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Hierarchical Least Squares Optimal Control of 2-D Systems

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Abstract— **An indefinite least squares approach to discrete-time linear quadratic control of two-dimensional systems of Roesser type is presented. Initial and final boundary states are constrained to lie in affine subspaces. By introducing a hierarchical decomposition technique, the problem is converted to a collection of similar smaller size problems. Successive use of the decomposition technique renders computational feasibility on substantially larger coordinate grids than without decomposition. Necessary and sufficient conditions for existence of a unique optimal solution are provided in terms of the smaller size problems.**

Keywords: 2-D systems, linear quadratic control, optimization, indefinite least squares.

I. Introduction

Non-recursive as well as recursive approaches to linear quadratic optimal control of discrete-time 2-D systems have been considered in the literature. In [1], [2] the problem is solved in closed-form by direct optimization with respect to the control sequence. See also [3] and references therein. Dynamic programming and Riccati equation approaches are studied in, for instance, [4], [5], [6].

Solving the linear quadratic control problems via optimization with respect to the complete control sequence, generally leads to system matrices whose dimensions grow proportionally with the number stages over which optimization takes place, that is, the product of horizontal and vertical stages. On large grids this may quickly result in numerical infeasibility. To alleviate the growth rate we introduce an alternative method that employs hierarchical decomposition of the control problem into smaller ones. By appropriate decomposition the growth rate can be reduced to at most linear in the sum of horizontal and vertical stages.

The 2-D system considered is of Roesser type. In contrast to most approaches, we allow for an indefinite cost criterion. Since indefinite linear optimal control problems do not always have solutions, conditions for existence of an optimal solution are provided in terms of the smaller size problems. Both initial and final boundary states are constrained to lie in affine subspaces.

The decomposition idea originates from [7], where it appears in a sign-definite cost form. It has since been extended to the indefinite cost case in [8].

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II. LQ-control of Roesser system with initial and final state constraints

Consider a discrete time-invariant Roesser type system [9]

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i,j)$$
$$e(i,j) = C \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Du(i,j) \tag{1}$$

where i is the horizontal coordinate, and j the vertical coordinate. Introduce the cost

$$\mathscr{J} = \sum_{i=0}^{\mathfrak{m}-1} \sum_{j=0}^{\mathfrak{n}-1} e(i,j)^T H_\phi e(i,j) + \sum_{j=0}^{\mathfrak{n}-1} (Z_h x^h(\mathfrak{m},j))^T H_\psi (Z_h x^h(\mathfrak{m},j))$$
$$+ \sum_{i=0}^{\mathfrak{m}-1} (Z_v x^v(i,\mathfrak{n})))^T H_\psi (Z_v x^v(i,\mathfrak{n})) \tag{2}$$

where Z_v and Z_h are given matrices, and H_ϕ and H_ψ are symmetric and nonsingular matrices, not necessarily sign-definite. Let

$$\xi_0^h = \begin{bmatrix} x^h(0,0)^T & \cdots & x^h(0,\mathfrak{n}-1)^T \end{bmatrix}^T$$
$$\xi_0^v = \begin{bmatrix} x^v(0,0)^T & \cdots & x^v(\mathfrak{m}-1,0)^T \end{bmatrix}^T$$
$$\xi_f^h = \begin{bmatrix} x^h(\mathfrak{m},0)^T & \cdots & x^h(\mathfrak{m},\mathfrak{n}-1)^T \end{bmatrix}^T$$
$$\xi_f^v = \begin{bmatrix} x^v(0,\mathfrak{n})^T & \cdots & x^v(\mathfrak{m}-1,\mathfrak{n})^T \end{bmatrix}^T$$

be vectors of initial horizontal and vertical states, respectively final horizontal and vertical states. Moreover, let

$$\Pi \begin{bmatrix} \xi_0^{hT} & \xi_0^{vT} \end{bmatrix}^T = y_a \tag{3}$$

$$\Upsilon \begin{bmatrix} \xi_f^{hT} & \xi_f^{vT} \end{bmatrix}^T = y_f \tag{4}$$

be a constraints on the initial and final states, respectively. Here Π and Υ are given matrices and y_a and y_f given vectors. The initial and final state constraints are said to be *feasible*, or briefly, that the boundary constraints are feasible, if for any initial state satisfying (3) there exists at least one control sequence $\{u(i,j)\}$ which brings the system from that initial state to a final state satisfying (4). We note that the initial states satisfy the constraints (3), if and only if,

$$\begin{bmatrix} \xi_0^{hT} & \xi_0^{vT} \end{bmatrix}^T = \Pi^\dagger y_a + W\vartheta \tag{5}$$

where $(\cdot)^\dagger$ denotes the Moore-Penrose inverse, W is a basis of $\ker \Pi$, and ϑ is a free parameter. When the initial states are sharply assigned both W and ϑ become vacuous, and we assume that Π then is chosen to be the identity matrix.

The state trajectory is given by

$$\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} = \sum_{\alpha=0}^{i} A^{i-\alpha,j} \begin{bmatrix} 0 \\ x^v(\alpha,0) \end{bmatrix} + \sum_{\beta=0}^{j} A^{i,j-\beta} \begin{bmatrix} x^h(0,\beta) \\ 0 \end{bmatrix}$$
$$+ \sum_{(0,0)\le(\alpha,\beta)\le(i,j)} M(i-\alpha,j-\beta)u(\alpha,\beta) \tag{6}$$

where the initial states are assumed to satisfy (3), and where the transition matrix is defined as

$$A^{0,0} = I\,, \quad A^{1,0} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \quad A^{0,1} = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

$$A^{i,j} = \begin{cases} A^{1,0}A^{i-1,j} + A^{0,1}A^{i,j-1}\,, & (i,j) > (0,0) \\ 0\,, & i < 0 \vee j < 0 \end{cases}$$

The matrices $M(i,j)$ are defined as

$$M(i,j) = A^{i-1,j}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} + A^{i,j-1}\begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{7}$$

Consider the following LQ optimal control problem.

Problem 1: Find, if there exist, initial states ξ_0^h, ξ_0^v satisfying the constraint (3), and a control sequence $\{u(i,j)\}$ such that, subject to the system dynamics (1), and the terminal state constraint (4), the indefinite cost (2) is minimized.

