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Modal Participation Factors Revisited: One Definition Replaced by Two

Eyad H. Abed, Munther A. Hassouneh and Wael A. Hashlamoun

***Abstract*— This paper presents a new fundamental approach to modal participation analysis of linear time-invariant systems, leading to new insights and new formulas for modal participation factors. Modal participation factors were introduced over a quarter century ago as a way of measuring the relative participation of modes in states, and of states in modes, for linear time-invariant systems. Participation factors have proved their usefulness in the field of electric power systems and in other applications. However, in the current understanding, it is routinely taken for granted that the measure of participation of *modes in states* is identical to that for participation of *states in modes*. Here, a new analysis using averaging over an uncertain set of system initial conditions yields the conclusion that these quantities should not be viewed as interchangeable. In fact, it is proposed that a *new definition and calculation* replace the existing ones for *state in mode participation factors*, while the previously existing participation factors definition and formula should be retained but viewed only as *mode in state participation factors*. Examples are used to illustrate the issues addressed and the results obtained.**

I. Introduction

This paper presents new concepts, results, and formulas in the subject of modal participation analysis of linear time-invariant systems. This topic is an important component of the Selective Modal Analysis (SMA) framework introduced by Perez-Arriaga, Verghese and Schweppe [8], [13] in the early 1980s. A main construct in SMA is the concept of modal participation factors (or simply participation factors). Participation factors are scalars intended to measure the relative contribution of system modes to system states, and of system states to system modes, for linear systems. The work of these authors has had a major impact especially in applications to electric power systems, where participation factors as they were originally introduced have become a routine tool for the practitioner and researcher alike.

Since their introduction, participation factors have been employed widely in electric power systems and other applications. They have been used for stability analysis, order reduction, sensor and actuator placement, and coherency and clustering studies (e.g., [8], [13], [9], [2], [6], [3], [10]). Several researchers have also considered alternate ways of viewing modal participation factors (e.g., [12], [4], [11]).

We study linear time-invariant continuous-time systems

$$\dot{x} \quad = \quad Ax(t) \qquad (1)$$

where $x \in R^n$ and A is a real $n \times n$ matrix. We make the blanket assumption that A has a set of n distinct eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. The solution of Eq. (1) then takes the form of a sum of modal components:

$$x(t) \quad = \quad \sum_{i=1}^{n} e^{\lambda_i t} c^i \qquad (2)$$

where the c^i are constant vectors determined by the initial condition x^0 and by the right and left eigenvectors of A.

In their study of modal participation for the system (1), the authors of [8], [13] selected particular initial conditions and introduced definitions motivated by the calculation of relative state and mode contributions using those initial conditions. In this paper, we take a different approach, building on our previous work [1], in which definitions of modal participation factors are formulated by averaging relative contributions of modes in states and states in modes over an uncertain set of initial conditions. In this approach, we consider initial conditions to be unknown, and we take the view that performing some sort of average over all possible initial conditions should give a more reliable result than focusing attention on one particular possible initial condition. The uncertainty in initial condition can be taken as set-theoretic (unknown but bounded) or probabilistic.

The main contribution of this paper is to reveal a previously unknown dichotomy in modal participation analysis. To wit, although the definitions obtained in [8], [13], and which have been in wide use since their introduction, give identical values for measures of participation of modes in states and for participation of states in modes, these are in fact better viewed as fundamentally different, and should be calculated using two distinct formulas. Summarizing, the main contribution of this paper is as follows: *we propose replacing the existing definition of participation factors with two separate definitions that yield distinct numerical values for participation of modes in states and for participation of states in modes. In this paper, the currently used participation factors measuring participation of* ***states in modes*** *are replaced with a new first-principles definition, a particular instance of which is an explicit formula given in Section V. In addition, we show that our formula for participation factors measuring participation of* ***modes in states*** *agrees with the commonly used participation factors formula under*

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E. H. Abed is with the Department of Electrical and Computer Engineering and the Institute for Systems Research, University of Maryland, College Park, MD 20742 USA (email: abed@umd.edu).

