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Iterative Learning Control with High-Order Internal Model for Linear Time-Varying Systems

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Abstract— **In this work we focus on iterative learning control (ILC) for iteratively varying reference trajectories which are described by a high-order internal model. The high-order internal model (HOIM) is formulated as a polynomial between two consecutive iterations. The classical ILC with iteratively invariant reference trajectories, on the other hand, is a special case of HOIM where the polynomial renders to a unity coefficient, in other words, the 0th order internal model. By inserting the polynomial (HOIM) into the past control input of the ILC law, and designing appropriate learning control gains, the learning convergence in the iteration axis can be guaranteed for continuous-time linear time varying (LTV) systems. The initial condition, P-type and D-type ILC, and possible extension to nonlinear cases are also explored.**

I. INTRODUCTION

In the past few years, iterative learning control (ILC) methods have been extended to tracking tasks with iteratively varying reference trajectory, for instance see [1] - [4]. To deal with the iteratively varying reference, an effective approach is to incorporate a high-order internal model (HOIM), which produces the iteratively varying reference trajectory, into the iterative learning law. A survey on HOIM applications to a variety of ILC problems was given in [5].

It should be noted that most HOIM-based ILC methods were proposed to deal with discrete-time linear time varying (LTV) processes. By lifting or supervector transformation, the discrete-time LTV process in a finite time interval can be remodeled as an augmented matrix P between the input and output vectors $(\mathbf{u}, \mathbf{y})$, that is $\mathbf{y} = P\mathbf{y}$, where every element in the matrix P has a fixed value. As such, the input-output relationship of the LTV process becomes a static mapping. A direct benefit from the static mapping is the simplicity of ILC design, which can not only focus on the learning performance along the iteration axis, but also eliminate the needs for initial state resetting (or identical initialization condition) and λ-norm that are two fundamental ILC issues in the time domain [6], [7], [5]. As a consequence, the HOIM-based ILC can be designed and analyzed using transfer function approach.

However, the lifting or supervector methods are not applicable to continuous-time LTV processes. The aim of this work is to explore the implementation of HOIM in the ILC design for continuous-time processes, and provide the analysis of ILC convergence with HOIM. Through analysis and discussions, we make three conclusions.

First,HOIM can be achieved simply by a stable polynomial, and the same polynomial applies to both the iteratively varying reference trajectory and the ILC law. The classical ILC with iteratively invariant reference trajectory is a special case of HOIM where the polynomial renders to a unity coefficient, in other words, the classical ILC uses a 0th order internal model.

Second, the identical initialization condition, under which the initial state values are kept the same for all iterations, are no longer adequate or suitable. Instead, the initial resetting condition, under which the initial state values are set to zero for all iterations, would be adequate.

Third, the λ-norm is needed in the analysis of HOIM-based ILC, analogous to classical ILC. The λ-norm, or time-weighted norm, is used in ILC to eliminate the influence of the state dynamics. As a result, the reference trajectory can be defined on an interval of arbitrary length.

In continuous time, the HOIM-based ILC is in essence a high-order ILC. The focus of this work is to extend the classical time-domain ILC design and analysis methods to iteratively varying reference trajectory generated by a known HOIM.

It is worth to mention, that HOIM-based ILC is different from many existing high-order ILC schemes. HOIM-based ILC can be viewed as a kind of high-order ILC in which the past control inputs are specifically weighted accordingly to the HOIM coefficients. On the other hand, many high-order ILC methods are developed aiming to improve the learning performance with iteratively invariant reference trajectories [8], [9]. Therefore those high-order ILC schemes either only employ a single past control input, or assigns weights for multiple past control inputs in terms of a particular control objective, such as the transient response along the iteration axis.

The paper is organized as follows. In Section II, the ILC problem with HOIM is formulated. In Section III, the convergence analysis of HOIM-based P-type ILC is conducted for continuous-time LTV systems. In Section IV, we explore the extension to HOIM-based D-type ILC, and possible extension to plants with nonlinear factors. In Section V, two numerical examples are given. Finally in Section VI, we given a conclusion.

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II. Problem Formulation

Consider a class of LTV systems

$$\left\{ \begin{array}{l} \dot{\mathbf{x}}_i(t) = A(t)\mathbf{x}_i(t) + B(t)\mathbf{u}_i(t) \\ \mathbf{y}_i(t) = C(t)\mathbf{x}_i(t) + D(t)\mathbf{u}_i(t) \end{array} \right. \tag{1}$$

where the subscript "i" denotes the ith learning iteration, $t \in [0,T]$, $A(t) \in \mathcal{C}([0,T], \mathcal{R}^{n\times n})$, $B(t) \in \mathcal{C}([0,T], \mathcal{R}^{n\times q})$, $C(t) \in \mathcal{C}([0,T], \mathcal{R}^{q\times n})$, $D(t) \in \mathcal{C}([0,T], \mathcal{R}^{q\times q})$, $\mathbf{u}$ and $\mathbf{y}$ are input and output vectors, respectively.

Denote $\mathbf{y}_{r,i}(t)$ the iteratively varying reference trajectory. The tracking error is defined to be $\mathbf{e}_i(t) = \mathbf{y}_{r,i}(t) - \mathbf{y}_i(t)$.