To put on more compact form the dependence of the output $e(i,j)$ on the initial state and the control, define the vector of all output values $e(i,j)$ listed in some given order, augmented with terms associated with the cost at the final states, say,

$$\begin{aligned} \bar{e} = \big[&e(0,0)^T, e(0,1)^T, e(0,2)^T, \cdots, e(\mathfrak{n}_1 - 1, \mathfrak{n}_2 - 1)^T, \\ &x^h(\mathfrak{m},0)^T Z_h^T, x^h(\mathfrak{m},1)^T Z_h^T, \cdots, x^h(\mathfrak{m},\mathfrak{n}-1)^T Z_h^T, \\ &x^v(0,\mathfrak{n})^T Z_v^T, x^v(1,\mathfrak{n})^T Z_v^T, \cdots, x^v(\mathfrak{m}-1,\mathfrak{n})^T Z_v^T\big]^T \end{aligned} \tag{8}$$

Similarly, define the vector

$$\bar{u} = \big[\vartheta^T, u(0,0)^T, u(1,0)^T, u(2,0)^T, \cdots, u(\mathfrak{n}_1 - 1, \mathfrak{n}_2 - 1)^T\big]^T$$

of all control values $u(i,j)$, together with ϑ, where ϑ is the free parameter describing the initial states satisfying the constraint (3). For simplicity we shall call $\bar{u}$ a *control sequence*.

$\check{A}$:

(9)

	x^h					x^v				
e	00	01	02	03	$\cdots$	00	10	20	30	$\cdots$
00	CA_h^{00}	0	0	0	$\cdots$	CA_v^{00}	0	0	0	$\cdots$
01	CA_h^{01}	CA_h^{00}	0	0	$\cdots$	CA_v^{01}	0	0	0	$\cdots$
02	CA_h^{02}	CA_h^{01}	CA_h^{00}	0	$\cdots$	CA_v^{02}	0	0	0	$\cdots$
03	CA_h^{03}	CA_h^{02}	CA_h^{01}	CA_h^{00}	$\cdots$	CA_v^{03}	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	
10	CA_h^{10}	0	0	0	$\cdots$	CA_v^{10}	CA_v^{00}	0	0	$\cdots$
11	CA_h^{11}	CA_h^{10}	0	0	$\cdots$	CA_v^{11}	CA_v^{01}	0	0	$\cdots$
12	CA_h^{12}	CA_h^{11}	CA_h^{10}	0	$\cdots$	CA_v^{12}	CA_v^{02}	0	0	$\cdots$
13	CA_h^{13}	CA_h^{12}	CA_h^{11}	CA_h^{10}	$\cdots$	CA_v^{13}	CA_v^{03}	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The (augmented) output sequence $\bar{e}$ may be expressed as

$$\bar{e} = \bar{A}y_a + \bar{B}\bar{u}\,, \quad \bar{A} = \check{A}\Pi^\dagger \tag{10}$$

where the entries of $\check{A}$ and $\bar{B}$ are obtained directly from (6), (4) and the output equation of (1). A partial listing of the entries is shown in (9) and (11), where A_h^{ij} and A_v^{ij} define a partition $[A_h^{ij}\ A_v^{ij}] := A^{ij}$ with the number of columns of A_h^{ij} and A_v^{ij} equal to $\dim x^h$ and $\dim x^v$, respectively. Columns and rows associated with the last two terms of (8) are not shown, but their construction follow the same pattern, except that C is replaced by $Z = \begin{bmatrix} Z_h & Z_v \end{bmatrix}$. The matrices K_{ij} in the first column of $\bar{B}$ represent the influence of the free parameter ϑ on the state $x(i,j)$, and are given by

$$K_{ij} = \sum_{\alpha=0}^{i} A^{i-\alpha,j}\begin{bmatrix} 0 \\ W_v \end{bmatrix} + \sum_{\beta=0}^{j} A^{i,j-\beta}\begin{bmatrix} W_h \\ 0 \end{bmatrix} \tag{12}$$

where $\begin{bmatrix} W_h^T & W_v^T \end{bmatrix}^T = W$, partitioned compatibly with $x^h(i,j)$ and $x^v(i,j)$.

The vector of final boundary state values may be expressed as

$$\begin{bmatrix} \xi_f^h \\ \xi_f^v \end{bmatrix} = \hat{A}\Pi^\dagger y_a + \hat{B}\bar{u} \tag{13}$$

where for zero control $\bar{u} = 0$, $\hat{A}$ is the matrix mapping an initial state $[\xi_0^{h^T}\ \xi_0^{v^T}]^T$ to the corresponding final state $[\xi_f^{h^T}\ \xi_f^{v^T}]^T$. The block row of $\hat{A}$ associated with component $\xi_f^h(j) = x^h(\mathfrak{m},j)$ is obtained directly from (6) by taking $i = \mathfrak{m}$, and collecting the $x^v(\alpha,0) \to x^h(\mathfrak{m},j)$ relating parts of the transition matrices appearing in the first sum of (6), and the $x^v(0,\beta) \to x^h(\mathfrak{m},j)$ relating parts of the transition matrices appearing in the second sum of (6). Similarly $\hat{B}$ is the matrix that for zero initial state maps an input sequence $\bar{u}$ to the corresponding final state $[\xi_f^{h^T}\ \xi_f^{v^T}]^T$. The block row of $\hat{B}$ associated with the component $\xi_f^h(j) = x^h(\mathfrak{m},j)$ is obtained directly from (6), by taking $i = \mathfrak{m}$, and collecting the upper, $u(\alpha,\beta) \to x^h(\mathfrak{m},j)$, parts of the M-matrices appearing in the third sum of (6), and the corresponding upper part of the initial state control matrix W of (5), ordered to match the ordering of input terms in $\bar{u}$. The block rows of $\hat{A}$ and $\hat{B}$ associated with the component $\xi_f^v(i) = x^v(i,\mathfrak{n})$ are obtained in the same way by collecting appropriate lower parts of the transition matrices, the M-matrices and the matrix W.

