2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Adaptive observer design for a class of nonlinear time-delay systems in lower-triangular form

Salim Ibrir

Abstract— **A simple nonlinear observer with a dynamic gain is proposed for a class of bounded-state nonlinear systems subject to state delay. By saturating the states of the system nonlinearities, we show that the observer exists whatever the delay is. Furthermore, it will be highlighted that the observer design is free from any preliminary analysis of the time-delay system such as estimating the Lipschitz constants of nonlinearities. The proposed design encompasses a wide class of nonlinear and time-delay systems written in triangular form and generalizes previous results on delayless nonlinear systems.**

Index Terms— **Nonlinear observers; Time-delay Systems; Adaptive observers.**

I. Introduction

TIME-delay often appears in many control systems such as chemical reactions, mechanical systems, electrical circuits [1], high-speed networks, and many other process control systems [2], [3], [4], [5]. This delay is due to several reasons such as communication protocols, transmission, transportation or inertia effects. Unlike systems governed by ordinary differential equations, delay systems called also hereditary or systems with aftereffects, are infinite dimensional in nature and time-delay is, in many cases, a source of instability. The stability issue and the performance of control systems with delay are, therefore, both of theoretical and practical importance. As a dual problem, observer design for time-delay systems turns out a stabilization issue since the observation error dynamics must be stabilized using only partial state measurements which is, in the general case, a quite hard problem as compared with full-state feedback stabilization issues. For further results on observation of time-delay systems, we refer the reader to [6], [7], [8], [9] and the references therein.

Referring to many results on observation of time-delay systems, conditions under which the observer/filter exists can be classified into two main categories: delay-independent conditions and delay-dependent ones. However, delay-dependent results reveal less conservative than delay-independent conditions. Nevertheless, in most cases, a small delay is tolerable to maintain stability by output feedbacks.

In the present paper, a new adaptive observer is proposed for bounded-state nonlinear systems written in triangular form. By appropriate selection of a parameter-dependent Riccati equation, we show that we can update the observer vector gain by updating just one parameter of the ARE. Even though the observer design methodology does not necessitate any preliminary analysis of the system, the knowledge of both the system delay and the domain of variation of the system states remain necessary to build the nonlinear observer. This assumption seems to be a natural assumption in practice since the observation is usually made in some compact and large set called the observation domain. We show that the saturated-state observer can reproduce the behaviors of the original states by injection of a time-varying gain. Furthermore, it will be outlined that the observer always exists without any condition on the size of the delay or the form of nonlinearities. Finally, extension of the obtained results to linear systems with lower-triangular nominal matrices is given.

Salim Ibrir is with University of Trinidad and Tobago, Pt. Lisas Campus, P.O. Box. 957, Esperanza Road, Brechin Castle, Couva, Trinidad, W.I., email: salim.ibrir@utt.edu.tt

Throughout this paper, we note by $\mathbb{R}$ the set of real numbers. The notation $A > 0$ (resp. $A < 0$) means that the matrix A is positive definite (resp. negative definite). A' is the matrix transpose of A. "$\star$" is used to notify an element which is induced by transposition. $\triangleq$ stands for an equality by definition. $\lambda_k(A)$ stands for the k-th eigenvalue of the matrix A. $\|\cdot\|$ is the Euclidean norm, $\|\cdot\|_\infty$ is the infinity norm, and $\mathrm{Spec}(A)$ stands for the set of eigenvalues of the matrix A. C_n^k is the binomial coefficient.

II. System description

Consider the time-delay nonlinear system given in lower triangular form:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) + f_1(x_1(t), x_1(t-\tau), u(t)),\\
\dot{x}_2(t) &= x_3(t) + f_2(x_1(t), x_2(t), x_1(t-\tau), x_2(t-\tau), u(t)),\\
&\vdots\\
\dot{x}_i(t) &= x_{i+1}(t) + f_i(x_1(t), \cdots, x_i(t), x_1(t-\tau), \cdots,\\
&\qquad\qquad x_i(t-\tau), u(t)),\\
&\vdots\\
\dot{x}_n(t) &= f_n(x_1(t), \cdots, x_n(t), x_1(t-\tau), \cdots,\\
&\qquad\qquad x_n(t-\tau), u(t)),\\
y(t) &= x_1(t),
\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathscr{U} \subset \mathbb{R}^m$ is a bouded control input and $y(t) \in \mathbb{R}$ is the system output. We assume that the delay τ is constant and $x(t) = \phi(t)$ for $t \le \tau$. In matrix notation, system (1) takes the form

$$\begin{aligned}
\dot{x}(t) &= A\,x(t) + f(x(t), x(t-\tau), u(t)),\\
y(t) &= C\,x(t),
\end{aligned} \tag{2}$$

978-1-4244-4524-0/09/$25.00 ©2009 AACC

where

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad C \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}' \in \mathbb{R}^n,$$

$$f(x(t), x(t-\tau), u(t)) \triangleq \begin{bmatrix} f_1(x_1(t), x_1(t-\tau), u(t)) \\ f_2(x_1(t), x_2(t), x_1(t-\tau), x_2(t-\tau), u(t)) \\ \vdots \\ f_n(x(t), x(t-\tau), u(t)) \end{bmatrix} \in \mathbb{R}^n. \tag{3}$$

To complete the system description, the following assumptions are considered.

Assumption 1: The nonlinearity $f(x(t), x(t-\tau), u(t))$ is smooth and well-defined for all $x(t) \in \mathbb{R}^n$ with $f(0,0,0) = 0$.

