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Stabilization of Linear Uncertain Delay Systems Irrespective of Bounds of Uncertain Variations

Tomoaki Hashimoto and Takashi Amemiya

Abstract— **This paper investigates the problem of designing a linear memoryless state feedback control to stabilize a class of linear uncertain systems with state delays. Each uncertain parameter and each delay under consideration may take arbitrarily large values. In such a situation, the locations of uncertain entries in the system matrices play an important role. It has been shown that it is a necessary and sufficient condition for the stabilization of time-varying or time-invariant uncertain systems without delays to have a particular geometric configuration called an ASC or a GASC, respectively. However, those results are inapplicable to systems that contain delays in the state variables. The objective of this paper is to show that if time-varying uncertain systems with time-varying delays or time-invariant uncertain systems with time-invariant delays have an ASC or a GASC, respectively, then the systems are stabilizable no matter how large the bounds of delays and uncertain parameters may be. However, we restrict our attentions to 3-dimensional systems for simplicity. The results shown here imply that the stabilizability conditions are not deteriorated by the existence of time delays.**

I. INTRODUCTION

This paper examines the stabilization problem of uncertain delay systems by means of linear memoryless state feedback control. It is useful to classify the existing results on the stabilization of uncertain systems into two categories. The first category includes several results which provide the stabilizability conditions depending on the bounds of uncertain parameters. The results in the second category provide the stabilizability conditions that are independent of the bounds of uncertain parameters but which depend on their locations. This paper specifically addresses the second category.

In the second category, the stabilization problem of linear uncertain systems without delays was studied by Wei [1] [2]. The stabilizability conditions have a particular geometric configuration with respect to the permissible locations of uncertain entries. Using the concept of antisymmetric stepwise configuration (ASC) [1] and generalized antisymmetric stepwise configuration (GASC) [2], Wei proved that a linear time-varying or time-invariant uncertain system is stabilizable independently of the given bounds of uncertain variations using linear state feedback control if and only if the system has an ASC or a GASC, respectively. The allowable uncertainty locations of GASC are more numerous than those of ASC, which means that the stabilizability conditions of linear time-varying uncertain systems are stricter than those of linear time-invariant uncertain systems. Wei derived the successful results on the stabilization problem of systems without delays, however, his both methods in [1] and [2] are inapplicable to systems that contain delays in the state variables.

On the one hand, based on the properties of an M-matrix, Amemiya [3] developed the conditions called a triangular configuration for the stabilization of linear time-varying uncertain systems with time-varying delays using linear memoryless state feedback control. The conditions obtained in [3] show a similar configuration to an ASC, but the allowable uncertainty locations are fewer than in an ASC by one step. In [4], the conditions of a triangular configuration [3] were further developed into the conditions of a triangular ASC. It was shown in [4] that if a linear time-varying uncertain delay system has a triangular ASC, then the system is stabilizable via linear memoryless state feedback control. Namely, the allowable uncertainty locations for the stabilization of the systems with delays were increased to those for the stabilization of the systems without delays. However, so far obtained results are valid in only the case where the uncertainty configurations are restricted to the triangular forms.

Our objective is to show that if a linear time-varying uncertain delay system has all admissible ASCs including not only triangular ASCs but also all other ASCs, then the system is stabilizable. In [4], the stabilization problem discussed here has been reduced to finding the proper manners of constructing the Vandermonde matrix as a variable transformation. To achieve our objective, we must find the proper variable transformation for each form of ASCs. The difficulty of solving this problem is the diversity of the classification of the proof. For simplicity, it is shown in this paper that if 3-dimensional linear time-varying uncertain delay systems have an arbitrary ASC, then the systems are stabilizable via linear control.

So far, we have discussed the stabilization problem of linear time-varying uncertain systems with time-varying delays. However, the stabilization problem of linear time-invariant uncertain systems with time-invariant delays still has not been addressed in the second category. For that reason, this paper also investigates the stabilization problem of linear time-invariant uncertain delay systems. The previous approach such as M-matrix method is useless for developing

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T. Hashimoto is with the Department of Electrical & Electronic Engineering, Division of Science and Technology, Graduate School, Shinshu University, 4-17-1, Wakazato, Nagano, 380-8553, Japan info@thashi.net or thashi@shinshu-u.ac.jp

T. Amemiya is with the Department of Business Administration and Information, Faculty of Business Administration and Information, Setsunan University, 17-8, Ikedanaka-machi, Neyagawa, Osaka, 572-8508, Japan amemiya@kjo.setsunan.ac.jp

the stabilizability conditions for this problem. Then, a novel approach is provided here, which is meaningful progress.