Remark 1: If the initial states are sharply assigned, W and ϑ are vacuous. The corresponding elements in the above matrices are then omitted. If the terminal cost in (2) is zero, that is, $Z_h = 0$ and $Z_v = 0$, the corresponding parts of $\bar{A}$, $\bar{B}$, $\bar{e}$ are vacuous, and are omitted.

The cost criterion (2) may now be written as the indefinite quadratic form

$$\mathscr{J} = \langle \bar{e}, \bar{e} \rangle := \bar{e}^T H_\sigma \bar{e} \tag{14}$$

where H_σ is the symmetric, nonsingular matrix

$$H_\sigma = \operatorname{diag}\left(H_\phi, \cdots, H_\phi, H_\psi, \cdots, H_\psi\right)$$

For a tall matrix M and a square, symmetric and invertible matrix H, define the following pseudo-inverse associated with the signature matrix H

$$M^\natural := (M^T H M)^{-1} M^T H \tag{15}$$

(See also Appendix.)

The following theorem solves Problem 1 by giving conditions for existence of a control sequence minimizing the indefinite quadratic form (14) under the constraint (4). It also gives explicit formulas for the solution. For simplicity only the case of a unique solution is considered.

Theorem 1: Assume that the boundary constraints are feasible. Let K be a basis matrix for the null space of $\Gamma := \Upsilon\hat{B}$. Subject to the system dynamics (1), and the terminal state constraint (4), a unique control sequence $\bar{u}$ minimizing the indefinite cost (2) then exists, if and only if,

$$K^T \bar{B}^T H_\sigma \bar{B} K > 0 \tag{16}$$

In this case the unique optimal control sequence and the optimal (augmented) output sequence are given by

$$\bar{u}^o = P y_a + Q y_f\,, \qquad \bar{e}^o = U y_a + V y_f \tag{17}$$

$\bar{B}$: (11)

e	ϑ	u: 00	10	20	30	$\cdots$	01	11	21	31	$\cdots$	02	12	22	32	$\cdots$
00	CK_{00}	D	0	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	0	$\cdots$
01	CK_{01}	CM_{01}	0	0	0	$\cdots$	D	0	0	0	$\cdots$	0	0	0	0	$\cdots$
02	CK_{02}	CM_{02}	0	0	0	$\cdots$	CM_{01}	0	0	0	$\cdots$	D	0	0	0	$\cdots$
03	CK_{03}	CM_{03}	0	0	0	$\cdots$	CM_{02}	0	0	0	$\cdots$	CM_{01}	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	
10	CK_{10}	CM_{10}	D	0	0	$\cdots$	0	0	0	0	$\cdots$	0	0	0	0	$\cdots$
11	CK_{11}	CM_{11}	CM_{01}	0	0	$\cdots$	CM_{10}	D	0	0	$\cdots$	0	0	0	0	$\cdots$
12	CK_{12}	CM_{12}	CM_{02}	0	0	$\cdots$	CM_{11}	CM_{01}	0	0	$\cdots$	CM_{10}	D	0	0	$\cdots$
13	CK_{13}	CM_{13}	CM_{03}	0	0	$\cdots$	CM_{12}	CM_{02}	0	0	$\cdots$	CM_{11}	CM_{01}	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	

where the matrices P, Q, U and V are defined as

$$
\begin{aligned}
P &= -(I - K(\bar{B}K)^{\natural}\bar{B})\Gamma^{\dagger}\Upsilon\hat{A}\Pi^{\dagger} - K(\bar{B}K)^{\natural}\bar{A} \\
Q &= (I - K(\bar{B}K)^{\natural}\bar{B})\Gamma^{\dagger} \\
U &= (I - \bar{B}K(\bar{B}K)^{\natural})(\bar{A} - \bar{B}\Gamma^{\dagger}\Upsilon\hat{A}\Pi^{\dagger}) \\
V &= (I - \bar{B}K(\bar{B}K)^{\natural})\bar{B}\Gamma^{\dagger}
\end{aligned}
$$

Proof: By (10), $\bar{e} = \bar{A}y_a + \bar{B}\bar{u}$. Recall from (13) that $\begin{bmatrix}\xi_f^h \\ \xi_f^v\end{bmatrix} = \hat{A}\Pi^{\dagger}y_a + \hat{B}\bar{u}$. Hence the terminal constraint may be written $\Upsilon\hat{B}\bar{u} = y_f - \Upsilon\hat{A}\Pi^{\dagger}y_a$. With $\beta = \bar{A}y_a$, $\Theta = -\bar{B}$, $\gamma = y_f - \Upsilon\hat{A}\Pi^{\dagger}y_a$ and $\mu = \bar{u}$, an application of Lemma 5 (i) in Appendix then yields

$$
\begin{aligned}
u^o = \mu_o &= (I - K(\bar{B}K)^{\natural}\bar{B})\Gamma^{\dagger}\gamma - K(\bar{B}K)^{\natural}\beta \\
&= (I - K(\bar{B}K)^{\natural}\bar{B})\Gamma^{\dagger}(y_f - \Upsilon\hat{A}\Pi^{\dagger}y_a) \\
-K(\bar{B}K)^{\natural} &= (-(I - K(\bar{B}K)^{\natural}\bar{B})\Gamma^{\dagger}\Upsilon\hat{A}\Pi^{\dagger} - K(\bar{B}K)^{\natural}\bar{A})y_a \\
&\quad + (I - K(\bar{B}K)^{\natural}\bar{B})\Gamma^{\dagger}y_f \\
&= Py_a + Qy_f
\end{aligned} \tag{18}
$$

where P and Q are as defined in the theorem. Similarly, by (36) in Lemma 5 (i) we have

$$
\begin{aligned}
\bar{e}^o = \beta - \Theta\mu_o &= (I - \bar{B}K(\bar{B}K)^{\natural})\beta + (I - \bar{B}K(\bar{B}K)^{\natural})\bar{B}\Gamma^{\dagger}\gamma \\
&= (I - \bar{B}K(\bar{B}K)^{\natural})\bar{A}y_a + (I - \bar{B}K(\bar{B}K)^{\natural})\bar{B}\Gamma^{\dagger}(y_f - \Upsilon\hat{A}\Pi^{\dagger}y_a) \\
&= (I - \bar{B}K(\bar{B}K)^{\natural})(\bar{A} - \bar{B}\Gamma^{\dagger}\Upsilon\hat{A}\Pi^{\dagger})y_a + (I - \bar{B}K(\bar{B}K)^{\natural})\bar{B}\Gamma^{\dagger}y_f \\
&= U_{\mathfrak{n}}y_a + V_{\mathfrak{n}}y_f
\end{aligned}
$$

where U and V are as defined in the theorem. □

III. Example 1

To illustrate Theorem 1, consider a problem with systems matrices $A_{11} = A_{12} = A_{22} = 1$, $A_{21} = -1$, $B_1 = B_2 = 1$, $C = \begin{bmatrix}1 & 0 \\ 0 & 1 \\ 0 & 0\end{bmatrix}$, $D = \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix}$, and an indefinite cost criterion

$$
\mathscr{J} = \sum_{i=0}^{2}\sum_{j=0}^{1} e(i,j)^T \operatorname{diag}(-1,1,1)e(i,j)
$$