Assumption 2: For all $t \geq 0$ and initial condition $x_0 \in \mathscr{M} \subset \mathbb{R}^n$, the state vector $x(t)$ is well-defined and globally bounded under the excitation of $u \in \mathscr{U}$.

Assumption 3: For all $t \geq 0$, the system input $u(t) \in \mathscr{U} \in \mathbb{R}^m$ is globally bounded.

Assumption 4: For all $t \geq 0$, the delay τ is known and constant.

In the sequel, for simplicity of notations, $u(t)$, $x(t)$ and $x(t-\tau)$ will be noted by u, x and x^τ, respectively.

III. Observer design

Before giving the main result of this paper, let us present some preliminary results.

A. Preliminary results

The following technical Lemma is necessary for the proof of the main statement.

Lemma 1: Let $P(\gamma)$ and $\tilde{P}$ be the solutions of the Algebraic Riccati Equations (AREs):

$$\begin{aligned} &P(\gamma)A' + AP(\gamma) - P(\gamma)C'CP(\gamma) + Q(\gamma) = 0, \\ &\tilde{P}A' + A\tilde{P} - \tilde{P}C'C\tilde{P} + \tilde{Q} = 0, \end{aligned} \tag{4}$$

respectively, where A and C are given in observable canonical form as in (3), $\tilde{Q}$ is any symmetric positive-definite matrix verifying the following matrix inequality:

$$\Lambda\tilde{Q} + \tilde{Q}\Lambda > 0; \quad \Lambda = \mathrm{diag}\big(1, 2, 3, \cdots, n\big), \tag{5}$$

and

$$Q(\gamma) = \gamma^2 D(\gamma)\tilde{Q}D(\gamma). \tag{6}$$

Then, $P(\gamma)$ is positive-definite for $\gamma > 0$ and has the following properties:

$i)$ for any $\gamma > 0$, the Cholesky decomposition of $P(\gamma)$ is given by:

$$P(\gamma) = R(\gamma)R'(\gamma), \quad R(\gamma) = \sqrt{\gamma}D(\gamma)\widetilde{R}, \tag{7}$$

where

$$D(\gamma) = \mathrm{diag}\Big(1, \gamma, \gamma^2, \cdots, \gamma^{n-1}\Big), \tag{8}$$

and $\tilde{P} = \widetilde{R}\widetilde{R}'$ is the solution of the ARE:

$$\tilde{P}A' + A\tilde{P} - \tilde{P}C'C\tilde{P} + \tilde{Q} = 0. \tag{9}$$

$ii)$ For any $\gamma > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma) > 0. \tag{10}$$

$iii)$ For any $\gamma > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}P^{-1}(\gamma) < 0. \tag{11}$$

$iv)$ For any lower triangular matrix $L \in \mathbb{R}^{n\times n}$ whose entries are all constants, we can always find two constants c_1 and c_2 such that

$$\sup_{\gamma\geq 1}\Big\|D^{-1}(\gamma)\,L\,D(\gamma)\Big\| \leq c_1 + \frac{c_2}{\gamma}. \tag{12}$$

Remark 1: In the particular case where

$$\tilde{Q} = \mathrm{diag}\big(C_n^1, C_n^2, \cdots, C_n^n\big),$$

the ARE (4) coincides with the ARE used in [10], [11], and the matrix $A - P(\gamma)C'C$ has an eigenvalue of multiplicity n equal to $-\gamma$, i.e.,

$$\mathrm{Spec}\Big(A - P(\gamma)C'C\Big) = \Big\{-\gamma, -\gamma, \cdots, -\gamma\Big\}. \tag{13}$$

The properties $i) - iv)$ of Lemma 1 do not require the matrix $Q(\gamma)$ to be diagonal as it has been required in the design of the ARE-based differentiation observers, see [10]. Therefore, more freedom in assigning the eigenvalues of $A - P(\gamma)C'C$ is tolerated. Explicit formulae that gives $Q(\gamma)$ in terms of the coefficients of the characteristic polynomial (not on the roots of this polynomial) is given in [12].

Proof of Lemma 1. Since $Q(\gamma)$ is symmetric and positive-definite for all $\gamma > 0$, then the matrix $P(\gamma)$ is the only solution of the ARE (4) which is always symmetric and positive-definite for $\gamma > 0$.

To prove $i)$ it is sufficient to prove that $P(\gamma) = \gamma D(\gamma)\tilde{P}D(\gamma)$. Pre- and post multiplying the ARE (9) by $\gamma D(\gamma)$, then we have:

$$\begin{aligned} &\gamma^2 D(\gamma)\tilde{P}A'D(\gamma) + \gamma^2 D(\gamma)A\tilde{P}D(\gamma) \\ &- \gamma^2 D(\gamma)\tilde{P}C'C\tilde{P}D(\gamma) + \gamma^2 D(\gamma)\tilde{Q}D(\gamma) = 0. \end{aligned} \tag{14}$$

where $\tilde{Q} = Q(\gamma)\big|_{\gamma=1}$. Using the following properties: $D(\gamma)A' = \gamma\, A'D(\gamma)$, $\gamma D(\gamma)A = AD(\gamma)$, $CD(\gamma) = C$, $D(\gamma)C' = C'$, $\gamma^2 D(\gamma)\tilde{Q}D(\gamma) = Q(\gamma)$ then, the first ARE in (4) can be rewritten as

$$\begin{aligned} &\Big[\gamma D(\gamma)\tilde{P}D(\gamma)\Big]A' + A\Big[\gamma D(\gamma)\tilde{P}D(\gamma)\Big] \\ &- \Big[\gamma D(\gamma)\tilde{P}D(\gamma)\Big]C'C\Big[\gamma D(\gamma)\tilde{P}D(\gamma)\Big] + Q(\gamma) = 0. \end{aligned} \tag{15}$$