The objective in this paper is to show that a linear time-varying or time-invariant uncertain delay system with an arbitrary ASC or GASC is stabilizable, respectively, using linear memoryless state feedback control. That means the stabilizability conditions of linear uncertain delay systems are not degraded by the existence of delays that are allowed to take unlimited large values. In this paper, an ASC and a GASC under consideration are free from the restriction of triangular configuration, but the systems addressed here are restricted to 3-dimensional systems. However, the intrinsic nature of high dimensional structures is adequately reflected to that of 3-dimensional structures.

This paper is organized as follows. Some notations and terminology are given in Sec. II. The systems considered here are defined in Sec. III. In Sec. IV, some preliminary results are introduced to state the present problem. The main results are provided in Sec. V. Sec. VI and Sec. VII are devoted to the proof of the main results. In Sec. VIII, we discuss the difference of the control design process between LMI approach [5]-[7] and our approach, and compare the property of the controller designed by each approach. Finally, some concluding remarks are presented in Sec. IX.

II. NOTATIONS AND TERMINOLOGY

First, some notations and terminology used in the subsequent description are given. For $a, b \in \mathbb{R}^m$ or $A, B \in \mathbb{R}^{n\times m}$, every inequality between a and b or A and B such as $a > b$ or $A > B$ indicates that it is satisfied componentwise by a and b or A and B. If $A \in \mathbb{R}^{n\times m}$ satisfies $A \geq 0$, A is called a non-negative matrix. The transpose of $A \in \mathbb{R}^{n\times m}$ is denoted by A'. For $a = (a_1, ..., a_m)' \in \mathbb{R}^m$, $|a| \in \mathbb{R}^m$ is defined as $|a| = (|a_1|, .., |a_m|)'$. Also for $A = (a_{ij}) \in \mathbb{R}^{n\times m}$, $|A|$ denotes a matrix with $|a_{ij}|$ as its (i, j) entries. Let $diag\{\cdots\}$ denote a diagonal matrix. Let $[a, b]$, a,$b \in \mathbb{R}$ be an interval in $\mathbb{R}$. The set of all continuous or piecewise continuous functions with domain $[a, b]$ and range $\mathbb{R}^n$ is denoted, respectively, by $\mathcal{C}^n[a, b]$ or $\mathcal{D}^n[a, b]$. We denote it simply by $\mathcal{C}^n$ or $\mathcal{D}^n$ if the domain is $\mathbb{R}$.

The notation for a class of functions is introduced below. Let $\xi(\mu) \in \mathcal{C}^1$ and let $m \in \mathbb{R}$ be a constant. If $\xi(\mu)$ satisfies the conditions

$$\limsup_{|\mu|\to\infty} \left|\frac{\xi(\mu)}{\mu^m}\right| < \infty, \quad \limsup_{|\mu|\to\infty} \left|\frac{\xi(\mu)}{\mu^{m-a}}\right| = \infty \tag{1}$$

for any positive scalar $a \in \mathbb{R}$, then $\xi(\mu)$ is called a function of order m, and we denote this as follows:

$$Ord(\xi(\mu)) = m. \tag{2}$$

The set of all $\mathcal{C}^1$ functions of order m is denoted by $O(m)$,

$$O(m) = \left\{\xi(\mu)|\xi(\mu) \in \mathcal{C}^1,\ Ord(\xi(\mu)) = m\right\}. \tag{3}$$

A real square matrix all of whose off-diagonal entries are non-positive is called an M-matrix if it is non-singular and its inverse matrix is non-negative. The set of all M-matrices is denoted by $\mathcal{M}$.

III. SYSTEM DESCRIPTION

Let n be a fixed positive integer. The system considered here is given by a delay differential equation defined on $x \in \mathbb{R}^n$ for $t \in [t_0, \infty)$ as follows:

$$\begin{aligned}\dot{x}(t) \quad &= \quad A^0x(t) + \Delta A^1(t)x(t) + \sum_{i=1}^{r} \Delta A^{2i}(t)x(t - \tau_i(t)) \\ &\qquad +(b + \Delta b(t))u(t)\end{aligned} \tag{4}$$

