there exist a set of symmetric and positive-definite matrices $X = (X(1), \cdots, X(N))$, $X(i) \in \mathbb{R}^{n\times n}$ a set of matrices $Y = (Y(1), \cdots, Y(N))$, $Y(i) \in \mathbb{R}^{m\times n}$ and sets of positive scalars $\varepsilon_A = (\varepsilon_A(1), \cdots, \varepsilon_A(N))$ and $\varepsilon_B = (\varepsilon_B(1), \cdots, \varepsilon_B(N))$ such that the following set of coupled LMIs (13) holds for each $i \in \mathscr{S}$, with

$$\begin{aligned} \widehat{J}(i) &= A(i)X(i) + X(i)A^\top(i) + B(i)Y(i) + B^\top(i)Y^\top(i) \\ &- (N_i - 1)\,\underline{\lambda}_i X(i) + \varepsilon_A(i)D_A(i)D_A^\top(i) + \varepsilon_B(i)D_B(i)D_B^\top(i) \\ \mathcal{X}_i(X) &= \mathrm{diag}\left[X(1), \cdots, X(i-1), X(i+1), \cdots, X(N)\right] \\ \mathcal{S}_i(X) &= \left[\sqrt{\overline{\lambda}_i}X(i), \cdots, \sqrt{\overline{\lambda}_i}X(i), \sqrt{\overline{\lambda}_i}X(i),\right. \\ &\left. \cdots, \sqrt{\overline{\lambda}_i}X(i)\right] \end{aligned}$$

The stabilizing controller gain is given by $K(i) = Y(i)X^{-1}(i)$, $i \in \mathscr{S}$. Moreover the cost (5) satisfies the following for all admissible uncertainties:

$$J(x_0, r_0) \leq tr\left[X^{-1}(r_0)x_0x_0^\top\right] \tag{16}$$

To compute the stabilizing controller that gives the optimal cost the optimization problem $\mathbb{P}$ can be used with the appropriate changes in the LMIs.

Remark 3.2: For the comparison purpose of the two cases we considered in this paper, it is important to notice that more knowledge we have on the transition jump rates, the less conservatism will be. Therefore, it is evident that the results for the case of totally unknown transition jump rates we have some conservatism. But we should mention that without the knowledge of the transition jump rates the literature results will not help to get the solution and ours represent and alternate to overcome this situation.

IV. NUMERICAL EXAMPLE

To illustrate the effectiveness of the proposed results, let us consider a stochastic system of the form (2) with three modes with the following data:

- mode # 1:

$$A(1) = \begin{bmatrix} -1.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & -1.0 & -1.0 \end{bmatrix}, B(1) = \begin{bmatrix} 0.3 & 0.0 \\ 0.0 & 0.1 \\ 0.2 & 1.0 \end{bmatrix},$$

$$D_A(1) = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}, E_A(1) = \begin{bmatrix} 0.3 & 0.2 & 0.1 \end{bmatrix},$$

$$D_B(1) = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.1 \end{bmatrix}, E_B(1) = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix},$$

$$R_1(1) = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, R_2(1) = \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 4.0 \end{bmatrix},$$

$$W(1) = \begin{bmatrix} 0.1 & 0.0 & 0.0 \\ 0.0 & 0.2 & 0.0 \\ 0.0 & 0.0 & -0.1 \end{bmatrix}$$

- mode # 2:

$$A(2) = \begin{bmatrix} 1.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & -1.0 \end{bmatrix}, B(2) = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 0.0 \\ 0.1 & 0.2 \end{bmatrix},$$

$$D_A(2) = \begin{bmatrix} 0.2 \\ 0.1 \\ 0.3 \end{bmatrix}, E_A(2) = \begin{bmatrix} 0.3 & 0.2 & 0.1 \end{bmatrix},$$

$$D_B(2) = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.2 \end{bmatrix}, E_B(2) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix},$$

$$R_1(2) = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.2 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, R_2(2) = \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 2.0 \end{bmatrix},$$

$$W(2) = \begin{bmatrix} -0.1 & 0.0 & 0.0 \\ 0.0 & -0.2 & 0.0 \\ 0.0 & 0.0 & 0.1 \end{bmatrix}$$

- mode # 3:

$$A(3) = \begin{bmatrix} 1.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 1.0 & 1.0 \end{bmatrix}, B(3) = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 0.0 \\ 0.1 & -0.2 \end{bmatrix},$$

$$D_A(3) = \begin{bmatrix} -0.2 \\ 0.1 \\ 0.3 \end{bmatrix}, E_A(3) = \begin{bmatrix} 0.3 & -0.2 & 0.1 \end{bmatrix},$$

$$D_B(3) = \begin{bmatrix} 0.3 \\ 0.1 \\ -0.2 \end{bmatrix}, E_B(3) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix},$$

$$R_1(3) = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.5 \end{bmatrix}, R_2(3) = \begin{bmatrix} 4.0 & 0.0 \\ 0.0 & 2.0 \end{bmatrix},$$

$$W(3) = \begin{bmatrix} -0.1 & 0.0 & 0.0 \\ 0.0 & -0.2 & 0.0 \\ 0.0 & 0.0 & -0.1 \end{bmatrix}$$

The transition jump rates matrix is given by:

$$\Lambda = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1.1 & 1.1 \\ 1 & 1 & -2 \end{bmatrix}.$$