Now we give the HOIM that generates the iteratively varying reference trajectory $\mathbf{y}_{r,i}(t)$. First, following the notations of [5], introduce a new shift operator, w, with the property

$$w^{-1}\mathbf{u}_i(t) = \mathbf{u}_{i-1}(t). \tag{2}$$

Definition 1: The iteratively varying reference trajectory is generated by the following HOIM

$$\mathbf{y}_{r,i+1} = H(w^{-1})\mathbf{y}_{r,i} \tag{3}$$

where $H(w^{-1})$ is the internal model described by a polynomial

$$H(w^{-1}) = h_1 + h_2 w^{-1} + \cdots + h_m w^{-m+1} \tag{4}$$

h_j are coefficients of a stable polynomial $w^m - h_1 w^{m-1} - \cdots - h_m$,and this mth order stable polynomial is the characteristic equation of the HOIM (3).

From (3) and (4), the reference trajectory with the HOIM is

$$\begin{aligned} \mathbf{y}_{r,i+1}(t) =& h_1\mathbf{y}_{r,i}(t) + h_2\mathbf{y}_{r,i-1}(t) \\ &+ \cdots + h_m\mathbf{y}_{r,i-m+1}(t). \end{aligned} \tag{5}$$

Note that m initial trajectories $\mathbf{y}_{r,0}(t)$, $\cdots$, $\mathbf{y}_{r,1-m}(t)$ are required to determine the regressor (5).

Remark 1: The interatively invariant reference trajectory is a special case of HOIM (5)

$$\mathbf{y}_{r,i+1}(t) = h_1\mathbf{y}_{r,i}(t)$$

where $h_1 = 1$, hence is a first order HOIM, as the characteristic polynomial between $\mathbf{y}_{r,i+1}$ and $\mathbf{y}_{r,i}$ is $w - h_1$.

Remark 2: In fact, $\mathbf{y}_{r,i+1} = H(w^{-1})\mathbf{y}_{r,i}$ is a general expression for HOIM. In existing works, a HOIM is expressed as a rational function [2]

$$\mathbf{y}_{r,i+1} = \frac{N(w^{-1})}{F(w^{-1})}\mathbf{y}_{r,i} \tag{6}$$

where $N(w^{-1}) = n_1 w^{-1} + n_2 w^{-2} + \cdots + n_l w^{-l}$ and $F(w^{-1}) = 1 + f_1 w^{-1} + f_2 w^{-2} + \cdots + f_m w^{-m} (l < m)$. Rewrite (6) as

$$F(w^{-1})\mathbf{y}_{r,i+1} = N(w^{-1})\mathbf{y}_{r,i}, \tag{7}$$

we have

$$\begin{aligned} &(1 + f_1 w^{-1} + f_2 w^{-2} + \cdots + f_m w^{-m})\mathbf{y}_{r,i+1} \\ &= (n_1 w^{-1} + n_2 w^{-2} + \cdots + n_l w^{-l})\mathbf{y}_{r,i}. \end{aligned} \tag{8}$$

Collecting terms according to the power of w, and using the factor $w^{-1}\mathbf{y}_{r,i+1} = \mathbf{y}_{r,i}$, we obtain

$$\begin{aligned} \mathbf{y}_{r,i+1} =& \big[- f_1 + (n_1 - f_2)w^{-1} + \cdots \\ &+ (n_l - f_{l+1})w^l - f_{l+2}w^{-l-1} - \cdots \\ &- f_m w^{-m+1}\big]\mathbf{y}_{r,i} \\ =& H(w^{-1})\mathbf{y}_{r,i} \end{aligned} \tag{9}$$

which is the same as (5) by defining $h_1 = -f_1$, $h_2 = n_1 - f_2$, $\cdots$, $h_m = f_m$.

Now we consider the initial condition. For HOIM, we need the initial resetting condition defined below

Assumption 1:

$$\mathbf{x}_i(0) = 0 \quad for \quad i = 1, 2, \cdots. \tag{10}$$

Note the difference between the initial resetting condition (10) and the identical initialization condition

$$\mathbf{x}_{i+1}(0) = \mathbf{x}_i(0) \quad for \quad i = 1, 2, \cdots.$$

The latter is widely assumed and accepted in ILC problems.

The control objective is to design an iterative learning law, $\mathbf{u}_i(t)$, such that as $i \to \infty$, the system output $\mathbf{y}_i(t)$ in (1) tracks the desired output trajectory $\mathbf{y}_{r,i}(t)$(5) as closely as possible $\forall t \in [0,T]$. To achieve this control objective, the simplest ILC law is a P-type ILC with mth-order internal model

$$\begin{aligned} \mathbf{u}_{i+1}(t) =& h_1\mathbf{u}_i(t) + h_2\mathbf{u}_{i-1}(t) + \cdots h_m\mathbf{u}_{i-m+1}(t) \\ &+ \gamma_1\mathbf{e}_i(t) + \gamma_2\mathbf{e}_{i-1}(t) + \gamma_m\mathbf{e}_{i-m+1}(t) \end{aligned} \tag{11}$$

or equivalently

$$\mathbf{u}_{i+1} = H(w^{-1})\mathbf{u}_i + \Gamma(w^{-1})\mathbf{e}_i \tag{12}$$

where $\Gamma(w^{-1}) = \gamma_1 + \gamma_2 w^{-1} + \cdots + \gamma_m w^{-m+1}$, and γ_j are learning gains.

Remark 3: The ILC law (11) is similar to the high-order ILC law. In most high-order ILC, the coefficients of $H(w^{-1})$ is chosen such that [8], [9]

$$\sum_{j=1}^{m} h_j = 1. \tag{13}$$

This relation is necessary for ILC to track iteratively invariant reference. In the ideal case when the past tracking errors $\mathbf{e}_j$ are zero and the past control inputs $\mathbf{u}_j$ converge to the designed one $\mathbf{u}_r$ for j up to i, then from (11)

$$\mathbf{u}_{i+1} = \sum_{j=1}^{m} h_j\mathbf{u}_r = \mathbf{u}_r$$

and the ideal control input retains for the new iteration. As for the HOIM-based ILC, the relation (13) does not hold in general.