with an initial curve $\phi \in \mathcal{D}^n[t_0 - \tau_0, t_0]$. Here, A^0, $\Delta A^1(t)$, $\Delta A^{2i}(t)$ $(i = 1, ..., r)$ are all real $n \times n$ matrices, where r is a fixed positive integer; also, A^0 is a known constant matrix. Furthermore, $\Delta A^1(t)$ and $\Delta A^{2i}(t)$ $(i = 1, ..., r)$ are uncertain coefficient matrices and may vary with $t \in [t_0, \infty)$. Other variables are as follows: $u(t) \in \mathbb{R}$ is a control variable, $b \in \mathbb{R}^n$ is a known constant vector, and $\Delta b(t) \in \mathbb{R}^n$ is an uncertain coefficient vector which may vary with $t \in [t_0, \infty)$. In addition, all $\tau_i(t)$ $(i = 1, ..., r)$ are piecewise continuous functions and are uniformly bounded, i.e., for a non-negative constant τ_0 they satisfy

$$0 \leq \tau_i(t) \leq \tau_0 \quad (i = 1, ..., r) \tag{5}$$

for all $t \geq t_0$. The upper bound τ_0 can be arbitrarily large and is not necessarily assumed to be known.

It is assumed that all entries of $\Delta A^1(t)$, $\Delta A^{2i}(t)$, and $\Delta b(t)$ are piecewise continuous functions and are uniformly bounded, i.e., for a non-negative constant matrices ΔA^{10}, $\Delta A^{2i0} \in \mathbb{R}^{n\times n}$, and for a non-negative constant vector $\Delta b^0 \in \mathbb{R}^n$, they satisfy

$$\left|\Delta A^1(t)\right| \leq \Delta A^{10}, \left|\Delta A^{2i}(t)\right| \leq \Delta A^{2i0}, \left|\Delta b(t)\right| \leq \Delta b^0 \tag{6}$$

for all $t \geq t_0$. The upper bound of each entry can independently take an arbitrarily large value, but each is assumed to be known.

Assumption 1: Because the system must be controllable, we assume that the pair (A^0, b) of the nominal system is a controllable pair and is in the controllable canonical form.

$$A^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{7}$$

Definition 1: We call system (4) defined above a linear time-varying uncertain delay (LTVUD) system. When all entries of ΔA^{2i0} $(i = 1, ..., r)$ are equal to zero, that is, the system contains no delays, we call this system a linear time-varying uncertain (LTVU) system. When all time-varying uncertain entries and time-varying delays of LTVUD system are time-invariant, we call this system a linear time-invariant uncertain delay (LTIUD) system. When all time-varying uncertain entries of LTVU system are time-invariant, we call this system a linear time-invariant uncertain (LTIU) system. We call the system satisfying Assumption 1 a standard system. Note that all LTVUD, LTVU, LTIUD, and LTIU systems belong to a class of standard systems.

Definition 2: An LTVUD system is said to be time-varying delay-independently stabilizable via linear control if there exists a linear memoryless state feedback control $u(t) = g'x(t)$, $g \in \mathbb{R}^n$ such that the equilibrium point $x = 0$ of the resulting closed-loop system is uniformly and asymptotically stable for all admissible uncertain delays and uncertain parameters.

Definition 3: An LTVU system is said to be quadratically stabilizable via linear control if there exists a linear state feedback control $u(t) = g'x(t)$, a positive definite matrix $Q \in \mathbb{R}^{n\times n}$, and a positive constant $\alpha \in \mathbb{R}$ such that the following condition holds: for all admissible uncertain parameters,

$$\begin{aligned} L(x) &= x'[A'(t)Q + QA(t)]x + 2x'Qb(t)g'x \\ &\leq -\alpha \|x\|^2, \end{aligned} \tag{8}$$

where $L(x)$ is the time derivative of the quadratic Lyapunov function $V(x) = x'Qx$ along the trajectories of the closed-loop system, and $A(t)$, $b(t)$ are defined as $A(t) = A^0 + \Delta A^1(t)$, $b(t) = b + \Delta b(t)$.

Definition 4: An LTIUD system is said to be time-invariant delay-independently stabilizable via linear control if there exists a linear state feedback control $u(t) = g'x(t)$ such that the characteristic equation of the closed-loop system,

$$\det\left(sI - A^0 - \Delta A^1 - \sum_{i=1}^{r} \Delta A^{2i} e^{-s\tau_i} - (b + \Delta b)g'\right) = 0 \tag{9}$$

is Hurwitz invariant, i.e., all the roots of (9) are in the strict left half of the complex plane for all admissible uncertain delays and uncertain parameters.

Definition 5: An LTIU system is said to be stabilizable via linear control if there exists a linear state feedback control $u(t) = g'x(t)$ such that the characteristic equation of the closed-loop system,

$$\det\left(sI - A^0 - \Delta A^1 - (b + \Delta b)g'\right) = 0 \tag{10}$$

is Hurwitz invariant, i.e., all the roots of (10) are in the strict left half of the complex plane for all admissible uncertain parameters.