The criterion implies a negative weight -1 on the horizontal state, and unit positive weights on the vertical state and the control. The initial and final states are fixed to $\xi_0^h = [1\ 1]^T$, $\xi_0^v = [1\ 1\ 1]^T$, and $\xi_f^h = [5\ 0]^T$, $\xi_f^v = -[0\ 2\ 4]^T$. We may therefore take $\Pi = I$ and $\Upsilon = I$. It may be verified that these constraints are feasible.

Setting up the coefficient matrices in (10) yields $\bar{A}$ and $\bar{B}$ of dimensions 18×5 and 18×6, respectively. The matrices $\hat{A}$ and $\hat{B}$ in (13) become

$$
\hat{A} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ -3 & 1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \\ 1 & -1 & -1 & -2 & 1 \end{bmatrix}, \quad
\hat{B} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & -1 & 1 & 0 \\ -1 & -2 & 1 & -1 & -1 & 1 \end{bmatrix}
$$

Using Theorem 1 to compute P and Q in the optimal solution (17) results in

$$
P = \begin{bmatrix} -0.4271 & -0.0199 & -0.8854 & -0.0625 & -0.0521 \\ -0.3125 & -0.0057 & -0.0625 & -0.8750 & -0.0625 \\ -0.2604 & 0.0256 & -0.0521 & -0.0625 & -0.8854 \\ 1.4271 & -0.4347 & -0.1146 & 0.0625 & 0.0521 \\ 0.8854 & -0.1960 & 0.1771 & -0.1875 & 0.0104 \\ 0.5208 & -0.0511 & 0.1042 & 0.1250 & -0.2292 \end{bmatrix} \tag{19}
$$

$$
Q = \begin{bmatrix} 0.2604 & -0.0795 & 0.0473 & -0.1222 & -0.0246 \\ 0.3125 & -0.0227 & -0.0341 & 0.1080 & -0.1023 \\ 0.4271 & 0.1023 & -0.0133 & 0.0142 & 0.1269 \\ -0.2604 & 0.2614 & 0.2254 & -0.2415 & -0.1572 \\ -0.0521 & 0.2159 & -0.0095 & 0.2244 & -0.1951 \\ 0.1458 & 0.2955 & 0.0265 & -0.0284 & 0.2462 \end{bmatrix} \tag{20}
$$

For the given initial states this yields the optimal control sequence $\{u(0,0),\ u(1,0),\ u(2,0),\ u(0,1),\ u(1,1),\ u(2,1)\}$={0.1979, 0.4375, 0.3646, 0.8021, 0.7604, 0.2708}. The optimal cost is $\mathscr{J} = -17.6146$.

IV. Restriction to subintervals

The number of rows of $\bar{A}$ and $\bar{B}$, as well as the number of columns of $\bar{B}$ grow proportionally with the product $\mathfrak{mn}$ of horizontal and vertical stages. To alleviate this growth we shall decompose the problem into smaller ones, consisting of an overlying problem that generates values of the optimal state trajectory at the boundaries of sufficiently small subintervals of the coordinate grid, and into subproblems generating restrictions of the optimal control to the subintervals. If the overlying problem fails to be computational tractable, a similar decomposition of this may attempted, and so forth. It may be shown that with appropriate choice of the sizes of the successive subdivisions, the growth rate of the number of rows and columns of the matrices $\bar{A}$ and $\bar{B}$ associated with the corresponding sub- and overlying problems, is no more than linear in $\mathfrak{m}+\mathfrak{n}$, instead of linear in $\mathfrak{mn}$, as without decomposition. Owing to the indefinite cost, also existence of an optimal solution must be ensured. To be convenient, the existence test should be consistently decomposed into existence tests for the smaller size problems that actually are to be solved.

The subproblems will be treated in this section, while the overlying problem is considered in Section V. For a corresponding decomposition and test of existence of solution for the one-dimensional case, see [8].

Assume that $\mathfrak{m} = \mathfrak{m}_2\mathfrak{m}_1$ and $\mathfrak{n} = \mathfrak{n}_2\mathfrak{n}_1$, and denote by

$$\mathfrak{D} = \{(i,j) \,:\, 0 \le i \le \mathfrak{m},\, 0 \le j \le \mathfrak{n}\} \tag{21}$$

the full grid. Divide $\mathfrak{D}$ into $\mathfrak{m}_2\mathfrak{n}_2$ rectangular blocks

$$\mathfrak{D}_{st} = \{(i,j) \,:\, s\mathfrak{m}_1 \le i \le (s+1)\mathfrak{m}_1,\, t\mathfrak{n}_1 \le j \le (t+1)\mathfrak{n}_1\}$$
$$0 \le s \le \mathfrak{m}_2 - 1,\;\; 0 \le t \le \mathfrak{n}_2 - 1 \tag{22}$$

of $\mathfrak{m}_1$ horizontal stages and $\mathfrak{n}_1$ vertical stages. By $\breve{\mathscr{D}}$ ($\breve{\mathscr{D}}_{st}$) we shall mean the array of grid points obtained by removing from $\mathscr{D}$ ($\mathscr{D}_{st}$) the rightmost column and uppermost row.