M. A. Hassouneh is with the Institute for Systems Research and the Department of Mechanical Engineering, University of Maryland, College Park, MD 20742 USA (email: munther@umd.edu).

W. A. Hashlamoun was visiting the Institute for Systems Research, University of Maryland, College Park, MD 20742 USA. He is now at the Department of Electrical and Computer Engineering, Birzeit University, Birzeit, Palestine (email: hwael@birzeit.edu).

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reasonable assumptions on the allowed uncertainty in the system initial conditions. Thus, a dichotomy is proposed in the calculation of participation factors.

The paper proceeds as follows. In Section II, the original definitions of modal participation factors are recalled from [8], [13]. In Section III, basic examples are used to illustrate the need for an approach that yields distinct formulas for measuring the two main types of modal participation. In Section IV, the initial condition uncertainty approach we introduced to this topic in [1] is recalled and applied to define mode-in-state participation factors. In Section V, the proposed new definition of state-in-mode participation factors is given, and an explicit formula is derived under a simplifying assumption on the initial condition uncertainty. Due to space constraints, here we provide the new analytical formula for state-in-mode participation factors only for the case of a mode associated with a real eigenvalue λ_i. The formula for the case of a mode associated with a complex conjugate pair of eigenvalues also follows along the same lines and will be reported in detail elsewhere [5].

II. Original Definitions of Participation Factors

In this section, the original definitions of modal participation factors are recalled from [8], [13]. Consider the linear system (1), repeated here for convenience:

$$\dot{x} \quad = \quad Ax(t) \tag{3}$$

where $x \in R^n$, and A is a real $n \times n$ matrix. The authors of [8], [13] also make the blanket assumption that A has n distinct eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Let $(r^1, r^2, \ldots, r^n)$ be right eigenvectors of the matrix A associated with the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, respectively. Let $(l^1, l^2, \ldots, l^n)$ denote left (row) eigenvectors of the matrix A associated with the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, respectively. The right and left eigenvectors are taken to satisfy the normalization [7]

$$l^i r^j \quad = \quad \delta_{ij} \tag{4}$$

where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The solution to (3) starting from an initial condition $x(0) = x^0$ is

$$x(t) \quad = \quad e^{At} x^0 \tag{5}$$

Since the eigenvalues of A are distinct, A is similar to a diagonal matrix. Using this, Eq. (5) can be rewritten in the form

$$x(t) = \sum_{i=1}^{n} (l^i x^0) e^{\lambda_i t} r^i. \tag{6}$$

From (6), $x_k(t)$ is given by

$$x_k(t) = \sum_{i=1}^{n} (l^i x^0) e^{\lambda_i t} r^i_k. \tag{7}$$

A. Relative participation of the i-th mode in the k-th state

To determine the relative participation of the i-th mode in the k-th state, the authors of [8], [13] select an initial condition $x^0 = e^k$, the unit vector along the k-th coordinate axis. As seen next, this choice is convenient in that it results in a simple formula for mode-in-state participation factors. With this choice of x^0, the evolution of the k-th state becomes

$$x_k(t) \quad = \quad \sum_{i=1}^{n} (l^i_k r^i_k) e^{\lambda_i t} =: \sum_{i=1}^{n} p_{ki} e^{\lambda_i t}. \tag{8}$$

The quantities

$$p_{ki} := l^i_k r^i_k \tag{9}$$

are found to be unit-independent, and are taken in [8], [13] as measures of the relative participation of the i-th mode in the k-th state.