Let $D \in \mathbb{R}^{n\times(n+1)}$ be defined as follows:

$$D = \begin{pmatrix} \Delta A^{30} & \Delta b^0 \end{pmatrix}, \tag{11}$$

where $\Delta A^{30} = \Delta A^{10} + \sum_{i=1}^{r} \Delta A^{2i0}$.

Definition 6: A standard system is said to have an anti-symmetric stepwise configuration (ASC) if D satisfies the following condition:

1) If $h \geq k+2$ and $d_{kh} \neq 0$, then $d_{uv} = 0$ for all $u \geq v$, $u \leq h-1$ and $v \leq k+1$.

Note that $d_{kh} (h \geq k+2)$ is the entry which exists in the upper part of the superdiagonal entries.

Definition 7: A standard system is said to have a generalized antisymmetric stepwise configuration (GASC) if D satisfies all the following conditions:

1) If $h \geq k+2$ and $d_{kh} \neq 0$, then $d_{uv} = 0$ for all $u \geq v$, $u \leq h-1$ and $v \leq k$.
2) $\det(D^r) = d_{12}d_{23}\cdots d_{nn+1}$, where D^r is the right submatrix of D defined as follows:

$$D^r := \begin{bmatrix} d_{12} & d_{13} & \cdots & d_{1n+1} \\ d_{22} & d_{23} & \cdots & d_{2n+1} \\ \vdots & \vdots & \vdots & \vdots \\ d_{n2} & d_{n3} & \cdots & d_{nn+1} \end{bmatrix}. \tag{12}$$

Definition 8: Let k be an integer satisfying $0 \leq k \leq n$. For this k, let $\Omega^l(k) = \{E = (e_{ij}) \in \mathbb{R}^{n\times(n+1)}\}$ be a set of matrices with the following properties:

(i) If $1 \leq j \leq k+1$, then $e_{ij} = 0$ for $j-1 \leq i \leq 2k-j$;
(ii) If $k+2 \leq j \leq n+1$, then $e_{ij} = 0$ for $2k-j+1 \leq i \leq j-1$.

Similarly, let $\Omega^u(k) = \{F = (f_{ij}) \in \mathbb{R}^{n\times(n+1)}\}$ be a set of matrices with the following properties:

(i) If $1 \leq j \leq k+1$, then $f_{ij} = 0$ for $j-1 \leq i \leq 2k-j+1$;
(ii) If $k+2 \leq j \leq n+1$, then $f_{ij} = 0$ for $2k-j+2 \leq i \leq j-1$.

A standard system is said to have a triangular antisymmetric stepwise configuration (TASC) if D satisfies the following condition: for fixed k,

$$D \subset \Omega^l(k) \ \text{ or } \ D \subset \Omega^u(k). \tag{13}$$

IV. PROBLEM STATEMENT

The stabilization problem of linear uncertain systems without delays was studied in [1], [2]. The stabilization problem of LTVUD systems was studied in [4].

Lemma 1 ([1]): An LTVU system is quadratically stabilizable via linear control if and only if the system has an ASC.

Lemma 2 ([2]): An LTIU system is stabilizable via linear control if and only if the system has a GASC.

Lemma 3 ([4]): An LTVUD system is time-varying delay-independently stabilizable via linear control if the system has a TASC.

The stabilization problem of LTIUD systems still has not been addressed. Our purpose is to show that if an LTVUD system has an ASC, then the system is time-varying delay-independently stabilizable via linear control and that if an LTIUD system has a GASC, then the system is time-invariant delay-independently stabilizable via linear control. To obtain our goal, we must find the proper manners of constructing the variable transformations for all possible ASCs or GASCs. The difficulty of solving this problem is the diversity of the classifications for each configuration. For that reason, we restrict our attention to 3-dimensional systems. Let $*$ denote the permissible location of an uncertain entry. For $D \in \mathbb{R}^{3\times 4}$, all possible ASCs are as follows:

$$(i)\begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \quad (ii)\begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(iii)\begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \quad (iv)\begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \end{pmatrix}$$

Moreover, for $D \in \mathbb{R}^{3\times 4}$, all possible GASCs are as follows: In addition to ASCs (i)-(iv),

$$(v)\begin{pmatrix} 0 & 0 & 0 & * \\ 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The objective of this paper is to show that if 3-dimensional LTVUD or LTIUD system has an ASC or a GASC, then the system is time-varying or time-invariant delay-independently stabilizable via linear control, respectively.