The horizontal (vertical) state values at the left/right (bottom/top) edges of block $\mathfrak{D}_{st}$ are collectively denoted

$$\tilde{x}^h(s,t) = \left[x^h(s\mathfrak{m}_1, t\mathfrak{n}_1)^T, \cdots, x^h(s\mathfrak{m}_1, (t+1)\mathfrak{n}_1 - 1)^T\right]^T \tag{23}$$

$$\tilde{x}^v(s,t) = \left[x^v(s\mathfrak{m}_1, t\mathfrak{n}_1)^T, \cdots, x^h((s+1)\mathfrak{m}_1 - 1, t\mathfrak{n}_1)^T\right]^T \tag{24}$$

Consider the following restriction of the original problem to any rectangular block $\mathfrak{D}_{st}$.

Problem 2 (Subproblem (s,t)): Given any feasible sharply assigned boundary states $\tilde{x}^h(s,t)$, $\tilde{x}^v(s,t)$, $\tilde{x}^h(s+1,t)$, $\tilde{x}^v(s,t+1)$ for block $\mathfrak{D}_{st}$. Find, if there exists, an input sequence $u(i,j)$, $(i,j) \in \breve{\mathfrak{D}}_{st}$, such that, subject to the system dynamics (1) and given boundary conditions, the following indefinite cost is minimized.

$$\mathscr{J}_{st} = \sum_{i=s\mathfrak{m}_1}^{(s+1)\mathfrak{m}_1-1} \;\sum_{j=t\mathfrak{n}_1}^{(t+1)\mathfrak{n}_1-1} e(i,j)^T H_\phi e(i,j) \tag{25}$$

Suppose that a specific subproblem (s,t) has feasible boundary conditions and admits a unique optimal solution. This solution may then be computed by application of Theorem 1. Since the initial and final states of the subproblem are assigned sharply, the boundary constraints for the subproblem may be set up with $\Pi = I$ and $\Upsilon = I$. In particular, no terminal cost term is needed in (25). The corresponding matrices P, Q, U, V and K in Theorem 1 do not depend on the boundary conditions. Hence every subproblem (s,t) admits a unique optimal solution for any feasible boundary conditions.

Let (i_{k_a}, j_{k_a}) be a point either at the left or at the upper boundary of $\mathfrak{D}$, and let (i_{k_b}, j_{k_b}) be a point either at the right or at the lower boundary of $\mathfrak{D}$. Consider a staircase shaped path $\{(i_k, j_\ell)\}$ of adjacent points in $\mathfrak{D}$, going from (i_{k_a}, j_{k_a}) to (i_{k_b}, j_{k_b}), with any two successive points (i_k, j_ℓ), $(i_{k+1}, j_{\ell+1})$ related either as $(i_{k+1}, j_{\ell+1}) = (i_k, j_\ell) + (1,0)$ or $(i_{k+1}, j_{\ell+1}) = (i_k, j_\ell) + (0,-1)$. For simplicity we shall call such a path a *cut*. The local state values $x^h(i,j)$ and $x^x h(i,j)$ at a cut yield the current state of the full system in the following sense. Define the *cut state* $x_{\mathscr{C}}$ of a cut $\mathscr{C}$ to be the collection of the horizontal state values $x^h(i,j)$ at all but the uppermost grid points of the vertical edges of the cut, and of the vertical state values $x^v(i,j)$ at all but the rightmost grid points of the horizontal edges of the cut. Then the cut state, together with the control values $u(i,j)$ on, and north-east of the cut, completely determine $x^h(i,j)$ and $x^v(i,j)$ on, and north-east of the cut. In particular, the cut-state $x_{\mathscr{C}'}$ of some other cut $\mathscr{C}'$ lying north-east of $\mathscr{C}$ (possibly partly coinciding with $\mathscr{C}$) is completely determined by $x_{\mathscr{C}}$ and the control values at the grid points in $\mathscr{C} \setminus \mathscr{C} \cap \mathscr{C}'$ and the grid points strictly between the two cuts.

We shall call the set of points between two non-intersecting cuts an *interval* in $\mathfrak{D}$. The lower- and leftmost (upper- and rightmost) cut will be called the *lower boundary* (*upper boundary*) of the interval. A single block $\mathfrak{D}_{st}$ gives rise to a class of intervals of $\mathfrak{D}$ for which the lower and upper boundaries coincide except at $\mathfrak{D}_{st}$ were the lower boundary follows the left and lower edges of $\mathfrak{D}_{st}$, and the upper boundary follows the upper and right edges of $\mathfrak{D}_{st}$. Since the only state transitions that can occur in such an interval must take place within $\mathfrak{D}_{st}$, we shall identify this class of intervals with $\mathfrak{D}_{st}$ itself.

The following principle of optimality holds.

Principle of optimality: For a system of type (1), consider the optimal control problem $\mathscr{P}$ of finding a control sequence minimizing some cost functional J over a grid $\mathfrak{D}$. Suppose that the sequence $\{u(i,j)\}$ uniquely solves $\mathscr{P}$. Let $\mathfrak{I}$ be an interval (as defined above) in $\mathfrak{D}$, and consider the control problem $\mathscr{P}_{\mathfrak{I}}$ obtained by restricting the full control problem $\mathscr{P}$ and its cost functional to the interval $\mathfrak{I}$, with the state values at the lower and upper boundaries of $\mathfrak{I}$ set equal to the corresponding optimal state values of $\mathscr{P}$. Then the restriction of $\{u(i,j)\}$ to the interval $\mathfrak{I}$ (more precisely $\breve{\mathfrak{I}}$) uniquely solves the restricted optimal control problem $\mathscr{P}_{\mathfrak{I}}$.

Let $x^{ho}(i,j)$, $x^{vo}(i,j)$ and $u^o(i,j)$ be the optimal trajectories of the Problem 1. Consider the restriction of Problem 1 to an interval $\mathfrak{D}_{st}$, with the boundary states of the interval $\mathfrak{D}_{st}$ set equal to the corresponding optimal horizontal and vertical state values $x^{ho}(i,j)$, $x^{vo}(i,j)$ at the boundary points. Then (25) is the restriction to $\mathfrak{D}_{st}$ of the original cost criterion. By the above optimality principle the restriction of $u^o(i,j)$ to $\breve{\mathfrak{D}}_{st}$ is then the unique control minimizing (25). Consequently, subproblem (s,t) with the given boundary conditions admits a unique optimal solution, which, as noted earlier, may be computed by application of Theorem 1. We summarize these observations in the following Lemma

Lemma 2: Suppose that the original problem, Problem 1, admits a unique optimal solution. Then the following holds for $s = 0,1,\ldots,\mathfrak{n}_2 - 1$, $t = 0,1,\ldots,\mathfrak{n}_2 - 1$.