B. Relative participation of the k-th state in the i-th mode

The relative participation of the k-th state in the i-th mode is studied in [8], [13] by first applying the similarity transformation

$$z := V^{-1} x \tag{10}$$

to system (3), where V is the matrix of right eigenvectors of A:

$$V = [r^1 \ r^2 \ \cdots \ r^n] \tag{11}$$

and V^{-1} is the matrix of left eigenvectors of A:

$$V^{-1} = \begin{bmatrix} l^1 \\ \vdots \\ l^n \end{bmatrix}. \tag{12}$$

Then z obeys the dynamics

$$\dot{z}(t) \quad = \quad V^{-1} A V z(t) = \Lambda z(t), \tag{13}$$

where $\Lambda := \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, with initial condition $z^0 := V^{-1} x^0$. This implies that the evolution of the new state vector components z_i, $i = 1, \ldots, n$ is given by

$$z_i(t) \quad = \quad z^0_i e^{\lambda_i t} = l^i x^0 e^{\lambda_i t} = \left[\sum_{k=1}^{n} (l^i_k x^0_k) \right] e^{\lambda_i t}. \tag{14}$$

For a real eigenvalue λ_i, clearly $z_i(t)$ represents the evolution of the associated mode. If λ_i is not real, then the associated mode is sometimes taken to be $z_i(t)$, but can also be taken as the combination of $z_i(t)$ and its complex conjugate $\bar{z}_i(t)$, which reflects the influence of the eigenvalue $\bar{\lambda}_i$. In the latter approach, we view λ_i and $\bar{\lambda}_i$ as representing the same "complex frequency." In the past, the former convention was used in most publications. In this paper, we allow both interpretations, but we will find it convenient to use the latter point of view when deriving a new state-in-mode participation factors formula for the case of complex eigenvalues.

In order to determine the relative participation of the k-th state in the i-th mode, the authors of [8], [13] select an initial condition $x^0 = r^i$, the right eigenvector associated with λ_i. As seen next, this choice is convenient in that it results in a simple formula for state-in-mode participation factors. We will revisit this later using an uncertain initial condition, and obtain a different result. With this choice of initial condition ($x^0 = r^i$), the evolution of the i-th mode becomes

$$z_i(t) \quad = \quad \left[\sum_{k=1}^{n}(l_k^i r_k^i)\right] e^{\lambda_i t} = \left[\sum_{k=1}^{n} p_{ki}\right] e^{\lambda_i t}. \qquad (15)$$

Based on Equation (15) and that for the chosen initial condition $x_k^0 = r_k^i$, $k = 1,\ldots,n$, the authors of [8], [13] propose the formula

$$p_{ki} = l_k^i r_k^i \qquad (16)$$

as a measure of the relative participation of the k-th state in the i-th mode.

Note that Eqs. (9), (16) provide identical formulas for participation of modes in states and participation of states in modes, respectively. For this reason, the same notation p_{ki} was used for both types of participation factors until now.

III. Motivating Examples Showing Inadequacy of Participation Factors Formula as a Measure of State in Mode Participation

In this section, by way of motivation for the subsequent analysis, two examples are given that show the need for a new definition and a new formula for state in mode participation factors.

Example 1 Consider the two-dimensional system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where a, b and d are constants with $a \neq d$. The eigenvalues of A are $\lambda_1 = a$ and $\lambda_2 = d$. The right eigenvectors associated with λ_1 and λ_2 are

$$r^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } r^2 = \begin{bmatrix} 1 \\ \frac{d-a}{b} \end{bmatrix},$$

respectively. The left eigenvectors associated with λ_1 and λ_2 and satisfying the normalization (4) are

$$l^1 = \begin{bmatrix} 1 & \frac{b}{a-d} \end{bmatrix} \text{ and } l^2 = \begin{bmatrix} 0 & \frac{-b}{a-d} \end{bmatrix},$$

respectively.

Before calculating the participation factors measuring the influence of states x_1 and x_2 in mode 1,[§] we write the evolution of mode 1 explicitly. Using (14), we have

$$\begin{aligned} z_1(t) \quad &= \quad l^1 x^0 e^{\lambda_1 t} = \begin{bmatrix} 1 & \dfrac{b}{a-d} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} e^{\lambda_1 t} \\ &= \quad \left(x_1^0 + \frac{b}{a-d} x_2^0\right) e^{\lambda_1 t}. \end{aligned} \qquad (17)$$

Note that the evolution of mode 1 is influenced by both x_1^0 and x_2^0, with the relative degree of influence depending on the values of the system parameters a, b and d.