V. MAIN RESULTS

In this section, we state our main results.

Theorem 1: All LTVUD systems with ASCs (i)-(iv) are time-varying delay-independently stabilizable via linear control.

Proof: The proof is found in Sec. VI. ■

Theorem 2: All LTIUD systems with GASCs (i)-(v) are time-invariant delay-independently stabilizable via linear control.

Proof: The proof is found in Sec. VII. ■

We see that the number of permissible uncertainty locations for the stabilizable LTIUD systems is more than that for the stabilizable LTVUD systems. For 3-dimensional systems, it was shown that the stabilizability conditions of systems without delays are sufficient conditions for the stabilization of systems with delays. Therefore, we see that the stabilizability conditions of systems with time-varying or time-invariant delays are not degraded by the existence of time-varying or time-invariant delays, respectively. From this point of view, the main results are important. It can be considered that the intrinsic nature of high dimensional structures is adequately reflected to that of 3-dimensional structures.

VI. STABILIZATION OF LTVUD SYSTEMS

This section is devoted to the proof of Theorem 1. Because of Assumption 1, it is possible to choose $g \in \mathbb{R}^n$ so that all the eigenvalues of $(A^0 + bg')$ are real, negative and distinct. Let g be as such. In addition, let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be such eigenvalues of $(A^0 + bg')$. Let $T \in \mathbb{R}^{n\times n}$ be the Vandermonde matrix constructed using $\lambda_1, \lambda_2, \cdots, \lambda_n$ such that

$$T = \begin{pmatrix} 1 & 1 & . & . & . & 1 \\ \lambda_1 & \lambda_2 & . & . & . & \lambda_n \\ {\lambda_1}^2 & {\lambda_2}^2 & . & . & . & {\lambda_n}^2 \\ . & . & . & . & . & . \\ {\lambda_1}^{n-1} & {\lambda_2}^{n-1} & . & . & . & {\lambda_n}^{n-1} \end{pmatrix}. \tag{14}$$

This T is well known to be non-singular in view of the above assumption. Then, let $P \in \mathbb{R}^{n\times(n+1)}$ be defined as follows:

$$P = -\Lambda - \left|T^{-1}\right| \Delta A^{30} \left|T\right| - \left|T^{-1}\right| \Delta b^0 \left|g'\right| \left|T\right|, \tag{15}$$

where

$$\Lambda = T^{-1}(A^0 + bg')T = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}. \tag{16}$$

The following lemma has been shown in [4].

Lemma 4 ([4]): If there exists T in (14) which assures

$$P \in \mathcal{M}, \tag{17}$$

then an LTVUD system is time-varying delay-independently stabilizable via linear control.

Note that our problem has been reduced to finding T that enables P to satisfy condition (17). For LTVUD systems (i), (ii), and (iv), it has been shown in [4] that there exists T that assures $P \in \mathcal{M}$, because systems (i), (ii), and (iv) belong to Ω^u. In the subsequent discussion, we consider the possibility of choosing T that assures $P \in \mathcal{M}$ for system (iii).

Here, let μ be a positive number and let $\alpha_{i(i=1,2,3)}$ be all negative numbers that are different from one another. Let μ be much larger than all entries of ΔA^{30} and Δb^0. Let $\alpha_{i(i=1,2,3)}$ be used for distinguishing eigenvalues from one another.

For system (iii), the proper way of choosing $\lambda_{i(i=1,2,3)}$ are shown below.

$$\left\{ \begin{array}{l} \lambda_1 = \alpha_1 \mu^{-1} \in O(-1), \\ \lambda_2 = \alpha_2 \mu^{1} \in O(1), \\ \lambda_3 = \alpha_3 \mu^{2} \in O(2). \end{array} \right. \tag{18}$$

To complete the proof, it must be shown that if we choose $\lambda_{i(i=1,2,3)}$ as in (18), then T constructed by such $\lambda_{i(i=1,2,3)}$ assures $P \in \mathcal{M}$.