(i) Subproblem (s,t), Problem 2, admits a unique optimal control for any feasible boundary conditions $\tilde{x}^h(s,t)$, $\tilde{x}^v(s,t)$, $\tilde{x}^h(s+1,t)$, $\tilde{x}^v(s,t+1)$.

(ii) Let $\{x^{ho}(i,j)\}$, $\{x^{vo}(i,j)\}$ and $\{u^o(i,j)\}$ be the unique optimal horizontal and vertical state and control sequences of Problem 1. Let the horizontal and vertical boundary states of subproblem (s,t) be given by the full problem optimal horizontal and vertical state values $x^{ho}(i,j)$, $x^{vo}(i,j)$ at the boundary points. Then the unique optimal state and control sequences of subproblem (s,t) agrees with the restrictions of $\{x^{ho}(i,j)\}$, $\{x^{vo}(i,j)\}$ to $\mathfrak{D}_{st}$ and $\{u^o(i,j)\}$ to $\breve{\mathfrak{D}}_{st}$, respectively.

V. Block System

The the horizontal and the vertical state values at the edges of a block $\mathfrak{D}_{st}$ are related as

$$\begin{bmatrix} \tilde{x}^h(s+1,t) \\ \tilde{x}^v(s,t+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}^h(s,t) \\ \tilde{x}^v(s,t) \end{bmatrix} + \begin{bmatrix} \tilde{\tilde{B}}_1 \\ \tilde{\tilde{B}}_2 \end{bmatrix} \tilde{u}(s,t) \tag{26}$$

where the matrices $\tilde{A}_{11}, \ldots, \tilde{A}_{22}, \tilde{\tilde{B}}_1, \tilde{\tilde{B}}_2$ are obtained directly from (6), and $\tilde{u}(s,t)$ is a vector collecting all the control values $u(i,j)$, $(i,j) \in \breve{\mathscr{D}}_{st}$. The "$B$"-matrix in (26) has $n\mathfrak{n}_1$ rows, where n is that number of rows of the "A"-matrix of the original Roesser system. However, the number of its columns is in general much larger. In order to keep matrix dimensions smallest possible we therefore replace the "B"-matrix by a basis matrix $\begin{bmatrix}\tilde{B}_1\\ \tilde{B}_2\end{bmatrix}$ of the image space of $\begin{bmatrix}\tilde{\tilde{B}}_1\\ \tilde{\tilde{B}}_2\end{bmatrix}$. The system (26) is thus replaced by

$$\begin{bmatrix}\tilde{x}^h(s+1,t)\\ \tilde{x}^v(s,t+1)\end{bmatrix} = \begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{bmatrix}\begin{bmatrix}\tilde{x}^h(s,t)\\ \tilde{x}^v(s,t)\end{bmatrix} + \begin{bmatrix}\tilde{B}_1\\ \tilde{B}_2\end{bmatrix}\tilde{\omega}(s,t) \qquad (27)$$

Systems (26) and (27) are equivalent in the sense that for any control sequence $\{\tilde{u}(s,t)\}$ of (26) there exists a unique control sequence $\{\tilde{\omega}(s,t)\}$ of (27) yielding the same state trajectory. Vice versa, for any control sequence $\{\tilde{\omega}(s,t)\}$ of (27) there exists at least one control sequence $\{\tilde{u}(s,t)\}$ of (26) yielding the same state trajectory.

Application of Theorem 1 to subproblem (s,t), with boundary states given by $\tilde{x}^h(s,t)$, $\tilde{x}^v(s,t)$, $\tilde{x}^h(s+1,t)$ and $\tilde{x}^v(s,t+1)$, yields a corresponding output vector

$$\tilde{e}(s,t) = \tilde{U}\begin{bmatrix}\tilde{x}^h(s,t)\\ \tilde{x}^v(s,t)\end{bmatrix} + \tilde{V}\begin{bmatrix}\tilde{x}^h(s+1,t)\\ \tilde{x}^v(s,t+1)\end{bmatrix} \qquad (28)$$

of all output values generated across the block $\mathfrak{D}_{st}$. By (27) we may in fact write

$$\tilde{e}(s,t) = \tilde{C}\begin{bmatrix}\tilde{x}^h(s,t)\\ \tilde{x}^v(s,t)\end{bmatrix} + \tilde{D}\tilde{\omega}(s,t), \quad \text{where} \qquad (29)$$

$$\tilde{C} = \tilde{U} + \tilde{V}\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{bmatrix}, \quad \tilde{D} = \tilde{V}\begin{bmatrix}\tilde{B}_1\\ \tilde{B}_2\end{bmatrix}$$

Let the initial and final state constraints be as in (3) and (4). We are interested in minimizing the cost

$$\tilde{\mathscr{J}} = \sum_{s=0}^{\mathfrak{m}_2-1}\sum_{t=0}^{\mathfrak{n}_2-1}\tilde{e}(s,t)^T\tilde{H}_\varsigma\tilde{e}(s,t) + \sum_{t=0}^{\mathfrak{n}_2-1}(\tilde{Z}^h\tilde{x}(\mathfrak{m}_2,t))^T\cdot$$
$$\cdot\tilde{H}_\psi\tilde{Z}^h\tilde{x}(\mathfrak{m}_2,t) + \sum_{s=0}^{\mathfrak{m}_2-1}(\tilde{Z}^v\tilde{x}(s,\mathfrak{n}_2))^T\tilde{H}_\psi\tilde{Z}^v\tilde{x}(s,\mathfrak{n}_2) \qquad (30)$$

where the weight on the running cost term is the symmetric, invertible matrix $H_\varsigma = \operatorname{diag}\left(H_\phi, \cdots, H_\phi\right)$, with $\mathfrak{m}_1\mathfrak{n}_1$ number of H_ϕ terms.