Calculating the participation factors using the original definition, we find the participation factor for state x_1 in mode 1 is $p_{11} = l_1^1 r_1^1 = 1$, while the participation factor for state x_2 in mode 1 is $p_{21} = l_2^1 r_2^1 = 0$. Thus, the original definition of participation factors for state in mode participation indicates that state x_2 has zero influence on mode 1 regardless of the values of system parameters a, b and d. This is in stark contradiction to what we observed using the explicit formula (17), and begs for a re-examination of the basic formula for state-in-mode participation factors. ■

Example 2 Consider the two-dimensional system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -d & -d \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where d is a constant. The eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 1 - d$. The right eigenvectors associated with λ_1 and λ_2 are

$$r^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } r^2 = \begin{bmatrix} 1 \\ -d \end{bmatrix},$$

respectively. The left eigenvectors associated with λ_1 and λ_2 and satisfying the normalization (4) are

$$l^1 = \begin{bmatrix} \frac{-d}{1-d} & \frac{-1}{1-d} \end{bmatrix} \text{ and } l^2 = \begin{bmatrix} \frac{1}{1-d} & \frac{1}{1-d} \end{bmatrix},$$

respectively. Denote by V the matrix of right eigenvectors of A:

$$V = [r^1 \quad r^2] = \begin{bmatrix} 1 & 1 \\ -1 & -d \end{bmatrix}.$$

From the normalization condition (4), we can immediately write

$$V^{-1} = \begin{bmatrix} l^1 \\ l^2 \end{bmatrix} = \begin{bmatrix} \frac{-d}{1-d} & \frac{-1}{1-d} \\ \frac{1}{1-d} & \frac{1}{1-d} \end{bmatrix}.$$

The evolution of the modes can be obtained using the diagonalizing transformation $z := V^{-1}x$ as was done in (10)-(13). The system modes are found to be

$$\begin{aligned} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \quad &= \quad \begin{bmatrix} l^1 x^0 e^{\lambda_1 t} \\ l^2 x^0 e^{\lambda_2 t} \end{bmatrix} \\ &= \quad \begin{bmatrix} \left(\frac{-d}{1-d} x_1^0 - \frac{1}{1-d} x_2^0\right) e^{\lambda_1 t} \\ \frac{1}{1-d}\left(x_1^0 + x_2^0\right) e^{\lambda_2 t} \end{bmatrix}. \end{aligned} \qquad (18)$$

Based on the original definition of participation factors, the participation factor for state x_1 in mode 2 is $p_{12} = r_1^2 l_1^2 = \frac{1}{1-d}$, and the participation factor for state x_2 in mode 2 is $p_{22} = r_2^2 l_2^2 = \frac{-d}{1-d}$. Clearly, in general $p_{12} \neq p_{22}$. However, from (18) we have the equation

$$z_2(t) \quad = \quad \frac{1}{1-d}\left(x_1^0 + x_2^0\right) e^{\lambda_2 t} \qquad (19)$$

for the second mode $z_2(t)$, from which we observe that state x_1 and state x_2 *participate equally* in mode 2 since $z_2(t)$ depends on the initial condition x^0 through the sum $x_1^0 + x_2^0$. Again, we find that the state-in-mode participation factors as commonly calculated yield conclusions that are very much at odds with what one might consider reasonable based on explicit calculation of the evolution of system modes as they depend on initial conditions of the state variables. ■

[§]For simplicity, we use the terminology 'mode i' in place of 'the mode associated with eigenvalue λ_i.'

The inadequacy of the original state-in-mode participation factors formula has been demonstrated in the two examples above. This motivates the need for a new formula that better assesses the influence of system states on system modes.

