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Synchronization Preservation Under Linear Polynomial Modifications

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***Abstract*— Robustness and preservation of stability and synchronization in the presence of structural changes is an important issue in the study of chaotic dynamical systems. In this work we present a methodology to establish conditions for preservation of stability in dynamical system in terms of linear matrix polynomial evaluation. The idea is to construct a group of dynamical transformations under which stability is retained along the stable, unstable and synchronization manifolds using simultaneous Schur decompositions.**

I. INTRODUCTION

The study of stability preservation makes sense when it comes to chaos control problems. As a matter of fact, the generalized synchronization can even be derived for different systems by finding a diffeomorphic transformation such that the states of the slave system can be written as a function of the states of the master dynamics (see [1] and references therein). This result can be seen as a timely contribution; however, in accordance to the goal of keeping intact the stability under the transformation, a new question arises: how can stability be preserved under transformations suffered by a dynamical system? An answer to this question might allow us to ensure synchronization in strictly different systems, in the sense that stability of the error is preserved under the transformation. In the case of linear dynamical systems there exist several results of stability preservation, for instance in [2], [3]. We propose a methodology to extend some classic results of the dynamical systems theory by preserving the signature of the real parts of the eigenvalues of an underlying Jacobian matrix. This methodology is based on the use of matrix theory tools, specifically, simultaneous Schur decomposition, the closure under product and sum of positive definite diagonal matrices and the eigenvalue sign-preservation for both real and complex diagonal matrices under matrix multiplication. We present a modified Chen attractor to show the preservation of the synchronization manifold.

II. MATHEMATICAL PRELIMINARIES

In this section we present the necessary definitions and results that will allow us to prove the main propositions of this paper.

Definition 1: *The group of matrices $A_1,\ A_2,\ \ldots,\ A_n$ is said to be Schur simultaneously decomposable if there exists a unitary matrix U, where $UU^\top = U^\top U = I$, such that $A_1 = UT_1U^\top,\ A_2 = UT_2U^\top,\ \ldots,\ A_n = UT_nU^\top$, where T_i are upper triangular matrices.*

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For the following discussion consider the dynamical system described by $\dot{x} = f(x)$, where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous differentiable function of its argument. Let $A = \left.\frac{\partial f}{\partial x}\right|_{x_0}$ be the Jacobian matrix associated with f evaluated at an equilibrium point x_0.

Lemma 1: *Consider the linear polynomial $P_n(A) = \sum_{i=0}^{n} M_iA^i$. Where A and M_i for all i are simultaneously Schur decomposable matrices of our dynamical system's associated Jacobian matrix $A = UT_AU^\top$ such that $M_i = UD_{M_i}U^\top$ where D_{M_i} $i = 1,\ldots,n$ are positive definite diagonal matrices. Any matrix polynomial $P_n(A)$ which fulfills $\sigma(P_n(A)) = \sigma(A)$, where $\sigma(P_n(A))$ and $\sigma(A)$ are the spectra of $P_n(A)$ and A, respectively, will preserve the system's hyperbolicity.*

III. LOCAL STABLE-UNSTABLE MANIFOLD THEOREM EXTENSION

The following proposition is a simple extension of the Local Stable-Unstable Manifold Theorem for the action of matrix polynomials $P_n(X) = \sum_{i=0}^{n} M_iX^i$ on the matrix A and the vector field $f(x)$ where A, the system's linear coefficients matrix, may be decomposed as $A = UT_AU^\top$ with T_A an upper triangular matrix, $UU^\top = U^\top U = I$. $M_i \in \Delta_{pd}; i = 1,\ldots,n$, where Δ_{pd} is the set of diagonalizable matrices whose real coefficients are all positive and the set Ψ as

$$\Psi = \{P_n(X)\,|\mathrm{sgn}(\sigma(P_n(X))) = \mathrm{sgn}(\sigma(X))\} \qquad (1)$$

where $\sigma(P_n(X))$ and $\sigma(X)$ are the spectra of $P_n(X)$ and X, respectively.

This proposition is an alternative result to proposition 4.2 presented in [4].

Proposition 1: Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system $\dot{x} = f(x) = Ax + g(x)$. Suppose that $f(0) = 0$ and that $A = Df(0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part, i.e., the origin is an hyperbolic fixed point. Then for each polynomial $P_n(A) \in \Psi$, as defined in (1), there exists a k-dimensional differentiable manifold S_P tangent to the stable subspace E_P^S and and $n-k$ dimensional differentiable manifold W_P tangent to the unstable subspace E_P^W of the linear system $\dot{x} = P_n(A)x$ at 0.

In consequence this transformation preserves hyperbolic points in nonlinear systems and dimension of the stable and unstable manifolds, i.e, an hyperbolic nonlinear system

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($\dot{x} = Ax + g(x)$) is mapped in a hyperbolic nonlinear systems ($\dot{x} = P_n(A)x + g(x)$), and $\dim S = \dim S_P$ and $\dim W = \dim W_P$.