The row dimension of the matrices $\tilde{C}$ and $\tilde{D}$ will in general be much larger than their column dimensions. They can, however, be replaced by matrices $\hat{C}$, $\hat{D}$ obtained by a factorization

$$\begin{bmatrix}\hat{C} & \hat{D}\end{bmatrix}^T\hat{H}_\zeta\begin{bmatrix}\hat{C} & \hat{D}\end{bmatrix} = \begin{bmatrix}\tilde{C} & \tilde{D}\end{bmatrix}^T H_\zeta\begin{bmatrix}\tilde{C} & \tilde{D}\end{bmatrix} \qquad (31)$$

After such replacement the row dimension of $[\tilde{C}\ \tilde{D}]$ is at most equal to its column dimension, which in turn is less or equal to $2(\mathfrak{n}_1 \dim x^h + \mathfrak{m}_1 \dim x^v)$. By approriate choice of the size of the blocks (i.e of $\mathfrak{m}_1$ and $\mathfrak{n}_1$) the latter expression in turn becomes much smaller than the row dimensions of $\tilde{C}$ and $\tilde{D}$. Note that the cost (30) remains unchanged under this size reduction. In the sequel we shall always assume that this size reduction of the output has been carried out.

The block system of equations (27), (29) is similar to the original system (1), but describes only transitions from edges to edges of the blocks $\mathfrak{D}_{st}$, in terms of the condensed control sequence $\{\tilde{\omega}(s,t)\}$.

The optimal control problem of minimizing, subject to (27), the cost criterion (30) under the initial and final state constraints (3) and (4), may be formulated as a new problem.

Problem 3 (Overlying problem): Find, if such exists, an input sequence $\tilde{\omega}(s,t)$, $s = 0,1,\ldots,\mathfrak{m}_2-1$, $t = 0,1,\ldots,\mathfrak{n}_2-1$, such that, subject to the system dynamics (27) and the boundary constraints (3) and (4), the indefinite cost criterion (30) is minimized.

The following lemma relates this overlying problem to the original problem.

Lemma 3: Suppose that Problem 1 admits a unique minimizing control sequence $\{u^o(i,j)\}$. Then Problem 3 also admits a unique minimizing sequence $\{\tilde{\omega}(s,t)\}$, and min $\tilde{\mathscr{J}}$ = min $\mathscr{J}$, where min $\tilde{\mathscr{J}}$ is the minimal value of the cost criterion of Problem 3, and min $\mathscr{J}$ is the minimal value of the cost criterion of Problem 1.

Proof: Cf. [8]. □

Verification that Problem 1 admits an optimal solution may be done directly by testing positive definiteness of the matrix (16). However, for large coordinate grids this may be computationally infeasible. We shall therefore derive an existence test that is delegated to the computationally less demanding subproblems and overlying problem. The result is given by the following theorem, which may be regarded a converse of Lemma 2 combined with Lemma 3.

Theorem 4: Suppose that the overlying problem, Problem 3 and any of the subproblems, Problem 2, admit unique optimal solutions. Then the original, full size problem, Problem 1, also has a unique optimal solution.

Proof: Cf. [8]. □

Irrespectively of any prior knowledge of existence of a solution to the original problem, the subproblem and the overlying problem may be formulated. Based on Lemma 2, Lemma 3, Theorem 4, existence of a solution to the original problem can then be checked on the computationally more feasible sub- and overlying problems. Similarly, the solution of the original problem can be obtained from the subproblem and the overlying problem. We summarize this in the following procedure.

DECOMPOSITION PROCEDURE:

STEP 1 Set up a subproblem, Problem 2, and test (eg. by (16)) whether this (and thereby any of the subproblems) admits a unique solution for any feasible boundary conditions. If so, proceed to Step 2; otherwise stop, since then no unique solution to the original problem exists either [Lemma 2].

STEP 2 For the subproblem set up, apply Theorem 1 to compute the corresponding matrices P, Q, U, V mapping the boundary conditions to its optimal control and optimal output. Note that the boundary conditions need not be specified at this stage.

STEP 3 Set up the block system (27), (29) associated with the overlying problem, Problem 3, using the matrices U and V, and system matrices of the original system. Test whether the overlying problem admits a unique solution (eg. by (16)). If so, proceed to Step 4; otherwise stop, since no unique solution to the original problem exists either [Lemma 3].

STEP 4 By Theorem 4, Problem 1 now has a unique optimal solution. Use (31) to reduce the row size of the output matrices of the overlying problem. Then apply Theorem 1, to the row size reduced overlying problem to obtain its unique optimal state trajectory $\{\tilde{x}^{ho}(s,t)\}$, $\{\tilde{x}^{vo}(s,t)\}$. By Lemma 3 and its proof, for each block (s,t), $\tilde{x}^{ho}(s,t)$, $\tilde{x}^{vo}(s,t)$, $\tilde{x}^{ho}(s+1,t)$, $\tilde{x}^{vo}(s,t+1)$ then coincide with the values at boundaries of block $\mathscr{D}_{st}$ of the optimal state trajectory of the original problem.

STEP 5 For subproblem (s,t), Problem 2, set the boundary state values to $\{\bar{x}^{ho}(s,t)\}$, $\{\bar{x}^{vo}(s,t)\}$, $\{\bar{x}^{ho}(s+1,t)\}$, $\{\bar{x}^{vo}(s,t+1)\}$, respectively. These are clearly feasible boundary conditions. By Theorem 1 the optimal control of subproblem (s,t) is then given by

$$u^o(s,t) = P\begin{bmatrix}\bar{x}^{ho}(s,t)\\ \bar{x}^{vo}(s,t)\end{bmatrix} + Q\begin{bmatrix}\bar{x}^{ho}(s+1,t)\\ \bar{x}^{vo}(s,t+1)\end{bmatrix} \quad (32)$$

Compute (32) for each subproblem.

STEP 6 Obtain the optimal control of the original problem by collecting the control values of $u^o(s,t)$ for $(s,t) = (0,0),(1,0),(2,0)\ldots,(\mathfrak{m}_2-1,\mathfrak{n}_2-1)$

If the overlying problem is too large to be handle numerically it may be decomposed further into a new overlying problem and subproblems. It can be shown that by appropriate application of such successive decompositions the growth rate of the number of rows and columns of all involved matrices $\bar{B}$ and $\bar{A}$ will be at most linear in $\mathfrak{m}+\mathfrak{n}$, the sum of number of horizontal and vertical stages of the problem.