IV. Initial Condition Uncertainty Approach to Derivation of Mode-in-State Participation Factors

For systems operating near equilibrium, it is often reasonable to view the system initial condition as being an uncertain vector in the vicinity of the system equilibrium point. In this paper, and in the authors' previous work [1], we approach the problem of measuring modal participation by averaging relative contributions over an uncertain set of initial conditions.

Next, we recall from our previous work [1] a basic definition of relative participation of a mode in a state. This definition involves taking an average over system initial conditions of a measure of the relative influence of a particular system mode on a system state. The initial condition uncertainty can be taken as set-theoretic or probabilistic. In the set-theoretic formulation, the participation factor measuring relative influence of the mode associated with λ_i on state x_k is defined as

$$p_{ki} \quad := \quad \underset{x^0 \in \mathcal{S}}{\text{avg}} \frac{(l^i x^0) r_k^i}{x_k^0} \tag{20}$$

whenever this quantity exists. Here, $x_k^0 = \sum_{i=1}^{n} (l^i x^0) r_k^i$ is the value of $x_k(t)$ at $t = 0$, and "$\text{avg}_{x^0 \in \mathcal{S}}$" is an operator that computes the average of a function over a set $\mathcal{S} \subset R^n$ (representing the set of possible values of the initial condition x^0). We assume that the initial condition uncertainty set $\mathcal{S}$ is symmetric with respect to each of the hyperplanes $\{x_k = 0\}$, $k = 1, \ldots, n$.

In the definition in [1] that starts with a probabilistic description of the uncertainty in the initial condition x^0, the average in (20) is replaced by a mathematical expectation. The general formula for the participation factor p_{ki} measuring participation of mode i in state x_k becomes

$$p_{ki} \quad := \quad E\left\{ \frac{(l^i x^0) r_k^i}{x_k^0} \right\} \tag{21}$$

where the expectation is evaluated using some assumed joint probability density function $f(x^0)$ for the initial condition uncertainty (of course, this definition applies only when the expectation exists).

Expanding the inner product term in (21), we find

$$\begin{aligned} p_{ki} &= E\left\{ \sum_{j=1}^{n} \frac{(l_j^i x_j^0) r_k^i}{x_k^0} \right\} \\ &= E\left\{ \frac{(l_k^i x_k^0) r_k^i}{x_k^0} \right\} + E\left\{ \sum_{j=1,\ j\neq k}^{n} \frac{(l_j^i x_j^0) r_k^i}{x_k^0} \right\} \\ &= l_k^i r_k^i + \sum_{j=1,\ j\neq k}^{n} l_j^i r_k^i E\left\{ \frac{x_j^0}{x_k^0} \right\}. \end{aligned} \tag{22}$$

The second term in (22) vanishes when the components of the initial condition vector $x_1^0, x_2^0, \ldots, x_n^0$ are independent with zero mean [1]. Therefore, under the assumption that the initial condition components $x_1^0, x_2^0, \ldots, x_n^0$ are independent with zero mean, the participation of the i-th mode in the k-th state is given by the same expression originally introduced by Perez-Arriaga, Verghese and Schweppe [8], [13]:

$$p_{ki} \quad = \quad l_k^i r_k^i. \tag{23}$$

This result can also be obtained using the set-theoretic averaging formula (20) [1].

Remark 1: (*Alternate Definition of Mode-in-State Participation Factor for a Complex Mode*) For a complex eigenvalue λ_i, the associated "mode" is taken above as the term containing $e^{\lambda_i t}$ in the system response (2). However, we can alternately view this mode as consisting of the combined contributions from λ_i and its complex conjugate eigenvalue $\bar{\lambda}_i$. This viewpoint is easily seen to lead, under the same symmetry hypotheses as above, to the following alternate expression for the participation factor of the mode associated with λ_i and $\bar{\lambda}_i$ in state x_k:

$$\tilde{p}_{ki} \quad = \quad 2\text{Re}\,\{l_k^i r_k^i\}. \tag{24}$$

V. New Definition of Participation Factors Measuring Participation of States in Modes

In this section, a new definition and calculation are given for participation factors measuring contribution of states in modes. The probabilistic approach presented in the previous section is used, where the initial condition is assumed to satisfy a joint probability density function. In order to obtain an explicit formula from the new general definition of state-in-mode participation factors, we find that it is necessary to make an assumption on the probability distribution of the initial condition which is more constraining than what was needed in the analysis above for mode-in-state participation factors. Thus, the explicit formula derived in this section should be viewed in the pragmatic sense that it provides an easy to use expression that reflects initial condition uncertainty. Other assumed forms of uncertainty may not lead to explicit formulas, although a formula requiring numerical evaluation of integrals can always be obtained from the definition. The explicit formula obtained here differs from the single formula (16) that is currently used to measure both state-in-mode participation and mode-in-state participation, while the currently used formula (16) is retained here as a measure of mode-in-state participation (noting that the alternate formula (24) can also be used for the case of a complex mode). This dichotomy represents a significant departure from current practice. We will also use the new formula to revisit the examples of Section III.

Consider the general linear time-invariant continuous-time system given in (3), repeated here for convenience:

$$\dot{x} \quad = \quad Ax(t) \tag{25}$$

Recall from Section II the expression

$$z_i(t) \quad = \quad e^{\lambda_i t} l^i x^0 = e^{\lambda_i t} \sum_{j=1}^{n} (l^i_j x^0_j). \qquad (26)$$

This equation shows the contribution of each component x^0_j, $j = 1, \dots, n$ of the initial state x^0 to $z_i(t)$. Recall also that for the case of a real eigenvalue λ_i, $z_i(t)$ is identically the i-th mode, while, for a complex eigenvalue λ_i, the associated mode can be taken as $z_i(t)$ or as the combination of $z_i(t)$ and its conjugate: $z_i(t) + \bar{z_i}(t) = 2\mathrm{Re}\ z_i(t)$. The following general definition of state-in-mode participation factors is obtained by averaging the relative contribution of x^0_k in the i-th mode and evaluating the result at $t = 0$. In this definition, we take the mode associated with a complex eigenvalue as $2\mathrm{Re}\ z_i(t)$, i.e., the combination of modal components due to the eigenvalue and its conjugate. Had we decided to view the mode associated with a complex eigenvalue λ_i as $z_i(t)$ alone, we would use the first expression in the definition below for both the case of a real and a complex eigenvalue. However, the derivation following the basic definition below of a simple final formula would become unwieldy for the complex eigenvalue case.

Definition 1: For a linear time-invariant continuous-time system (25), the participation factor for the k-th state in the i-th mode is

$$\pi_{ki} \quad := \quad \begin{cases} E\left\{\frac{l^i_k x^0_k}{z^0_i}\right\} & \text{if } \lambda_i \text{ is real} \\ E\left\{\frac{(l^i_k + \bar{l}^i_k) x^0_k}{z^0_i + \bar{z}^0_i}\right\} & \text{if } \lambda_i \text{ is complex} \end{cases} \qquad (27)$$

whenever the expectation exists.

Note that in (27), the notation z^0_i means $z_i(t = 0) = l^i x^0$ and the bar denotes complex conjugation. Also, analogous to the approach in Section IV, the quantities being evaluated represent the contribution of state x_k to a mode divided by the total mode evaluated at time $t = 0$.

As was emphasized in the original work of [8], [13], definitions of modal participation factors should be independent of the units in which the system state variables are measured. This is indeed the case for the newly defined state in mode participation factors π_{ki}. This can be shown readily, using an argument similar to that in [1]. Thus, we have the following important statement.

Proposition 1: The state in mode participation factors π_{ki} introduced in Definition 1 are invariant to arbitrary changes in the units of the system state variables x_k.