T and T^{-1} are given as follows:

$$T = \begin{pmatrix} O(0) & O(0) & O(0) \\ O(-1) & O(1) & O(2) \\ O(-2) & O(2) & O(4) \end{pmatrix}, \tag{19}$$

$$T^{-1} = \begin{pmatrix} O(0) & O(-1) & O(-3) \\ O(-2) & O(-1) & O(-3) \\ O(-4) & O(-3) & O(-4) \end{pmatrix}. \tag{20}$$

Considering $\Delta b^0 = 0$ for system (iii), we obtain

$$\begin{aligned} P &= -\Lambda - \left|T^{-1}\right| \left| \begin{pmatrix} 0 & 0 & O(0) \\ 0 & 0 & 0 \\ O(0) & O(0) & O(0) \end{pmatrix} \right| |T| \\ &= -\Lambda - \left| \begin{pmatrix} O(-2) & O(2) & O(4) \\ O(-3) & O(0) & O(2) \\ O(-4) & O(-2) & O(0) \end{pmatrix} \right|. \end{aligned} \tag{21}$$

For the evaluation of condition $P \in \mathcal{M}$, Proposition 3 in [4] is useful. Considering that Λ is a diagonal matrix whose entries belong to $O(-1)$, $O(1)$, and $O(2)$, we have

$$\begin{aligned} -1 &> -2, \\ 1 &> 0, \\ 2 &> 0, \\ -1 &> 4 - 2 + (-4) = -2, \\ 1 &> 2 - 2 + (-2) = -2, \\ -1 &> \max\{2, (4-2-2)\} - 1 \\ &\quad + \max\{-3, (2-4-2)\} = -2. \end{aligned} \tag{22}$$

Hence, we see from inequalities (22) that $P \in \mathcal{M}$.

Therefore, the proof of Theorem 1 is completed.

VII. STABILIZATION OF LTIUD SYSTEMS

This section is devoted to the proof of Theorem 2. First, we present a stabilizability criterion for LTIUD systems in a general setting. Next, we consider the possibility of choosing the feedback gain g such that the closed-loop system satisfies such a stabilizability criterion.

Let $\mathbb{C}_+$ be the closed right half of the complex plane. It is well known [8] that (9) is Hurwitz invariant if and only if

$$\det\left(I-(sI-A)^{-1}(\Delta A-\sum_{i=1}^{r}\Delta A^{2i}e^{-s\tau_i})\right)\neq 0 \quad (23)$$

for all $s\in\mathbb{C}_+$, where $A:=A^0+bg'$ and $\Delta A:=\Delta A^1+\Delta bg'$. Without loss of generality, (23) is rewritten as follows:

$$1+\frac{Y(s)}{X(s)}\neq 0, \quad (24)$$

where $X(s)=\det(sI-A)$. Note that the order of $X(s)$ with respect to s is higher than that of $Y(s)$. Because of Assumption 1, it is possible to choose $g\in\mathbb{R}^n$ so that all the roots of $X(s)$ are set in the strict left half plane. In such a situation, it is obvious from Nyquist stability criterion [9] that condition (23) holds if and only if the plot of $Y(j\omega)/X(j\omega)$ with ω varying from 0 to ∞ has no counter-clockwise encirclement of the point $(-1+j0)$, where $j^2=-1$. Then, we obtain the following lemma.

Lemma 5: An LTIUD system is time-invariant delay-independently stabilizable via linear control if and only if there exists $g\in\mathbb{R}^n$ such that all the roots of $X(s)$ are set in the strict left half plane and the following condition holds:

$$\left|\frac{Y(j\omega)}{X(j\omega)}\right|<1, \quad \text{for all } \omega\in\mathbb{R}. \quad (25)$$

Proof: Sufficiency: If condition (25) is satisfied, then the trajectory of nyquist plot is strictly inside the unit circle. Hence, there is no counter-clockwise encirclement of the point $(-1+j0)$. Clearly, it follows that an LTIUD system is time-invariant delay-independently stabilizable via linear control under the assumption that all the roots of $X(s)$ are set in the strict left half plane.

Necessity: If $X(s)$ has an unstable root, then the trajectory of nyquist plot must cross over the unit circle, because the nyquist plot must have at least one counter-clockwise encirclement of the point $(-1+j0)$ for the stability of closed-loop system. If there exists ω such that $|Y(j\omega)/X(j\omega)|>1$, then the nyquist plot cross over the unit circle. In these case, due to the term $e^{-j\omega\tau_i}$, the nyquist plot can have the arbitrary number of counter-clockwise encirclement of the point $(-1+j0)$. Hence, all the roots of $X(s)$ must be in the strict left half plane, and $|Y(j\omega)/X(j\omega)|\leq 1$ must be satisfied. Moreover, if there exists ω such that $|Y(j\omega)/X(j\omega)|=1$, then (23) has the root on the imaginary axis, that is, the system is unstable. Therefore, an LTIUD system is time-invariant delay-independently stabilizable via linear control only if all the roots of $X(s)$ are set in the strict left half plane and condition (25) is assured. ■

Using Lemma 5, we next consider the stabilizability of LTIUD systems (i)-(v). It is apparent from Theorem 1 that LTIUD systems (i)-(iv) are time-invariant delay-independently stabilizable via linear control. Then, it remains to show system (v) is also time-invariant delay-independently stabilizable via linear control. In fact, we can show that it is impossible to find the proper g that assures $P\in\mathcal{M}$ for system (v). In the following, we show that there exists the feedback gain g such that the closed-loop system satisfies condition (25).