Solving the LMIs (13) of Theorem 3.2, we get:

$$\varepsilon_A(1) = 0.2981, \varepsilon_A(2) = 0.0078, \varepsilon_A(3) = 0.0041,$$
$$\varepsilon_B(1) = 0.3872, \varepsilon_B(2) = 0.0078, \varepsilon_B(3) = 0.0079,$$
$$X(1) = \begin{bmatrix} 0.0489 & 0.0499 & -0.0661 \\ 0.0499 & 0.0703 & -0.0882 \\ -0.0661 & -0.0882 & 0.1537 \end{bmatrix},$$
$$X(2) = \begin{bmatrix} 0.0183 & 0.0301 & -0.0389 \\ 0.0301 & 0.0570 & -0.0705 \\ -0.0389 & -0.0705 & 0.1011 \end{bmatrix},$$
$$X(3) = \begin{bmatrix} 0.0163 & 0.0290 & -0.0247 \\ 0.0290 & 0.0596 & -0.0483 \\ -0.0247 & -0.0483 & 0.0469 \end{bmatrix},$$
$$Y(1) = \begin{bmatrix} -0.1430 & 0.0004 & -0.0889 \\ 0.0018 & -0.0249 & -0.2473 \end{bmatrix},$$
$$Y(2) = \begin{bmatrix} -0.0494 & -0.0000 & -0.0469 \\ 0.0012 & -0.0001 & -0.0939 \end{bmatrix},$$
$$Y(3) = \begin{bmatrix} -0.0248 & 0.0000 & -0.0236 \\ -0.0006 & -0.0001 & 0.0946 \end{bmatrix},$$

which give the following gains:

$$K(1) = \begin{bmatrix} -11.4000 & 4.3692 & -2.9726 \\ -0.2199 & -8.3511 & -6.4950 \end{bmatrix},$$
$$K(2) = \begin{bmatrix} -24.6693 & 4.9882 & -6.4804 \\ -4.7004 & -6.6619 & -7.3871 \end{bmatrix},$$
$$K(3) = \begin{bmatrix} -28.1027 & 5.4128 & -10.2224 \\ -6.3626 & -7.4688 & -13.0352 \end{bmatrix}.$$

For the totally unknown transition jump rate, if we assume that we have the following bounds:

$$\text{mode \# 1: } \underline{\lambda}_1 = 0.8, \bar{\lambda}_1 = 1.2,$$
$$\text{mode \# 2: } \underline{\lambda}_2 = 0.9, \bar{\lambda}_2 = 1.3,$$
$$\text{mode \# 3: } \underline{\lambda}_3 = 0.9, \bar{\lambda}_3 = 1.3.$$

and solving the LMIs of Corollary 3.2, we get:

$$\varepsilon_A(1) = 3.1378\ 10^{-4}, \varepsilon_A(2) = 3.1187\ 10^{-4},$$
$$\varepsilon_A(3) = 1.4261\ 10^{-4}, \varepsilon_B(1) = 0.0042,$$
$$\varepsilon_B(2) = 5.2255\ 10^{-4}, \varepsilon_B(3) = 3.0930\ 10^{-4},$$
$$X(1) = \begin{bmatrix} 0.0011 & -0.0001 & -0.0004 \\ -0.0001 & 0.0001 & 0.0001 \\ -0.0004 & 0.0001 & 0.0049 \end{bmatrix},$$
$$X(2) = \begin{bmatrix} 0.0006 & 0.0001 & -0.0001 \\ 0.0001 & 0.0001 & -0.0003 \\ -0.0001 & -0.0003 & 0.0014 \end{bmatrix},$$
$$X(3) = 10^{-3} \begin{bmatrix} 0.3782 & 0.0354 & -0.0603 \\ 0.0354 & 0.0752 & -0.1649 \\ -0.0603 & -0.1649 & 0.7205 \end{bmatrix},$$
$$Y(1) = \begin{bmatrix} -0.0375 & 0.0094 & 0.0687 \\ 0.0281 & -0.0227 & -0.2078 \end{bmatrix},$$
$$Y(2) = \begin{bmatrix} -0.0414 & 0.0000 & -0.0072 \\ 0.0171 & 0.0000 & -0.0143 \end{bmatrix},$$
$$Y(3) = \begin{bmatrix} -0.0227 & 0.0000 & -0.0039 \\ -0.0094 & -0.0000 & 0.0157 \end{bmatrix},$$

which give the following gains:

$$K(1) = \begin{bmatrix} -25.9610 & 66.6521 & 10.8253 \\ 2.4000 & -180.5475 & -39.4168 \end{bmatrix},$$
$$K(2) = \begin{bmatrix} -73.2249 & 6.7024 & -9.6712 \\ 32.2511 & -98.7444 & -30.6965 \end{bmatrix},$$
$$K(3) = \begin{bmatrix} -113.9071 & 22.0483 & -14.4114 \\ 49.4368 & -115.8798 & -42.2448 \end{bmatrix}.$$

V. Conclusion

This paper dealt with the guaranteed cost control problem of the stochastic class of systems with random abrupt changes. The transition jump rates are considered to be totally known and totally unknown but bounded with some known bounds in each mode. Sufficient conditions to compute the state feedback controller which guarantees that the closed-loop system is stochastically stable and give the optimal cost is designed using convex optimization problems. Results were developed for the two cases. The conditions we developed in this paper are tractable using commercial convex optimization tools.

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