Unfortunately, even under an assumption such as symmetry of the initial condition uncertainty, there is no single closed-form expression for the state in mode participation factors π_{ki}. To obtain a simple closed-form expression for the state in mode participation factors π_{ki} using Eq. (27), we need to find an assumption on the probability density function $f(x^0)$ governing the uncertainty in the initial condition x^0 that allows us to explicitly evaluate the integrals inherent in the definition.

Due to space constraints, here we will give the derivation of a particular analytical formula for state-in-mode participation factors only for the case of a real eigenvalue λ_i. The derivation for a complex eigenvalue proceeds in the same way (both are based on Definition 1 above and Lemma 1 below) and will be presented in detail elsewhere [5].

In the remainder of this section, we assume that the probability density function $f(x^0)$ is such that the components $x^0_1, x^0_2, \dots, x^0_n$ are jointly uniformly distributed over the unit sphere in R^n centered at the origin:

$$f(x^0) = \begin{cases} k & ||x^0|| \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad (28)$$

(This is the same as assuming a uniform distribution in an ellipsoid that is centered at the origin and symmetric with respect to the coordinate hyperplanes in the original state variable units, a physically palatable assumption and independent of units by construction.) Here, the constant k is chosen to ensure the normalization condition

$$\int_{||x^0|| \le 1} f(x^0) dx^0 = 1. \qquad (29)$$

The value of the constant k can be determined by evaluating the integral in (29) using $f(x^0)$ given in (28):

$$\int_{||x^0|| \le 1} f(x^0) dx^0 = \int_{||x^0|| \le 1} k dx^0_1 dx^0_2 \dots dx^0_n = k V_n = 1 \quad (30)$$

where V_n is the volume of the unit sphere in R^n. The constant k is then given by

$$k = \frac{1}{V_n}. \qquad (31)$$

Next, the relative participation of the k-th state in the i-th mode is evaluated using Definition 1 for a real eigenvalue λ_i and under the assumption above on the distribution of the initial condition x^0. Before proceeding, we recall the relationship between x^0 and z^0:

$$x^0 \quad = \quad V z^0 = \sum_{j=1}^{n} r^j z^0_j. \qquad (32)$$

To determine the participation of the k-th state in the i-th mode, we substitute $x^0_k = \sum_{j=1}^{n} r^j_k z^0_j$ in (27):

$$\begin{aligned} \pi_{ki} &= E\left\{\frac{l^i_k x^0_k}{z^0_i}\right\} = E\left\{\frac{l^i_k \sum_{j=1}^{n} r^j_k z^0_j}{z^0_i}\right\} \\ &= E\left\{\frac{l^i_k r^i_k z^0_i}{z^0_i}\right\} + \sum_{j=1,\ j\neq i}^{n} l^i_k r^j_k E\left\{\frac{z^0_j}{z^0_i}\right\} \\ &= l^i_k r^i_k + \sum_{j=1,\ j\neq i}^{n} l^i_k r^j_k E\left\{\frac{z^0_j}{z^0_i}\right\}. \end{aligned} \qquad (33)$$

Note that the first term in (33) coincides with p_{ki}, the original participation factors formula. We will show that, in general, the second term in (33) does not vanish. This is true even in case the components $x^0_1, x^0_2, \dots, x^0_n$ representing the initial conditions of the state are assumed to be independent. This is due to the fact that the second term involves the components of z^0 (i.e., $z^0_1, z^0_2, \dots, z^0_n$) which need not be independent even under the assumption that the x^0_k are independent, due to the transformation $z^0 = V^{-1} x^0$. This was overlooked in [1], leading to the incorrect conclusion there that the second term in (33) vanishes.

The following Lemma will be used below.

Lemma 1: For vectors $a, b \in R^n$ with $b \neq 0$ we have

$$\int_{||x|| \leq 1} \frac{a^T x}{b^T x} d_n x = \frac{a^T b}{b^T b} V_n \tag{34}$$

where $d_n x$ denotes the differential volume element $dx_1 dx_2 \cdots dx_n$, and V_n is the volume of a unit sphere in R^n which is given by

$$V_n = \begin{cases} 2, & n = 1 \\ \pi, & n = 2 \\ \frac{2\pi}{n} V_{n-2}, & n \geq 3 \end{cases} \tag{35}$$

See [5] for a proof.