VI. EXAMPLE 2

Consider the example of Section III, but now with horizons $\mathfrak{m}=\mathfrak{n}=6$, and initial and final states $\xi_0^h = \xi_0^v = [1\ 1\ 1\ 1\ 1\ 1]^T$, and $\xi_f^h = [5\ 0\ 5\ 0\ 5\ 0]^T$, $\xi_f^v = -[0\ 2\ 4\ 0\ 2\ 4]^T$, respectively. Divide the grid into $(\mathfrak{m}_2 = 2)\cdot(\mathfrak{n}_2 = 3)$ blocks of size $(\mathfrak{m}_1 = 3)\times(\mathfrak{n}_1 = 2)$, with each block defining the horizon of the corresponding subproblem. Except for boundary state values the subproblems are identical to the problem of Section III. The matrices P and Q in the solution formula (32) for subproblems, are thus given by (19)-(20). The boundary states of the subproblems are obtained by setting up and solving the overlying problem on the $(\mathfrak{m}_2 = 2)\times(\mathfrak{n}_2 = 3)$ grid. The dimensions of the coefficient matrices $\bar{A}$ and $\bar{B}$ are for the overlying problem 54×12 and 54×24 respectively, and for the subproblems 18×5 and 18×6, respectively. Since the corresponding dimension for a direct solution of the full problem would be 18×12 and 108×36, respectively, this represents a substantial reduction of matrix dimensions in the solution formulas of Theorem 1.

As an example of size reduction at a larger grid we take $\mathfrak{m}=\mathfrak{n}=40$, $\mathfrak{m}_1=\mathfrak{n}_1=10$, $\mathfrak{m}_2=\mathfrak{n}_2=4$. To easily guarantee existence of a solution we change the signature matrix of the cost criterion from $\mathrm{diag}(-1,1,1)$ to $\mathrm{diag}(1,1,1)$. Table I shows the dimensions of matrices $\bar{A}$ and $\bar{B}$. Clearly, the size

	size $\bar{A}$	size $\bar{B}$	comp. time $\bar{B}K^\natural$
Full problem	4800×80	4800×1600	463.14 s
Subproblem	300×20	300×100	0.36 s
Overlying problem	624×80	624×304	0.55 s

TABLE I

reduction is even more advantageous in this case. To indicate reduction in computational time, the last column shows the time required to compute the pseudoinverse $\bar{B}K^\natural$ with Matlab on a 2.53 GHz Pentium 4, directly from (15).

VII. CONCLUSION

A pseudo-inverse approach to discrete-time finite horizon quadratic optimal control of a linear two-dimensional systems under an indefinite cost criterion has been proposed. To enhance computational tractability, a hierarchical decomposition technique is used to split the original problem into an equivalent set of reduced size problems defined on subgrids of the full coordinate grid. Owing to the indefinite cost an optimal solution does, however, not always exist. Precise conditions for existence of a unique optimal solution are derived, and the actual existence test, as well as computation of the optimal solution, are carried out on the reduced size problems.

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APPENDIX

Lemma 5: Consider an indefinite quadratic form in the vector μ

$$\langle \beta-\Theta\mu, \beta-\Theta\mu\rangle = (\beta-\Theta\mu)^T H(\beta-\Theta\mu) \quad (33)$$

where Θ is a given matrix, β a given vector, and H the symmetric, invertible matrix defining the indefinite inner product $\langle\cdot,\cdot\rangle$. Introduce a constraint

$$\Gamma\mu = \gamma \quad (34)$$

on μ, where Γ is a given matrix, and γ a given vector in $\mathrm{im}\,\Gamma$. Let K be basis matrix for $\ker\Gamma$.

(i) Subject to constraint (34) the quadratic form (33) has a unique stationary point, if and only if, $K^T\Theta^T H\Theta K$ is non-singular. In this case the unique stationary point is given by

$$\mu_o = (I-K(\Theta K)^\natural\Theta)\Gamma^\dagger\gamma + K(\Theta K)^\natural\beta \quad (35)$$

The value of $\beta-\Theta\mu$ at the stationary point μ_o is

$$(I-\Theta K(\Theta K)^\natural)\beta - (I-\Theta K(\Theta K)^\natural)\Theta\Gamma^\dagger\gamma \quad (36)$$

(ii) The unique stationary point in (i), of (33) restricted to (34), is a minimum point, if and only if, $K^T\Theta^T H\Theta K > 0$.

Proof: Cf. [8]. □

REFERENCES

[1] T. Kaczorek and J. Klamka, "Minimum energy control of 2-d linear systems with variable coefficients," *Int. J. Control*, vol. 44, pp. 645–650, 1986.

[2] T. Kaczorek, "The linear-quadratic optimal regulator for singular 2-d systems with variable coefficients," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 565–566, 1989.

[3] ——, *Positive 1D and 2D systems*. Springer, 2002.

[4] W. Marszalek and J. Sadecki, "Dynamic programming for 2-d discrete linear systems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 181–184, 1989.

[5] M. Rostan and E. Lee, "Optimal control (quadratic performance) for two-dimensional sytems," in *Proc. IEEE Conf. Decision Contr., Tampa, Florida*, 1989.

[6] M. Jagganathan and V. Syrmos, "A linear-quadratic optimal regulator for two-dimensional systems," in *Proc. 35th Conference on Decision and Control*, 1996.

[7] G. Marro, D. Prattichizzo, and E. Zattoni, "A nested computational approach to the discrete-time finite-horizon LQ control problem," *Siam J. Control Optim*, vol. 42, no. 3, pp. 1002–1012, 2003.

[8] P.-O. Nyman, "Generalized pseudo-inverse based nested discrete-time lq-control with indefinite cost," in *Proc. 44th IEEE Conference on Decision and Control and European Control Conference 2005*, 12-15 Dec. 2005.

[9] R. P. Roesser, "A discrete state-space model for linear image processing," *IEEE Trans. Automat. Contr*, vol. 20, no. 1, pp. 1–10, 1975.