Before analyzing the ILC law (11) in next section, let us give the definition and property of the λ-norm.

Definition 2: We define the λ norm as

$$\|f(\cdot)\|_\lambda = \sup_{t\in[0,t]} e^{-\lambda t}\|f(t)\|$$

A useful property associated with the λ-norm the following inequality

Property 1: λ norm has the next property

$$\begin{aligned}
&\sup_{t\in[0,T]} e^{-\lambda t}\int_0^t ||f(\cdot)||\, e^{a(t-\tau)}d\tau \\
&= \sup_{t\in[0,T]}\int_0^t e^{-\lambda\tau}\|f(\cdot)\|\, e^{(a-\lambda)(t-\tau)}d\tau \\
&\le \frac{1-e^{(a-\lambda)T}}{\lambda-a}\|f(\cdot)\|_\lambda
\end{aligned}$$

III. Convergence Analysis

In this section, we discuss the ILC convergence.

Theorem 1: For the plant (1) and reference trajectory (5), apply the P-type ILC with mth-order internal model (11). The learning convergence, $\|\mathbf{y}_{r,i}-\mathbf{y}_i\|\to 0$ as $i\to\infty$, is guaranteed, provided the learning gains γ_j are chosen such that for $\eta_j \triangleq \|h_jI-\gamma_jD\|$, $j=1,2,\cdots,m$, the polynomial

$$R(z) = z^m - \eta_1 z^{m-1} - \cdots - \eta_m$$

is asymptotically stable.

Proof:

Let us start from

$$\begin{aligned}
\mathbf{e}_{i+1} =&\mathbf{y}_{r,i+1}-\mathbf{y}_{i+1}\\
=&H(w^{-1})\mathbf{y}_{r,i}-H(w^{-1})\mathbf{y}_i-\mathbf{y}_{i+1}\\
&+H(w^{-1})\mathbf{y}_i\\
=&H(w^{-1})\mathbf{e}_i-((C\mathbf{x}_{i+1}+D\mathbf{u}_{i+1})\\
&-H(w^{-1})(C\mathbf{x}_i+D\mathbf{u}_i))\\
=&H(w^{-1})\mathbf{e}_i-D(\mathbf{u}_{i+1}-H(w^{-1})\mathbf{u}_i)\\
&-C(\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i).
\end{aligned}\tag{14}$$

Using relationship (5) and (11), we obtain

$$\begin{aligned}
\mathbf{e}_{i+1} =&h_1\mathbf{e}_i+h_2\mathbf{e}_{i-1}+\cdots+h_m\mathbf{e}_{i-m+1}\\
&-D(\gamma_1\mathbf{e}_i+\gamma_2\mathbf{e}_{i-1}+\cdots+\gamma_m\mathbf{e}_{i-m+1})\\
&-C([\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i]\\
=&(h_1I-\gamma_1D)\mathbf{e}_i+(h_2I-\gamma_2D)\mathbf{e}_{i-1}+\cdots\\
&+(h_mI-\gamma_mD)\mathbf{e}_{i-m+1}\\
&-C[\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i].
\end{aligned}\tag{15}$$

Taking norms on both sides of (15) we obtain

$$\begin{aligned}
\|\mathbf{e}_{i+1}\| \le&\|h_1I-\gamma_1D\|\,\|\mathbf{e}_i\|+\|h_2I-\gamma_2D\|\,\|\mathbf{e}_{i-1}\|\\
&+\cdots+\|h_mI-\gamma_mD\|\,\|\mathbf{e}_{i-m+1}\|\\
&+\|C\|\,\|\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i\|.
\end{aligned}\tag{16}$$

To evaluate the state-dependent term $\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i$ in (16), integrating $\dot{\mathbf{x}}_{i+1}-H(w^{-1})\dot{\mathbf{x}}_i$ and applying the state dynamics (1), we have, for $t\in[0,T]$,

$$\begin{aligned}
&\|\mathbf{x}_{i+1}(t)-H(w^{-1})\mathbf{x}_i(t)\|\\
&=\|\mathbf{x}_{i+1}(0)-H(w^{-1})\mathbf{x}_i(0)\|\\
&\quad+\textstyle\int_0^t[(A\mathbf{x}_{i+1}+B\mathbf{u}_{i+1})\\
&\quad-H(w^{-1})(A\mathbf{x}_i+B\mathbf{u}_i)]d\tau\|\\
&\le\|\mathbf{x}_{i+1}(0)-H(w^{-1})\mathbf{x}_i(0)\|\\
&\quad+\int_0^t[\|A\|\,\|\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i\|\\
&\quad+\|B\|\,\|\mathbf{u}_{i+1}-H(w^{-1})\mathbf{u}_i\|]d\tau.
\end{aligned}\tag{17}$$

According to *Assumption* 1 on the initial resetting condition, we have

$$\mathbf{x}_{i+1}(0)-H(w^{-1})\mathbf{x}_i(0)=0.$$

Applying the Bellman-Gronwall Lemma to (17), we have

$$\begin{aligned}
&\|\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i\|\\
\le&\textstyle\int_0^t\|B\|\,\|\mathbf{u}_{i+1}-H(w^{-1})\mathbf{u}_i\|e^{a(t-\tau)}d\tau\\
\le&\textstyle\int_0^t\|B\|\,[\|\gamma_1\|\,\|\mathbf{e}_i\|+\|\gamma_2\|\,\|\mathbf{e}_{i-1}\|+\cdots\\
+&\|\gamma_m\|\,\|\mathbf{e}_{i-m+1}\|]e^{a(t-\tau)}d\tau
\end{aligned}\tag{18}$$

for all $t\in[0,T]$, where $a=\|A\|_\infty$.