By comparing the last ARE with the first ARE of (4), we conclude that

$$P(\gamma) = \gamma D(\gamma)\tilde{P}D(\gamma). \tag{16}$$

Since $\tilde{P} = \widetilde{R}\widetilde{R}'$, this immediately implies that

$$P(\gamma) = \gamma D(\gamma)\widetilde{R}\widetilde{R}' D(\gamma). \tag{17}$$

and consequently, $R(\gamma) = \sqrt{\gamma}D(\gamma)\widetilde{R}$. This ends the proof of item i).

ii) We have for all $\gamma > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}(\gamma\, D(\gamma)) = \mathrm{diag}\big(1, 2\gamma, \cdots, n\,\gamma^{n-1}\big) = D(\gamma)\Lambda. \tag{18}$$

This gives

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}Q(\gamma) = \gamma D(\gamma)\Lambda\tilde{Q}D(\gamma) + \gamma D(\gamma)\tilde{Q}D(\gamma)\Lambda \tag{19}$$

Since $D(\gamma)$ and Λ commute then, $D(\gamma)\Lambda = \Lambda D(\gamma)$, which makes the derivative of γ equal to

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}Q(\gamma) = \gamma D(\gamma)\Big[\Lambda\tilde{Q} + \tilde{Q}\Lambda\Big]D(\gamma). \tag{20}$$

By assumption of Lemma 1, $\Lambda\tilde{Q}+\tilde{Q}\Lambda > 0$, then $\frac{\mathrm{d}}{\mathrm{d}\gamma}Q(\gamma) > 0$ for all $\gamma > 0$.

By differentiating the first ARE in (4) with respect to γ, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma)A' + A\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma)\\ &-\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma)C'CP(\gamma) - P(\gamma)C'C\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma) + \frac{\mathrm{d}}{\mathrm{d}\gamma}Q(\gamma) = 0.\end{aligned} \tag{21}$$

The last equation can be rewritten as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma)\Big(A - P(\gamma)C'C\Big)'\\ &+\Big(A - P(\gamma)C'C\Big)\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma) = -\frac{\mathrm{d}}{\mathrm{d}\gamma}Q(\gamma)\end{aligned} \tag{22}$$

Then, we conclude that the matrix derivative $\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma)$ verifies the Lyapunov equation (22) which implies that $\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma)$ is positive-definite. This ends the proof of ii).

The proof of iii) is easy since the derivative of the matrix inverse can be explicitly written as

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}P^{-1}(\gamma) = -P^{-1}(\gamma)\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma)P^{-1}(\gamma). \tag{23}$$

Using the result obtained in ii), we conclude that $\frac{\mathrm{d}}{\mathrm{d}\gamma}P^{-1}(\gamma) < 0$.

iv) Since each (i,j)-entry of the matrix $D^{-1}(\gamma)LD(\gamma)$ is in the form $\alpha_{i,j} + \frac{\beta_{i,j}}{\gamma}$, then we conclude that

$$\begin{aligned}&\sup_{\gamma\geq 1}\Big\|D^{-1}(\gamma)L\,D(\gamma)\Big\| \leq \sqrt{n}\sup_{\gamma\geq 1}\Big\|D^{-1}(\gamma)L\,D(\gamma)\Big\|_\infty\\ &\leq c_1 + \frac{c_2}{\gamma}.\end{aligned} \tag{24}$$

Before presenting the saturated-state observer, we need to introduce the following Lemma.

Lemma 2: Let v and w be two n-dimensional real-valued vectors and suppose that P is a real, symmetric and positive semi-definite matrix. Let $\bar{\sigma}_1$, $\bar{\sigma}_2$, $\cdots$, $\bar{\sigma}_n$ be different and positive saturation levels. Then, for any v, $w \in \mathbb{R}^n$, we have

$$\Big(\sigma(v) - \sigma(w)\Big)'P\Big(\sigma(v) - \sigma(w)\Big) \leq \Big(v - w\Big)'P\Big(v - w\Big), \tag{25}$$

where each component of the vector

$$\sigma(\cdot) = \begin{bmatrix} \sigma_1(v_1) & \sigma_2(v_2) & \cdots\sigma_n(v_n) \end{bmatrix}'$$

is defined as

$$\sigma_i(x_i) \triangleq \begin{cases} x_i & \text{if } , |x_i| \leq \bar{\sigma}_i \\ \bar{\sigma}_i \ \mathrm{sign}(x_i) & \text{otherwise,} \end{cases}, \quad 1 \leq i \leq n. \tag{26}$$

Proof. The vector of the saturation functions $\sigma(v)$ can be rewritten as

$$\sigma(v) = \Delta(v)\, v, \tag{27}$$

where $\Delta(v) = \mathrm{diag}\Big(\Delta_1(v_1), \cdots, \Delta_n(v_n)\Big)$ is $n{\times}n$ diagonal matrix whose elements are defined as

$$\begin{aligned}\Delta(v) &= \mathrm{diag}\Big(\Delta_1(v_1), \cdots, \Delta_n(v_n)\Big)\\ &= \begin{bmatrix} \frac{\sigma_1(v_1)}{v_1} & 0 & \cdots & 0 \\ \star & \frac{\sigma_2(v_2)}{v_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \cdots & \frac{\sigma_n(v_n)}{v_n} \end{bmatrix}.\end{aligned} \tag{28}$$