For system (v), $X(s)$ and $Y(s)$ are given as follows:

$$X(s)=s^3-g_3s^2-g_2s-g_1, \quad (26)$$

$$Y(s)=-b_1g_1s^2+\{-s^2+(g_3+b_1g_1)s\}\times\left(a_{22}^1+\sum_{i=1}^{r}a_{22}^{2i}e^{-s\tau_i}\right). \quad (27)$$

Therein, b_1 denotes the first entry of Δb. a_{22}^1 and a_{22}^{2i} represent the $(2,2)$ entry of ΔA^1 and ΔA^{2i}, respectively. Let a_{22} be defined as follows:

$$\begin{aligned}&\left(a_{22}^1+\sum_{i=1}^{r}a_{22}^{2i}e^{-j\omega\tau_i}\right)\\ &\leq\left(|a_{22}^1|+\sum_{i=1}^{r}|a_{22}^{2i}||e^{-j\omega\tau_i}|\right)\\ &\leq\left(|a_{22}^1|+\sum_{i=1}^{r}|a_{22}^{2i}|\right):=a_{22}.\end{aligned} \quad (28)$$

Then, we have the followings:

$$\begin{aligned}|Y(j\omega)| &\leq \left|b_1g_1\omega^2+\{\omega^2+(g_3+b_1g_1)j\omega\}a_{22}\right|\\ &= \left|(b_1g_1+a_{22})\omega^2+(g_3+b_1g_1)a_{22}\omega j\right|,\end{aligned} \quad (29)$$

$$\begin{aligned}|X(j\omega)| &= \left|-j\omega^3+g_3\omega^2-g_2j\omega-g_1\right|\\ &= \left|(g_3\omega^2-g_1)-(\omega^2+g_2)\omega j\right|.\end{aligned} \quad (30)$$

From (29) and (30), it follows that

$$\begin{aligned}\left|\frac{Y(j\omega)}{X(j\omega)}\right|^2 &\leq \frac{(b_1g_1+a_{22})^2\omega^4+(g_3+g_1b_1)^2a_{22}^2\omega^2}{(g_3\omega^2-g_1)^2+(\omega^2+g_2)^2\omega^2}\\ &= \frac{(b_1g_1+a_{22})^2\omega^4+(g_3+g_1b_1)^2a_{22}^2\omega^2}{\omega^6+(g_3^2+2g_2)\omega^4+(g_2^2-2g_1g_3)\omega^2+g_1^2}\end{aligned} \quad (31)$$

Noting the coefficients of ω^4 and ω^2, it is apparent that if

$$(b_1g_1+a_{22})^2\leq(g_3^2+2g_2), \quad (32)$$
$$(g_3+g_1b_1)^2a_{22}^2\leq(g_2^2-2g_1g_3), \quad (33)$$

then (numerator) $<$ (denominator) for all $\omega\in\mathbb{R}$, i.e., condition (25) holds.

Now, we show the proper way of choosing the eigenvalues $\lambda_{i(i=1,2,3)}$ of (A^0+bg') as follows:

$$\begin{cases}\lambda_1=\alpha_1\mu^{-2}\in O(-2),\\ \lambda_i=\alpha_i\mu^1\in O(1) \quad (i=2,3).\end{cases} \quad (34)$$

From the relations between the roots and the coefficients of the characteristic equation $\det(A^0+bg')$, g' is found to have the following order structure:

$$g'=\begin{bmatrix} O(0) & O(2) & O(1)\end{bmatrix}. \quad (35)$$

Note that the right hand sides of (32) and (33) are

$$(g_3^2+2g_2)\in O(2) \quad\text{and}\quad (g_2^2-2g_1g_3)\in O(4), \quad (36)$$

respectively, and both terms have positive values. Also, note that the left hand sides of (32) and (33) are

$$(b_1g_1+a_{22})^2\in O(0) \quad\text{and}\quad (g_3+g_1b_1)^2a_{22}^2\in O(2), \quad (37)$$

respectively. Comparing (36) with (37), we see that conditions (32) and (33) hold for sufficiently large μ. It is apparent that sufficiently large μ such that condition (25) holds always exists however large the given upper bounds of uncertainties b_1 and a_{22} might be. Therefore, using Lemma 5, we can conclude that system (v) is time-invariant delay-independently stabilizable via linear control.