Substituting (18) into (16), and noticing $\eta_j=\|h_jI-\gamma_jD\|$, it is easy to derive

$$\begin{aligned}
&\|\mathbf{e}_{i+1}\|\\
\le&\eta_1\|\mathbf{e}_i\|+\eta_2\|\mathbf{e}_{i-1}\|+\cdots+\eta_m\|\mathbf{e}_{i-m+1}\|\\
+&\textstyle\rho_1\int_0^t\|\mathbf{e}_i\|\,e^{a(t-\tau)}d\tau+\rho_2\int_0^t\|\mathbf{e}_{i-1}\|\,e^{a(t-\tau)}d\tau\\
+&\textstyle\cdots+\rho_m\int_0^t\|\mathbf{e}_{i-m+1}\|\,e^{a(t-\tau)}d\tau
\end{aligned}\tag{19}$$

where $\rho_j\triangleq\|\gamma_jCB\|$, $\forall k=1,\cdots,m$.

Next taking λ-norm on both sides of (19), and applying *Property 1*, we have $\sup_{t\in[0,T]}e^{-\lambda t}\rho_j\int_0^t\|\mathbf{e}_j\|\,e^{a(t-\tau)}d\tau\le\frac{1-e^{(a-\lambda)T}}{\lambda-a}\rho_j\|\mathbf{e}_j\|_\lambda\triangleq\delta_j(\lambda)$, where $\delta_j(\lambda)$ is a function of λ and can be made arbitrarily small or far below η_i with a sufficiently large λ.

Consequently we can derive the following relation from (19)

$$\begin{aligned}
\|\mathbf{e}_{i+1}\|_\lambda\le&(\eta_1+\delta_1)\|\mathbf{e}_i\|_\lambda+(\eta_2+\delta_2)\|\mathbf{e}_{i-1}\|_\lambda\\
&+\cdots+(\eta_m+\delta_m)\|\mathbf{e}_{i-m+1}\|_\lambda.
\end{aligned}\tag{20}$$

Since δ_j are arbitrarily small with sufficiently large λ, the learning convergence of the error sequence $\mathbf{e}_i$ (20) is dominated by the polynomial

$$R(z)=z^m-\eta_1z^{m-1}-\cdots-\eta_m$$

which is asymptotically stable if all eigenvalues of $R(z)=0$ are within the unit circle. As $i\to\infty$, $\mathbf{e}_i\to 0$ implies $\mathbf{y}_i\to\mathbf{y}_{r,i}$. ■

Remark 4: The ILC law (12) is in essence a high-order ILC. The HOIM is achieved through incorporating it in between $\mathbf{u}_{i+1}$ and $\mathbf{u}_i$. Controller gains γ_j are chosen to guarantee the learning convergence. Clearly, to implement an mth order HOIM, we need mth order ILC.

Remark 5: A more conservative design is to choose

$$\sum_{i=1}^m\eta_i<1,\tag{21}$$

which ensures the learning convergence [10].

Remark 6: In general the identical initialization condition $\mathbf{x}_{i+1}(0)=\mathbf{x}_i(0)$ does not meet the initial condition of HOIM $\mathbf{x}_{i+1}(0)-H(w^{-1})\mathbf{x}_i(0)=0$ which is required by (17). The identical initialization condition implies the initial condition of HOIM if $\sum_i^m h_j=1$. Thus, for HOIM based ILC, the initial setting condition defined in *Assumption 1* is required.

IV. Extensions

In this sesction we consider two possible extensions of HOIM based ILC.

A. Extension to D-type ILC

Consider a class LTV dynamics system with the relative degree of 1

$$\left\{\begin{array}{l}\dot{\mathbf{x}}_i(t)=A(t)\mathbf{x}_i(t)+B(t)\mathbf{u}_i(t)\\ \mathbf{y}_i(t)=C(t)\mathbf{x}_i(t),\end{array}\right. \tag{22}$$

when CB is of full rank. It is well known in ILC that a D-type learning law is required

$$\begin{aligned}\mathbf{u}_{i+1}(t)&=h_1\mathbf{u}_i(t)+h_2\mathbf{u}_{i-1}(t)+\cdots+h_m\mathbf{u}_{i-m+1}(t)\\ &+\gamma_1\dot{\mathbf{e}}_i(t)+\gamma_2\dot{\mathbf{e}}_{i-1}(t)+\cdots+\gamma_m\dot{\mathbf{e}}_{i-m+1}(t).\end{aligned} \tag{23}$$

The learning convergence can be derived analogous to the P-type.

Theorem 2: For the plant (22) and reference trajectory (5), apply the D-type ILC with mth-order internal model (22). The learning convergence, $\|\mathbf{y}_{r,i}-\mathbf{y}_i\|\to 0$ as $i\to\infty$, is guaranteed, provided $\mathbf{y}_{r,i}(0)=0$, and the learning gains γ_j are chosen such that for $\eta_j\triangleq\|h_jI-\gamma_jCB\|$, $j=1,2,\cdots,m$, the polynomial $R(z)=z^m-\eta_1z^{m-1}-\cdots-\eta_m1$ is asymptotically stable.