According to this notation, the matrix $\Delta(v)$ is positive definite and all its eigenvalues are less or equal to one. We have

$$\begin{aligned}&\Big(\sigma(v) - \sigma(w)\Big)'P\Big(\sigma(v) - \sigma(w)\Big) =\\ &\Big(v'\Delta(v) - w'\Delta(w)\Big)P\Big(\Delta(v)\,v - \Delta(w)\,w\Big)\\ &= \begin{bmatrix} v' & w' \end{bmatrix}\begin{bmatrix} \Delta(v) & \mathbf{0} \\ \mathbf{0} & \Delta(w) \end{bmatrix}\begin{bmatrix} P & -P \\ -P & P \end{bmatrix}\\ &\times\begin{bmatrix} \Delta(v) & \mathbf{0} \\ \mathbf{0} & \Delta(w) \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix}.\end{aligned} \tag{29}$$

Let us note

$$A = \begin{bmatrix} P & -P \\ -P & P \end{bmatrix},\ S = \begin{bmatrix} \Delta(v) & \mathbf{0} \\ \mathbf{0} & \Delta(w) \end{bmatrix}. \tag{30}$$

Then, by the use of Ostrowski Theorem [13], we have

$$\lambda_k(SAS') = \theta_k\,\lambda_k(A), \tag{31}$$

where $\lambda_1(SS') \leq \theta_k \leq \lambda_n(SS')$. Since $0 < \lambda_k(SS') \leq 1$, $\forall k$ then, from (31), we conclude that $0 \leq \theta_k \leq 1$, and hence,

$$\lambda_k(SAS') \leq \lambda_k(A),\ \forall k, \tag{32}$$

or equivalently, $SAS' \le A$, which means that

$$\Big(\sigma(v)-\sigma(w)\Big)' P\Big(\sigma(v)-\sigma(w)\Big) \le \begin{bmatrix} v' & w' \end{bmatrix} \begin{bmatrix} P & -P \\ -P & P \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \Big(v-w\Big)' P\Big(v-w\Big). \tag{33}$$

This ends the proof.

B. Adaptive observer design

The observer design methodology is basically founded on estimating the bounds of the system states and saturating the nonlinearities of the observer according to the estimated levels of variation of the true states. The idea of this new design is to adapt the observer gain without any care of the maximum levels that can reach the system states. The analysis of the observer is summarized in the following statement.

Theorem 1: Consider the time-delay nonlinear system (2) under Assumptions 1-4. For a given initial condition $x_0 \in \mathscr{M} \subset \mathbb{R}^n$, we assume that there exist positive saturation levels $\bar{\sigma}_1, \cdots, \bar{\sigma}_n$ such that

$$\forall t \ge 0, \ \ |x_i(t)| \le \bar{\sigma}_i, \ \ 1 \le i \le n, \tag{34}$$

Let

$$\begin{aligned} &\dot{\hat{x}} = A\hat{x} + f(\sigma(\hat{x}), \sigma(\hat{x}^\tau), u) - P(\gamma)C'(C\hat{x} - Cx), \\ &\dot{\gamma} = \hat{\gamma}\big|\hat{x}_1 - y\big|, \ \ \gamma(0) > 1, \ \hat{\gamma} > 0, \\ &\ \ P(\gamma)A' + AP(\gamma) - P(\gamma)C'CP(\gamma) + Q(\gamma) = 0, \\ &\ \ \tilde{P}A' + A\tilde{P} - \tilde{P}C'C\tilde{P} + \tilde{Q} = 0, \end{aligned} \tag{35}$$

be the adaptive observer where $\Lambda\tilde{Q} + \tilde{Q}\Lambda > 0$, $\Lambda = \mathrm{diag}(1, 2, \cdots, n)$, $Q(\gamma) = \gamma^2 D(\gamma)\tilde{Q}D(\gamma)$ and $\sigma(x) = [\sigma_1(x_1), \cdots, \sigma_n(x_n)]'$ is a vector of saturation functions defined as in (26). Then, the observation error $e = \hat{x} - x$ that results from (2) and (35) is asymptotically stable for all $\hat{x}(0) \in \mathbb{R}^n$.

Proof: The dynamic equation of the observation error is given by:

$$\begin{aligned} \dot{e} &= (A - P(\gamma)C'C)e \\ &\quad + f(\sigma(\hat{x}), \sigma(\hat{x}^\tau), u) - f(\sigma(x), \sigma(x^\tau), u). \end{aligned} \tag{36}$$

Let us assign the functional

$$\begin{aligned} V(e) &= e'P^{-1}(\gamma)e \\ &\quad + \frac{1}{2}\int_{t-\tau}^{t} e'(s)P^{-1}(\gamma)Q(\gamma)P^{-1}(\gamma)e(s)\,\mathrm{d}\,s \end{aligned} \tag{37}$$

to the dynamics (36). Then, we have

$$\begin{aligned} \dot{V} &= \dot{e}'P^{-1}(\gamma)e + e'P^{-1}(\gamma)\dot{e} + \frac{1}{2}e'P^{-1}(\gamma)Q(\gamma)P^{-1}(\gamma)e \\ &\quad - \frac{1}{2}e'^{\tau}P^{-1}(\gamma)Q(\gamma)P^{-1}(\gamma)e^{\tau} + \dot{\gamma}e'\frac{\mathrm{d}P^{-1}(\gamma)}{\mathrm{d}\gamma}e. \end{aligned} \tag{38}$$