Sketch of Proof:
Consider a matrix A with Schur decomposition $A = UT_AU^\top$ and the modifying matrix $M_i = UT_{D_i}U^\top$, $M \in \Delta_{pd}$. Then $P_n(A) = \sum_{i=1}^{n} M_iA^i = U\left(\sum_{i=1}^{n} D_{M_i}{T_A}^i\right)U^\top$ and the eigenvalues of the matrix M_iA^i are the product of the eigenvalues of matrices M_i and A^i.

Since each matrix $M_i \in \Delta_{pd}$, $i = 1, \ldots, n$ has all diagonal elements strictly positive and M_0 is selected appropriately so that $P_n(A) \in \Psi$, then the matrix $P_n(A)$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. Since the dimensions of each manifold have not changed, the result is a consequence of the Stable-Unstable Manifold Theorem and Lemma 1. ■

This proposition is a generalization of the results proposed in [5].

IV. PRESERVATION OF SYNCHRONIZATION IN MODIFIED SYSTEMS

For simplicity we will consider a first order polynomial, $P(A) = MA + M_0$, for the examples presented in the following sections.

Consider the following n-dimensional systems in a master-slave configuration

$$\begin{aligned} \dot{x} &= Ax + g(x) \\ \dot{y} &= Ay + f(y) + u(t) \end{aligned}$$

where $A \in R^{n \times n}$ is a constant matrix, $f, g : R^n \to R^n$ are continuous nonlinear functions and $u \in R^n$ is the control input.

Considering the error state vector $e = y - x \in R^n$, $f(y) - g(x) = L(x, y)$ and an error dynamics equation

$$\dot{e} = Ae + L(x, y) + u(t)$$

choosing $u(t) = Bv(t) - L(x, y)$, where B is a constant gain matrix which is selected such that (A, B) be controllable. Since the pair (A, B) is controllable a suitable choice for state feedback is a linear-quadratic state-feedback regulator [1], which minimizes the quadratic cost function.

This state-feedback law renders the error equation to $\dot{e} = (A - BK)e$, with $(A - BK)$ a Hurwitz matrix.

Now consider $M \in \Delta_{pd}$, and suppose that the following two n-dimensional systems are chaotic:

$$\begin{aligned} \dot{x} &= (MA + M_0)\, x + g(x) \\ \dot{y} &= (MA + M_0)\, y + f(y) + \widehat{u}(t) \end{aligned}$$

We have that $\widehat{u}(t) = -\left(MBK + M_0\right)e - L(x, y)$ stabilizes the zero solution of the error dynamics system, i.e., the resultant system

$$\dot{e} = (MA + M_0 - MBK - M_0)\, e$$

is asymptotically stable. The original control $u(t) = -BKe - L(x, y)$ is preserved in its linear part by the transformation $P(BK)$ and the new control is given by $\widehat{u}(t) = -\left(MBK + M_0\right)e - L(x, y)$. Therefore the controller $u(t)$ which achieves the synchronization in the two original systems is preserved under the transformation $P(BK)$ so that $\widehat{u}(t)$ achieves the synchronization in the two transformed systems.

Take for example the Chen attractor, whose Jacobian, $L(x, y)$ and modifying matrices are defined as follows

$$A = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix}, L(x, y) = \begin{pmatrix} 0 \\ -y_1y_3 + x_1x_3 \\ y_1y_2 - x_1x_2 \end{pmatrix},$$

$$M = \begin{pmatrix} 3 & -0.1 & 0 \\ -0.1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}, M_0 = \begin{pmatrix} 39 & 11 & 0 \\ 11 & -53 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

where M and M_0 were constructed using simultaneous Schur decomposition.

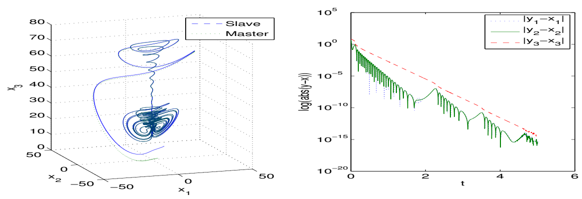

Fig. 1. Synchronization of modified Chen attractor and magnitude of error $|e| = |y - x|$ between master and slave systems in semi-logarithmic plot

In Figure 1 we have the transformed system's dynamics, which seem to preserve chaotic behavior, and the absolute error of the master/slave system configuration where there is clearly an effective convergence to zero.

V. CONCLUSIONS

It has been shown that a scheme consisting of a master/slave pair for which a constant state feedback achieves synchronization, the transformed master/slave/controller system preserves this characteristic. It is an attempt to study how a given collective dynamic can be preserved when changes occur in the dynamical system. From the results we may conclude that the fundamental properties of the synchronization manifold are preserved thus showing that robustness under the proposed transformations is a consequence of this methodology.

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