We now use Lemma 1 to simplify the expression (33) for the participation factor for the k-th state in the i-th mode:

$$\pi_{ki} = l_k^i r_k^i + \sum_{j=1,\ j \neq i}^{n} l_k^i r_k^j E \left\{ \frac{z_j^0}{z_i^0} \right\}. \tag{36}$$

Substituting $z_i^0 = l^i x^0$ into (36) yields

$$\pi_{ki} = l_k^i r_k^i + \sum_{j=1,\ j \neq i}^{n} l_k^i r_k^j E \left\{ \frac{l^j x^0}{l^i x^0} \right\} \tag{37}$$

Denote $a := (l^j)^T$ and $b := (l^i)^T$. Then $E\left\{ \frac{l^j x^0}{l^i x^0} \right\}$ is

$$E \left\{ \frac{l^j x^0}{l^i x^0} \right\} = \int_{||x^0|| \leq 1} \frac{l^j x^0}{l^i x^0} f(x^0)\ dx^0 = k \int_{||x^0|| \leq 1} \frac{a^T x^0}{b^T x^0}\ dx^0$$

Using Lemma 1 and the normalization condition $kV_n = 1$ from (31), this integral reduces to

$$E \left\{ \frac{l^j x^0}{l^i x^0} \right\} = \frac{a^T b}{b^T b} k V_n = \frac{a^T b}{b^T b}. \tag{38}$$

Substituting (38) into (37) yields an important result of this paper, a new formula for the participation factor for state x_k in mode i for the case of a real eigenvalue λ_i:

$$\pi_{ki} = l_k^i r_k^i + \sum_{j=1,\ j \neq i}^{n} l_k^i r_k^j \frac{l^j (l^i)^T}{l^i (l^i)^T}. \tag{39}$$

Remark 2: Another expression equivalent to (39) is

$$\pi_{ki} = \frac{(l_k^i)^2}{l^i (l^i)^T} = \frac{(l_k^i)^2}{\sum_{j=1}^{n} (l_j^i)^2} \tag{40}$$

This also follows easily from Lemma 1 (see [5]).

Next, we revisit Examples 1 and 2 using the newly derived formula (39) for state in mode participation factors, and compare the results to the participation factors obtained using the original definitions. Note that all eigenvalues in these examples are real, so the formula (39) applies as a (new) measure of state-in-mode participation factors π_{ki} (as does the equivalent formula (40)).

Example 1 Revisited

For Example 1, the participation factors for states x_1 and x_2 in mode 1 based on the new formula (39) are

$$\pi_{11} = \frac{(a-d)^2}{(a-d)^2 + b^2} \quad \text{and} \quad \pi_{21} = \frac{b^2}{(a-d)^2 + b^2},$$

respectively. The participation factors for states x_1 and x_2 in mode 1 based on the original formula are $p_{11} = 1$ and $p_{21} = 0$, respectively. Note that the coupling between states x_1 and x_2, not reflected in the original formula p_{ki}, is now reflected by the new participation factors π_{ki}.

Example 2 Revisited

For Example 2, the participation factors for states x_1 and x_2 in mode 2 based on the new formula (39) are

$$\pi_{12} = \frac{1}{2} \quad \text{and} \quad \pi_{22} = \frac{1}{2},$$

respectively. The results using the new formula more faithfully reflect the relative contributions of the initial conditions of the two state variables to the evolution of mode 2, which is given explicitly by the formula $z_2(t) = \frac{1}{1-d}\left(x_1^0 + x_2^0\right) e^{\lambda_2 t}$. Here it is clear that $z_2(t)$ is equally influenced by x_1^0 and x_2^0 since it depends on the initial condition x^0 through the sum $x_1^0 + x_2^0$.

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