Since $\dot{\gamma} > 0$ for all $t \ge 0$ and $\frac{\mathrm{d}P^{-1}(\gamma)}{\mathrm{d}\gamma} < 0$ for all $\gamma > 0$, see the result of Lemma 1, then

$$\begin{aligned} \dot{V}(e) &\le \dot{e}'P^{-1}(\gamma)e + e'P^{-1}(\gamma)\dot{e} \\ &\quad + \frac{1}{2}e'P^{-1}(\gamma)Q(\gamma)P^{-1}(\gamma)e - \frac{1}{2}e'^{\tau}P^{-1}(\gamma)Q(\gamma)P^{-1}(\gamma)e^{\tau} \\ &= e'\bigg(A'P^{-1}(\gamma) + P^{-1}(\gamma)A - 2C'C \\ &\quad + \frac{1}{2}P^{-1}(\gamma)Q(\gamma)P^{-1}(\gamma)\bigg)e \\ &\quad + 2e'P^{-1}(\gamma)\Big(f(\sigma(\hat{x}), \sigma(\hat{x}^\tau), u) - f(\sigma(x), \sigma(x^\tau), u)\Big) \\ &\quad - \frac{1}{2}e'^{\tau}P^{-1}(\gamma)Q(\gamma)P^{-1}(\gamma)e^{\tau}. \end{aligned} \tag{39}$$

Using the differential Mean-Value Theorem, we can write that

$$\begin{aligned} &f(\sigma(\hat{x}), \sigma(\hat{x}^\tau), u) - f(\sigma(x), \sigma(x^\tau), u) \\ &= \int_0^1 \frac{\partial f(\alpha, \beta, u)}{\partial \alpha}\bigg|_{\substack{\alpha=\alpha(\lambda)\\ \beta=\beta(\lambda)}} \Big(\sigma(\hat{x}) - \sigma(x)\Big)\mathrm{d}\lambda \\ &+ \int_0^1 \frac{\partial f(\alpha, \beta, u)}{\partial \beta}\bigg|_{\substack{\alpha=\alpha(\lambda)\\ \beta=\beta(\lambda)}} \Big(\sigma(\hat{x}^\tau) - \sigma(x^\tau)\Big)\mathrm{d}\lambda \end{aligned} \tag{40}$$

where $\alpha(\lambda) = \sigma(\hat{x}) - \rho(\lambda)(\sigma(\hat{x}) - \sigma(x))$, $\beta(\lambda) = \sigma(\hat{x}^\tau) - \rho(\lambda)(\sigma(\hat{x}^\tau) - \sigma(x^\tau))$, where

$$\rho(\lambda) = \begin{cases} \lambda & \text{if } , 0 \le \lambda \le 1, \\ 1 & \text{otherwise.} \end{cases} \tag{41}$$

Let us note

$$\frac{\partial f(\alpha, \beta, u)}{\partial \alpha}\bigg|_{\substack{\alpha=\alpha(\lambda)\\ \beta=\beta(\lambda)}} = \mathscr{L}_\alpha(\lambda), \frac{\partial f(\alpha, \beta, u)}{\partial \beta}\bigg|_{\substack{\alpha=\alpha(\lambda)\\ \beta=\beta(\lambda)}} = \mathscr{L}_\beta(\lambda) \tag{42}$$

This immediately gives

$$\begin{aligned} \dot{V}(e) &\le \int_0^1 e'\bigg(-C'C - \frac{1}{2}P^{-1}(\gamma)Q(\gamma)P^{-1}(\gamma)\bigg)e\,\mathrm{d}\lambda \\ &\quad + 2\int_0^1 e'P^{-1}(\gamma)\mathscr{L}_\alpha(\lambda)\big(\sigma(\hat{x}) - \sigma(x)\big)\,\mathrm{d}\lambda \\ &\quad + 2\int_0^1 e'P^{-1}(\gamma)\mathscr{L}_\beta(\lambda)\ \big(\sigma(\hat{x}^\tau) - \sigma(x^\tau)\big)\,\mathrm{d}\lambda \\ &\quad - \frac{1}{2}\int_0^1 e'^{\tau}\Big(P^{-1}(\gamma)Q(\gamma)P^{-1}(\gamma)\Big)e^{\tau}\ \mathrm{d}\lambda \end{aligned} \tag{43}$$