Proof: Let us first prove the convergence of $\dot{\mathbf{e}}_i$. Differentiating $\mathbf{e}_i$, and substituting the state dynamics (22) and control law (23), we can derive

$$\begin{aligned}\dot{\mathbf{e}}_{i+1}&=\dot{\mathbf{y}}_{r,i+1}-\dot{\mathbf{y}}_{i+1}\\ &=H(w^{-1})\dot{\mathbf{y}}_{r,i}-H(w^{-1})\dot{\mathbf{y}}_i-\dot{\mathbf{y}}_{i+1}+H(w^{-1})\dot{\mathbf{y}}_i\\ &=H(w^{-1})\dot{\mathbf{e}}_i-C[A(\mathbf{x}_{i+1}\\ &-H(w^{-1})\mathbf{x}_i)+(\mathbf{u}_{i+1}-H(w^{-1})\mathbf{u}_i)]\\ &=(h_1I-\gamma_1CB)\dot{\mathbf{e}}_i+(h_2I-\gamma_2CB)\dot{\mathbf{e}}_{i-1}+\cdots\\ &+(h_mI-\gamma_mCB)\dot{\mathbf{e}}_{i-m+1}-CA\big[\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i\big].\end{aligned} \tag{24}$$

Taking norms on both sides of (24) we obtain

$$\begin{aligned}\|\dot{\mathbf{e}}_{i+1}\|\le&\|h_1I-\gamma_1CB\|\,\|\dot{\mathbf{e}}_i\|\\ &+\|h_2I-\gamma_2CB\|\,\|\dot{\mathbf{e}}_{i-1}\|\\ &+\cdots+\|h_mI-\gamma_mCB\|\,\|\dot{\mathbf{e}}_{i-m+1}\|\\ &+\|CA\|\,\|\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i\|.\end{aligned} \tag{25}$$

Comparing the relation (25) with the relation (16), we can see the analogy except for the substitution of quantity $\mathbf{e_i}$ with $\dot{\mathbf{e}}_i$. The convergence of sequence $\dot{\mathbf{e}}_i$ can be concluded straightforward.

Next we investigate the convergence condition for $\mathbf{e}_i$. Since $\dot{\mathbf{e}}_i(t)\to 0\ \forall t\in[0,\ T]$ as $i\to\infty$, $\mathbf{e}_i(t)=0$ if $\mathbf{e}_i(0)=0$. From the initial resetting condition, $\mathbf{y}_i(0)=C\mathbf{x}_i(0)=0$. Thus it is necessary to have $\mathbf{y}_{r,i}(0)=0$ in order to achieve perfect tracking. In the case the given reference trajectory $\mathbf{y}_{r,i}(0)\neq 0$, we can apply an appropriate filter to $\mathbf{y}_{r,i}(t)$ such that the filtered reference trajectory starts from zero, and after a short interval approaches the original trajectory. One such design was given by [11]. ■

B. Possible Extension with Nonlinear Factors

To date, all HOIM works focus on LTI or LTV. It would be meaningful to explore possible extension to plants with nonlinear factors. Consider the following plant

$$\left\{\begin{array}{l}\dot{\mathbf{x}}_i(t)=\mathbf{f}(\mathbf{x}_i(t))+B\mathbf{u}_i(t)\\ \mathbf{y}_i(t)=C(t)\mathbf{x}_i(t)+D(t)\mathbf{u}_i(t)\end{array}\right. \tag{26}$$

Assumption 2: The nonlinear function $\mathbf{f}(\mathbf{x}_i(t))$ satisfies the *Lipschitz* continuity condition

$$\|f(\mathbf{x}_{i+1})-f(H(w^{-1})\mathbf{x}_i)\|\le L_f\|\mathbf{x}_{k+1}-H(w^{-1})\mathbf{x}_i\| \tag{27}$$

where L_f is the Lipschitz constant.
Comparing the plants (1) and (26), the only difference is the replacement of $A\mathbf{x}_i$ with $\mathbf{f}(\mathbf{x}_i)$ in the state dynamics. Therefore, following the derivation procedure in Theorem 1, we obtain the relation (16).

To evaluate the state-dependent term $\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i$ in (16), using *Assumption 1* we have

$$\begin{aligned}&\|\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i\|\\ =\ &\|\textstyle\int_0^t\big[(\mathbf{f}(\mathbf{x}_{i+1})+B\mathbf{u}_{i+1})-H(w^{-1})\\ &\cdot(\mathbf{f}(\mathbf{x}_i,t)+B\mathbf{u}_i)\big]d\tau\|\\ =\ &\|\textstyle\int_0^t\Big\{\big[\mathbf{f}(\mathbf{x}_{i+1})-\mathbf{f}(H(w^{-1})\mathbf{x}_i)\big]\\ &+\big[\mathbf{f}(H(w^{-1})\mathbf{x}_i)-H(w^{-1})\mathbf{f}(\mathbf{x}_i)\big]\\ &+B(\mathbf{u}_{i+1}-H(w^{-1})\mathbf{u}_i)\Big\}d\tau\|\\ \le\ &\textstyle\int_0^t\|\mathbf{f}(\mathbf{x}_{i+1},t)-H(w^{-1})\mathbf{f}(\mathbf{x}_i,t)\|d\tau\\ &+\textstyle\int_0^t\|B\|\|\mathbf{u}_{i+1}-H(w^{-1})\mathbf{u}_i\|d\tau\\ &+\textstyle\int_0^t\|\mathbf{f}(H(w^{-1})\mathbf{x}_i)-H(w^{-1}\mathbf{f}(\mathbf{x}_i))\|d\tau\end{aligned} \tag{28}$$