Consequently, the proof of Theorem 2 is completed.

VIII. DESIGN OF CONTROLLER

In this section, we discuss the property of both methods in the first category and in the second category for designing a stabilizing controller. For examples of the first category method and the second category method, we adopt the LMI approach and our approach presented here, respectively.

For a class of linear uncertain delay systems, the LMI condition derived from the Lyapunov-Krasovskii functional approach is a commonly used tool. In the following, we show the result obtained by LMI condition [5] for system (i)-(v) via example. For simplicity, let $r = 1$ in the subsequent discussion.

In most papers using the Lyapunov-Krasovskii approach, the uncertain parameters are assumed to satisfy the following condition:

$$\left[\begin{array}{ccc} \Delta A^1(t) & \Delta A^{21}(t) & \Delta b(t) \end{array} \right] = H\Delta(t) \left[\begin{array}{ccc} A^{10} & A^{20} & b^0 \end{array} \right], \tag{38}$$

where H, A^{10}, A^{20}, and b^0 are constant matrices such that $\Delta'(t)\Delta(t) \leq I$ is satisfied.

It has been shown in [5] that the systems can be stabilized if there exist symmetric positive-definite matrices X, Z, a matrix Y, and a positive scalar γ such that

$$\left[\begin{array}{ccc} \Sigma_{11} & 0 & \Sigma_{13} \\ 0 & -(1-\tau_d)Z & XA^{20'} \\ \Sigma_{13}' & A^{20}X & -\gamma I \end{array} \right] < 0, \tag{39}$$

where

$$\Sigma_{11} = A^0X + XA^{0'} + bY + Y'b' + Z + \gamma HH',$$
$$\Sigma_{13} = XA^{10'} + Y'b^{0'}.$$

The above inequality is exceptionally used to denote a negative definite matrix. τ_d is assumed to be a positive constant such that $\dot{\tau}(t) \leq \tau_d < 1$. Here, we assume that $\tau_d = 0.5$ for systems (i)-(iv) and $\tau_d = 0$ for system (v), although the method proposed here is free from the restriction of $\dot{\tau}(t)$.

We also assume that all the given upper bounds of uncertain entries $*$ in systems (i)-(v) take the same value, and we denote that value by ε. Using our method, we can design a stabilizing controller however large ε might be. However, LMI solver becomes infeasible, when the value of ε exceeds a certain value. Then, for comparing our method with LMI method, we first check the feasibility on LMI solver repeatedly with ε decreasing so that we identify the maximum value ε_{max}. Using the obtained solution in the case of ε_{max}, we have a stabilizing feedback gain by calculating $g' = YX^{-1}$. Next, for the given ε_{max}, we design a stabilizing feedback gain by our method. Both stabilizing feedback gains obtained using LMI method and our method are shown in the above Table. We see that the stabilizing feedback gain obtained using our method is much less than that obtained using LMI method.

LMI method	(i)	(ii)	(iii)	(iv)	(v)
ε_{max}	0.32	0.37	0.29	0.28	0.49
g_1	183.3	24.90	17.01	130.1	7.380
g_2	213.4	1351	34.62	3631	48.05
g_3	51.69	1194	18.50	1603	34.98
Our method	(i)	(ii)	(iii)	(iv)	(v)
g_1	156.7	0.0001	2.880	0.0002	0.3360
g_2	131.2	0.0033	24.16	0.0834	5.340
g_3	22.40	0.1143	19.33	7.511	4.567

IX. CONCLUSION

The stabilization problem of linear uncertain delay systems using linear memoryless state feedback control was investigated in this paper. Each uncertain parameter and each delay under consideration may take arbitrarily large values. It was shown that linear time-varying uncertain delay systems having all admissible ASCs are stabilizable, irrespective of the given bounds of uncertain entries and delays. For time-invariant systems, a novel stabilizability criterion was derived using Nyquist stability criterion. Moreover, it was shown that linear time-invariant uncertain delay systems having all admissible GASCs are stabilizable. However, the systems under consideration here are restricted to 3-dimensional linear uncertain delay systems. Nevertheless, we found a significant fact that the stabilizability conditions of systems with time-varying or time-invariant delays are not deteriorated by the existence of time-varying or time-invariant delays, respectively. It is highly desired that the obtained results for 3-dimensional systems are generalized to the ones for high-dimensional systems. This is a problem to be considered in the future.

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