Using the fact that for given vectors $\zeta_1 \in \mathbb{R}^n$, $\zeta_2 \in \mathbb{R}^n$ the following inequality $2\zeta_1'\zeta_2 \le \zeta_1'X\zeta_1' + \zeta_2'X^{-1}\zeta_2$ holds for any $X \in \mathbb{R}^{n\times n} > 0$. Then,

$$
\begin{aligned}
& 2\int_0^1 e' P^{-1}(\gamma)\mathscr{L}_\alpha(\lambda)\big(\sigma(\hat{x}) - \sigma(x)\big)\,\mathrm{d}\lambda \\
& \le \int_0^1 e' P^{-1}(\gamma) e\,\mathrm{d}\lambda \\
& + \int_0^1 \big(\sigma(\hat{x}) - \sigma(x)\big)' \mathscr{L}_\alpha'(\lambda) P^{-1}(\gamma)\mathscr{L}_\alpha(\lambda)\big(\sigma(\hat{x}) - \sigma(x)\big) \\
& \qquad \mathrm{d}\lambda
\end{aligned} \tag{44}
$$

and

$$
\begin{aligned}
& 2\int_0^1 e' P^{-1}(\gamma)\mathscr{L}_\beta(\lambda)\ \big(\sigma(\hat{x}^\tau) - \sigma(x^\tau)\big)\,\mathrm{d}\lambda \\
& \le \int_0^1 e' P^{-1}(\gamma)\, e\,\mathrm{d}\lambda \\
& + \int_0^1 \big(\sigma(\hat{x}^\tau) - \sigma(x^\tau)\big)' \mathscr{L}_\beta'(\lambda) P^{-1}(\gamma)\mathscr{L}_\beta(\lambda) \\
& \qquad \times \big(\sigma(\hat{x}^\tau) - \sigma(x^\tau)\big)\mathrm{d}\lambda
\end{aligned} \tag{45}
$$

Using the result of Lemma 2, we can write that

$$
\begin{aligned}
& 2\int_0^1 e' P^{-1}(\gamma)\mathscr{L}_\alpha(\lambda)\big(\sigma(\hat{x}) - \sigma(x)\big)\,\mathrm{d}\lambda \\
& + 2\int_0^1 e' P^{-1}(\gamma)\mathscr{L}_\beta(\lambda)\ \big(\sigma(\hat{x}^\tau) - \sigma(x^\tau)\big)\,\mathrm{d}\lambda \\
& \le 2\int_0^1 e' P^{-1}(\gamma) e\,\mathrm{d}\lambda + \int_0^1 e' \mathscr{L}_\alpha'(\lambda) P^{-1}(\gamma)\mathscr{L}_\alpha(\lambda) e\,\mathrm{d}\lambda \\
& + \int_0^1 e'^\tau \mathscr{L}_\beta'(\lambda) P^{-1}(\gamma)\mathscr{L}_\beta(\lambda)\, e^\tau\,\mathrm{d}\lambda
\end{aligned} \tag{46}
$$

This implies that

$$
\begin{aligned}
\dot{V}(e) & \le -\frac{1}{2}\int_0^1 e' \Big(P^{-1}(\gamma) Q(\gamma) P^{-1}(\gamma)\Big)\, e\,\mathrm{d}\lambda \\
& + 2\int_0^1 e' P^{-1}(\gamma) e\,\mathrm{d}\lambda + \int_0^1 e' \mathscr{L}_\alpha'(\lambda) P^{-1}(\gamma)\mathscr{L}_\alpha(\lambda) e\,\mathrm{d}\lambda \\
& + \int_0^1 e'^\tau \mathscr{L}_\beta'(\lambda) P^{-1}(\gamma)\mathscr{L}_\beta(\lambda)\, e^\tau\,\mathrm{d}\lambda \\
& - \frac{1}{2}\int_0^1 e'^\tau \Big(P^{-1}(\gamma) Q(\gamma) P^{-1}(\gamma)\Big) e'^\tau\ \mathrm{d}\lambda
\end{aligned} \tag{47}
$$

Let us define $\xi = D^{-1}(\gamma)e$, $\xi^\tau = D^{-1}(\gamma)e^\tau$. Using the result of Lemma 1, we have

$$
\begin{aligned}
P^{-1}(\gamma) &= \frac{1}{\gamma} D^{-1}(\gamma)\tilde{P}^{-1} D^{-1}(\gamma) \\
Q(\gamma) &= \gamma^2 D(\gamma)\tilde{Q} D(\gamma).
\end{aligned} \tag{48}
$$

From (47), we get

$$
\begin{aligned}
\dot{V}(e) & \le -\frac{1}{2}\int_0^1 \xi' \tilde{P}^{-1}\tilde{Q}\tilde{P}^{-1}\,\xi\,\mathrm{d}\lambda \\
& + \frac{2}{\gamma}\int_0^1 \xi' \tilde{P}^{-1}\xi\,\mathrm{d}\lambda + \frac{1}{\gamma}\int_0^1 \xi' D(\gamma)\mathscr{L}_\alpha'(\lambda) D^{-1}(\gamma)\tilde{P}^{-1} \\
& \qquad \times D^{-1}(\gamma)\mathscr{L}_\alpha(\lambda) D(\gamma)\,\xi\,\mathrm{d}\lambda \\
& + \frac{1}{\gamma}\int_0^1 \xi'^\tau D(\gamma)\mathscr{L}_\beta'(\lambda) D^{-1}(\gamma)\tilde{P}^{-1} \\
& \qquad \times D^{-1}(\gamma)\mathscr{L}_\beta(\lambda) D(\gamma)\,\xi^\tau\,\mathrm{d}\lambda \\
& - \frac{1}{2}\int_0^1 \xi'^\tau \tilde{P}^{-1}\tilde{Q}\tilde{P}^{-1}\,\xi^\tau\ \mathrm{d}\lambda
\end{aligned} \tag{49}
$$