Applying Bellman-Gronwall Lemma and *Lipschitz* continuity condition, we have

$$\begin{aligned}&\|\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i\|\\ \le&L_f\int_0^t\big[\|\mathbf{x}_{i+1}-H(w^{-1})\mathbf{x}_i\|d\tau\\ &+\|B\|\,\|\mathbf{u}_{i+1}-H(w^{-1})\mathbf{u}_i\|\big]d\tau+\epsilon\\ \le&\int_0^t\|B\|\|\mathbf{u}_{i+1}-H(w^{-1})\mathbf{u}_i\|e^{L_f(t-\tau)}d\tau+\epsilon\\ \le&\int_0^t\|B\|\big(\|\gamma_1\|\,\|\mathbf{e}_i\|+\|\gamma_2\|\,\|\mathbf{e}_{i-1}\|+\cdots\\ &+\|\gamma_m\|\,\|\mathbf{e}_{i-m+1}\|\big)e^{L_f(t-\tau)}d\tau+\epsilon,\end{aligned} \tag{29}$$

where $\epsilon\triangleq\int_0^T\|\mathbf{f}(H(w^{-1})\mathbf{x}_i)-H(w^{-1})\mathbf{f}(\mathbf{x}_i)\|dt$.

Comparing (29) with (18), we note a is replaced with L_f, and additional term δ. Thus, analogous to the derivations in (19) and (20) by applying the λ-norm, we obtain

$$\begin{aligned}\|\mathbf{e}_{i+1}\|_\lambda\le&(\eta_1+\delta_1)\|\mathbf{e}_i\|_\lambda+(\eta_2+\delta_2)\|\mathbf{e}_{i-1}\|_\lambda\\ &+\cdots+(\eta_m+\delta_m)\|\mathbf{e}_{i-m+1}\|_\lambda+\epsilon_\lambda,\end{aligned} \tag{30}$$

where

$$\delta_j(\lambda)=\frac{1-e^{(L_f-\lambda)T}}{\lambda-L_f}\rho_j,\ \ \varepsilon_\lambda=\sup_{t\in[0,T]}e^{-\lambda t}\varepsilon_i$$

and ρ_j is defined the same as in Theorem 1. When $\epsilon_\lambda=0$, the above relation (30) is exactly the same as (20), thus the gain design and learning convergence condition are the same as Theorem 1.

When $\epsilon_\lambda \neq 0$, there could be a steady state error as $i \to \infty$. The magnitude of the error depends on the choice of the control gains or the parameters η_j. For instance, by choosing gains γ_j to meet the convergence condition (21), then the steady state error is linearly dependent on ϵ_λ

$$\lim_{i\to\infty} \|\mathbf{e}_i\|_\lambda \leq \frac{\epsilon_\lambda}{1-\sum_{j=1}^{m}\eta_j}.$$

It is worth to explore the conditions such that ϵ_λ is either zero or sufficiently small. First, from the definition of ϵ_λ,a sufficient condition for $\epsilon_\lambda = 0$ is $\mathbf{f}(H(w^{-1})\mathbf{x}_i) - H(w^{-1})\mathbf{f}(\mathbf{x}_i) = 0$. By observation, the condition holds if $\mathbf{f}(\cdot)$ is a linear function, that is, $\mathbf{f}(\mathbf{x_i}) = \mathbf{x_i}$. Second $\mathbf{f}(\mathbf{x_i})$ could be globally nonlinear but its Taylor series expansion may give much smaller higher order terms around $\mathbf{x_i}$,for instance,$\mathbf{f}(\mathbf{x_i})$ can be expressed by

$$\mathbf{f}(\mathbf{x_i}) = \mathbf{x_i} + \delta_\mathbf{i}$$

where $||\delta_\mathbf{i}|| \ll ||\mathbf{x_i}||$. In such circumstance, the ILC will still be effective and produce small steady state errors.

Finally, the influence of ϵ_λ could be sufficiently small after applying the λ-norm. Let us evaluate ϵ_λ. According to $Property$ 1,

$$\begin{aligned}\varepsilon_\lambda = &\sup_{t\in[0,T]} e^{-\lambda t} L_f \int_0^T \|\mathbf{f}(H(w^{-1})\mathbf{x}_i) \\ &- H(w^{-1})\mathbf{f}(\mathbf{x}_i)\|dt \\ \leq &\frac{1-e^{-\lambda T}}{\lambda} L_f \|\mathbf{f}(H(w^{-1})\mathbf{x}_i) - H(w^{-1})\mathbf{f}(\mathbf{x}_i)\|_\lambda.\end{aligned}$$

On the other hand, note the GLC condition of $\mathbf{f}(\cdot)$,

$$\begin{aligned}&\|\mathbf{f}(H(w^{-1})\mathbf{x}_i) - H(w^{-1})\mathbf{f}(\mathbf{x}_i)\|_\lambda \\ &\leq \sum_{j=1}^{m} 2L_f|h_j|\|\mathbf{x}_{i-j+1}\|_\lambda.\end{aligned}$$