This implies that

$$
\begin{aligned}
\dot{V}(e) & \le \int_0^1 \Big(-\frac{1}{2}\lambda_{\min}\big(\tilde{P}^{-1}\tilde{Q}\tilde{P}^{-1}\big) \\
& + \frac{\lambda_{\max}(\tilde{P}^{-1})}{\gamma}\Big[2 + \big\|D^{-1}(\gamma)\mathscr{L}_\alpha(\lambda) D(\gamma)\big\|^2\Big]\Big)\|\xi\|^2\,\mathrm{d}\lambda \\
& + \int_0^1 \Big(-\frac{1}{2}\lambda_{\min}\big(\tilde{P}^{-1}\tilde{Q}\tilde{P}^{-1}\big) \\
& + \frac{\lambda_{\max}(\tilde{P}^{-1})}{\gamma}\big\|D^{-1}(\gamma)\mathscr{L}_\beta(\lambda) D(\gamma)\big\|^2\Big)\|\xi^\tau\|^2\,\mathrm{d}\lambda
\end{aligned} \tag{50}
$$

Since the matrixes $\mathscr{L}_\alpha(\lambda)$ and $\mathscr{L}_\beta(\lambda)$ are lower triangular matrices. Then, by the use of $iv)$ of Lemma 1, and using the fact that $0 \le \lambda \le 1$, we have

$$
\begin{aligned}
\big\|D^{-1}(\gamma)\mathscr{L}_\alpha(\lambda) D(\gamma)\big\|^2 &\le \left(c_1(\lambda) + \frac{c_2(\lambda)}{\gamma}\right)^2, \\
\big\|D^{-1}(\gamma)\mathscr{L}_\beta(\lambda) D(\gamma)\big\|^2 &\le \left(c_1^\tau(\lambda) + \frac{c_2^\tau(\lambda)}{\gamma}\right)^2,
\end{aligned} \tag{51}
$$

where $c_1(\lambda)$, $c_2(\lambda)$, $c_1^\tau(\lambda)$, and $c_2^\tau(\lambda)$ are bounded positive constants that depend on λ. Finally, we can write that

$$
\begin{aligned}
\dot{V}(e) & \le \int_0^1 \Bigg[-\frac{1}{2}\lambda_{\min}\big(\tilde{P}^{-1}\tilde{Q}\tilde{P}^{-1}\big) \\
& + \frac{\lambda_{\max}(\tilde{P}^{-1})}{\gamma}\left[2 + \left(c_1(\lambda) + \frac{c_2(\lambda)}{\gamma}\right)^2\right]\Bigg]\|\xi\|^2\,\mathrm{d}\lambda \\
& + \int_0^1 \Bigg[-\frac{1}{2}\lambda_{\min}\big(\tilde{P}^{-1}\tilde{Q}\tilde{P}^{-1}\big) \\
& + \frac{\lambda_{\max}(\tilde{P}^{-1})}{\gamma}\left(c_1^\tau(\lambda) + \frac{c_2^\tau(\lambda)}{\gamma}\right)^2\Bigg]\|\xi^\tau\|^2\,\mathrm{d}\lambda.
\end{aligned} \tag{52}
$$

Since γ is increasing whenever $\hat{x}_1 \neq y$ then, there exist a finite-time T whereby $-\frac{1}{2}\lambda_{\min}\big(\tilde{P}^{-1}\tilde{Q}\tilde{P}^{-1}\big) + \frac{\lambda_{\max}(\tilde{P}^{-1})}{\gamma}\left[2 + \left(c_1(\lambda) + \frac{c_2(\lambda)}{\gamma}\right)^2\right] < 0$, and $-\frac{1}{2}\lambda_{\min}\big(\tilde{P}^{-1}\tilde{Q}\tilde{P}^{-1}\big) + \frac{\lambda_{\max}(\tilde{P}^{-1})}{\gamma}\left(c_1^\tau(\lambda) + \frac{c_2^\tau(\lambda)}{\gamma}\right)^2 < 0$

and consequently, $\dot{V}(e) < 0$ for all $t \geq T$. Since the dynamics of the nonlinear observer can be rewritten as

$$\begin{aligned}\dot{\hat{x}} &= (A - P(\gamma)C'C)\hat{x} + f(\sigma(\hat{x}), \sigma(\hat{x}^\tau), u)\\ &\quad + P(\gamma)C'C\sigma(x),\ \dot{\gamma} = \hat{\gamma}\big|\hat{x}_1 - y\big|,\ \ \gamma(0) > 1,\ \hat{\gamma} > 0,\\ &\quad P(\gamma)A' + AP(\gamma) - P(\gamma)C'CP(\gamma) + Q(\gamma) = 0,\end{aligned} \tag{53}$$

The vector $f(\sigma(\hat{x}), \sigma(\hat{x}^\tau), u)$ can be seen as a bounded vector that perturbs the stable system $\dot{\hat{x}} = (A - P(\gamma)C'C)\hat{x}$. Then, it is easy to prove that the observer state trajectories are bounded for $0 \leq t \leq T$ whenever $u(t)$ and $\gamma(t)$ are bounded. This ends the proof.

In the following statement, we show that we can apply the previous results for linear systems whose states are not necessarily bounded. This result is given by the following statement.