Let σ denote the upper bound of the sequence $\|\mathbf{x_j}\|_\lambda$ for $j = 1, \cdots, i$, then

$$\varepsilon_\lambda \leq \frac{1-e^{-\lambda T}}{\lambda} 2mL_f^2\sigma \sum_{j=1}^{m}|h_j| = O(\lambda^{-1})$$

which can be arbitrarily small by a sufficiently large λ. The boundedness of $\|\mathbf{x}_j\|_\lambda$ can be derived by the induction. The boundedness of the control inputs $\mathbf{u}_j$ for $j \leq i$ is obvious because $\mathbf{u}_j$ is the finite sum of previous control inputs and tracking errors. The boundedness of $\|\mathbf{x}_i\|_\lambda$ is the result of the boundedness of the input $\|\mathbf{u}_i\|$ and the GLC condition of the dynamics (26). The boundedness of the $\|\mathbf{u}_i\|$ and $\|\mathbf{e}_i\|$ ensures the boundedness of the new control input $\|\mathbf{u}_{i+1}\|$, in the sequel the boundedness of the states $\|\mathbf{x}_{i+1}\|_\lambda$.

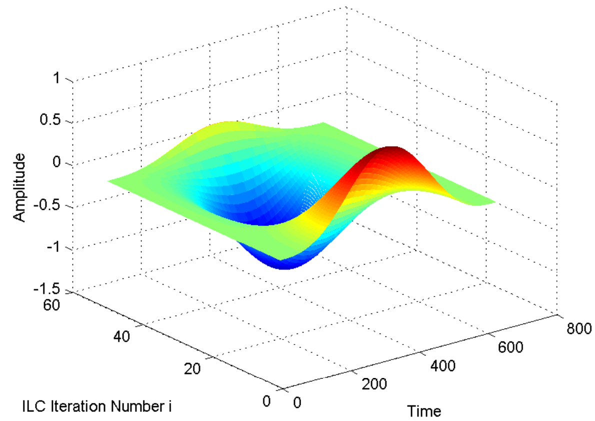

Fig. 1. Iteratively varying reference trajectory

V. Numerical Examples

A. LTV System

Consider the linear time-varying system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -1-10^{-3}t & -2-10^{-3}t \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) + u(t).\end{aligned} \quad (31)$$

The original reference trajectory $\mathbf{y}_r(t)$ and internal model are given by

$$\begin{aligned}\mathbf{y}_d(t) &= \sin^3 0.5t \quad 0 \leq t \leq 2\pi \\ H(w^{-1}) &= h_1 + h_2 w^{-1} = 2\cos(10T_s) - w^{-1}\end{aligned} \quad (32)$$

where $T_s = 0.01$ is a sample period , and $h_1 = 2\cos(10T_s)$ and $h_2 = -1$ are internal model coefficients.

Note that the reference trajectory with HOIM is in fact an oscillatory one because from

$$\mathbf{y}_{r,i+1}(t) = H(w^{-1})\mathbf{y}_{r,i}(t) \quad (33)$$

its characteristic equation is $w^2 - 2w\cos(10T_s) + 1$, which has two poles on the unit circle.Fig.1 demonstrates the reference trajectory in $3D$.

Case 1: We first use traditional iterative learning control . The learning algorithm is constructed by only one past control data,i.e.,$\mathbf{u}_{i+1}(t) = \mathbf{u}_i(t) + \gamma\dot{\mathbf{e}}(t)$.We choose the learning operator $\gamma = 0.9$ so that $\eta = ||1-\gamma CB|| = 0.1 < 1$.

Case 2: We now design a high order ILC controller

$$\mathbf{u}_{i+1}(t) = K_1\mathbf{u}_i(t) + K_2\mathbf{u}_{i-1}(t) + \gamma_1\mathbf{e}_i(t) + \gamma_2\mathbf{e}_{i-1}(t).$$

where K_1 and K_2 satisfied following conditions $K_1 + K_2 = I$ and $||K_1 - \gamma_1 D||_\infty + ||K_2 - \gamma_2 D||_\infty < 1$. Here,the control gains are chosen as $K_1 = 1.1, K_2 = -0.1, \gamma_1 = 1.4$ and $\gamma_2 = -0.15$.

Case 3: In this case, we use the HOIM control. The controller is

$$\mathbf{u}_{i+1}(t) = h_1\mathbf{u}_i(t) + h_2\mathbf{u}_{i-1}(t) + \gamma_1\mathbf{e}_i(t) + \gamma_2\mathbf{e}_{i-1}(t). \quad (34)$$

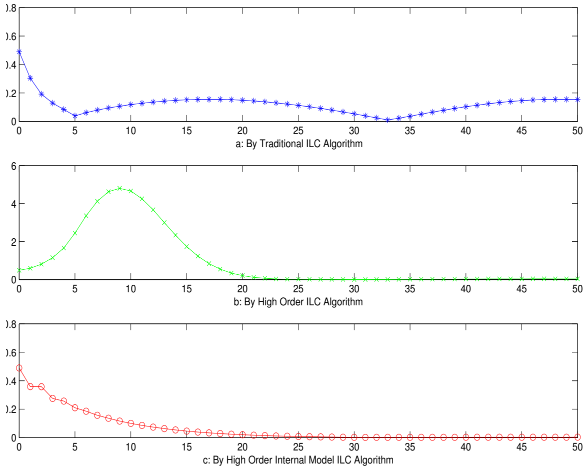

Fig. 2. The tracking error profile with various algorithm for LTV plant.

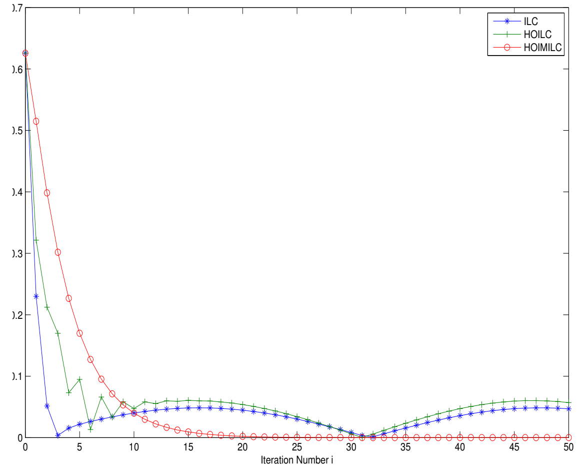

Fig. 3. The tracking error profile with various algorithm for nonlinear plant.

In close-loop system,the learning operators are designed as $\gamma_1 = 1.24$ and $\gamma_2 = -1.085$ so that $\eta_1 = ||h_1 I - \gamma_1 D|| = 0.75$ and $\eta_2 = ||h_2 I - \gamma_2 D|| = 0.085$, and hence,the polynomial $R(z) = z^2 - 0.75z - 0.085 = 0$ has two roots $z_{1,2} = 0.85, -0.1$ inside the unit circle.

The absolute maximum tracking error profile along the iteration domain is shown in Fig.2.

50th Trial	Case 1	Case 2	Case 3
Max Error	0.1552	0.0363	0.0031

We can see the fast learning convrgence rate, desptite the iteratively varying reference.

B. LTV with Nonlinear Factors and Disturbances

Consider the following nonlinear continuous-time system

$$\left\{ \begin{array}{rcl} \begin{bmatrix} \dot{\mathbf{x}}_{1i}(t) \\ \dot{\mathbf{x}}_{2i}(t) \end{bmatrix} & = & \begin{bmatrix} \sin(\mathbf{x}_{2i}(t)) & 1+\sin(\mathbf{x}_{1i}(t)) \\ -2-5t & -3-2t \end{bmatrix} \begin{bmatrix} \mathbf{x}_{1i}(t) \\ \mathbf{x}_{2i}(t) \end{bmatrix} \\ & & +\mathbf{u}_i(t) \\ \mathbf{y}_i(t) & = & \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 i(t) \\ \mathbf{x}_2 i(t) \end{bmatrix} + \mathbf{u}_i(t) \end{array} \right.$$

where time $t \in [0,1]$. The original reference tracking trajectory is

$$y_r(t) = 6t^5 - 15t^4 + 10t^3, \quad t \in [0,1].$$

Consider the same internal model $H(w^{-1})$ as (23). Chosen the same parameters as Section V-A. As shown in Fig.3, the HOIM-based ILC achieves satisfactory tracking performance in the presence of nonlinear factors .

VI. Conclusions

In this paper, a new ILC algorithm with HOIM is developed for a kind of iteratively varying reference trajectory. The problem with HOIM is first formulated in a simple and straightforward form. The learning convergence property and associated conditions are analyzed and made clearly. Both P-type and D-type ILC are discussed. The analytic and simulation results verify the learning convergence, that is the effectiveness of the proposed HOIM based ILC.

References

[1] S.S. Saab, W.G. Vogt and M.H. Mickle,"Learning control algorithm for tracking 'slowly' varying trajectories," *IEEE Transactions on Systems, Man, and Cybernetics - Part B*, Vol.27, no.4, pp.657-669, 1997.

[2] Y.Q. Chen and K.L. Moore, "Harnessing the nonrepetitiveness in iterative learning control," *Proceedings of 41st IEEE Conference and Decision and Control*, Las Vegas, pp.3350-3355, 2002.

[3] J.-X. Xu and J. Xu, "On iterative learning for different tracking tasks in the presence of time-varying uncertainties," *IEEE Transactions on Systems, Man, and Cybernetics - Part B*, Vol.34, no.1, pp.589-597, 2004.

[4] H.S. Ahn, K.L. Moore and Y.Q. Chen, Iterative Learning Control: Robustness and Monotonic Convergence for Interval Systems. Springer-Verlag, Communications and Control Engineering Series, ISBN: 978-1-84628-846-3, 2007.

[5] K.L. Moore, Y.Q. Chen and H.S. Ahn, "Iterative learning control: a tutorial and big picture View," *Proceedings of 45th IEEE Conference of Decision and Control*, pp.2352-2357, 2006.

[6] K.S. Lee, I.S. Chin and J.H. Lee, "Model predictive control technique combined with iterative learning for batch processes," *AIChE Journal*, Vol.45, no.10, pp.2175-2187, 2000.

[7] D.A. Bristow, M. Tharayil and A.G. Allyne, "A survey of iterative learning," *IEEE Control Systems Magazine*, Vol.26, no.3, pp.96-114, 2006.

[8] Z. Bien and I.M. Huh, "High-order Iterative Learning Control Algorithm," *IEE Proceedings Part-D: Control Theory and Applications*, Vol.136, no.3, pp.105-112, 1989.

[9] J.X. Xu and Y. Tan, Linear and nonlinear iterative learning control, Lecture notes in control and information sciences, Springer, 2003

[10] Y.Q. Chen,Z. Gong and C.Y. Wen,"Analysis of a High-order Iterative Learning Control Algorithm for Uncertain Nonlinear Systems with State Delay," *Automatica,* vol.34,no.3,pp.345-353,March,1998.

[11] M.X. Sun and D.W. Wang, "Iterative learning control with initial rectifying action," *Automatica*, Vol.38, pp.1177–1182, 2002.