Corollary 1: Consider the time-delay linear system

$$\begin{aligned}\dot{x}(t) &= (A + \Delta A)\, x(t) + A_d\, x(t - \tau) + \rho(u)\\ y(t) &= C\, x(t),\ x(t) = \psi(t),\ \ 0 \leq t \leq \tau,\end{aligned} \tag{54}$$

where $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{n}$ are defined as in (3). Let ΔA and A_d be known, real, and lower-triangular matrices and $\rho(u)$ is an input-injection vector of dimension n. Then the following system

$$\begin{aligned}&\dot{\hat{x}}(t) = (A + \Delta A)\, \hat{x}(t) + A_d\, \hat{x}(t - \tau) - P(\gamma)C'(C\hat{x} - y)\\ &\ + \rho(u),\\ &\dot{\gamma} = \hat{\gamma}\big|\hat{x}_1 - y\big|,\ \ \gamma(0) > 1,\ \hat{\gamma} > 0,\\ &P(\gamma)A' + AP(\gamma) - P(\gamma)C'CP(\gamma) + Q(\gamma) = 0,\end{aligned} \tag{55}$$

is an asymptotic observer for system (54).

Proof: The matrices ΔA and A_d can be seen as the matrix Jacobian. Therefore, the proof becomes straightforward as it was developed before.

IV. Example

Let us consider the time-delay system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= -\alpha\, x_1(t) - \beta\, x_2(t - \tau)\, \cos(x_2(t - \tau)),\\ y(t) &= x_1(t).\end{aligned} \tag{56}$$

The system nonlinearity is not globally Lipschitz, but the system states exhibit bounded behaviors for the initial condition $x_0 = [1\ \ -1]'$ with $\alpha = 3.5$, $\beta = 1.2$, $\tau = 0.5$. By tacking $\bar{\sigma}_1 = 4$ and $\bar{\sigma}_2 = 8$, the saturated-state observer

$$\begin{aligned}&\dot{\hat{x}}_1(t) = \hat{x}_2(t) + \ell_1(\gamma)(y(t) - \hat{x}_1(t)),\\ &\dot{\hat{x}}_2(t) = -\alpha\, \sigma_1(\hat{x}_1(t)) - \beta\, \sigma_2(\hat{x}_2(t - \tau))\, \cos(\sigma_2(\hat{x}_2(t - \tau))),\\ &\qquad + \ell_2(\gamma)(y(t) - \hat{x}_1(t)),\\ &\qquad \dot{\gamma} = 2|y(t) - \hat{x}_1(t)|\end{aligned} \tag{57}$$

converges as shown in Fig. 1 where $P(\gamma)C' = [\ell_1(\gamma)\ \ \ell_2(\gamma)]'$.

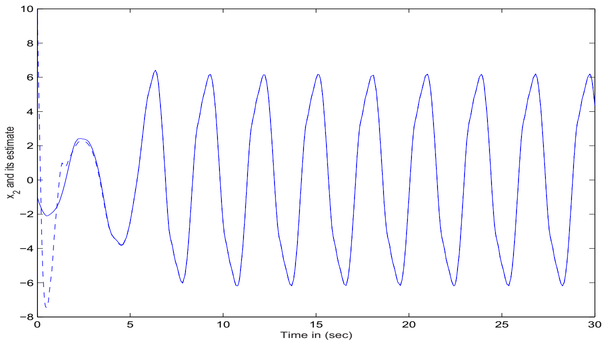

Fig. 1. The second state x_2 and its estimate

V. Conclusion

The results given in this paper can been seen as an extension of the previous results on observation of uniformly-observable nonlinear systems written is lower-triangular form [14]. The proposed design is related to neither delay-independent nor delay-dependent conditions whatever the size of the delay. We showed that by adaptation of the observer gain we could eliminate preliminary analysis steps and avoid excessive high-gain design. The proposed design is dedicated to a large class of bounded-state nonlinear systems that are not necessarily globally Lipschitz.

References

[1] A. Bellen, N. Guglielmi, and A. E. Ruehli, "Methods for linear systems of circuit delay differential equations of neutral type," *IEEE Transactions on Circuits and Systems–I: Fundamental Theory and Applications*, vol. 46, no. 1, pp. 212–216, January 1999.

[2] J. K. Hale and S. M. V. Lunel, *Introduction to functional differential equation*. Springer, 1993, new York.

[3] L. Dugard and E. I. Verriest, *Stability and control of time-delay systems*. London: Springer-Verlag, 1997.

[4] S.-I. Niculescu, *Delay effect on stability: a robust control approach*. Springer-Verlag, 2001.

[5] K. Gu, L. K. Vladimir, and J. Chen, *Stability of time-delay systems*. Birkhäuser, 2003.

[6] S. Ibrir, W. F. Xie, and C.-Y. Su, "Observer design for discrete-time systems subject to time-delay nonlinearities," *International Journal of Systems Science*, vol. **37**, no. 9, pp. 629–641, 2006.

[7] M. Boutayeb, "Observer design for linear time-delay systems," *Systems & Control Letters*, vol. **44**, no. 2, pp. 103–109, 2001.

[8] Z. Wang, D. P. Goodall, and K. J. Burnham, "on designing observers for time-delay systems with nonlinear disturbances," *International Journal of Control*, vol. **75**, no. 11, pp. 803–811, 2002.

[9] A. Germani, C. Manes, and P. Pepe, "A new approach to state observation of nonlinear systems with delayed output," *IEEE Transactions on Automatic Control*, vol. **47**, no. 1, pp. 96–101, 2002.

[10] S. Ibrir, "New differentiators for control and observation applications," in *Proceedings of the American Control Conference*, Arlington, VI, USA, 2001, pp. 2522–2527.

[11] ——, "On-line exact differentiation and notion of asymptotic algebraic observers," *IEEE Transactions on Automatic Control*, vol. **48**, no. 11, pp. 2055–2060, 2003.

[12] ——, "On observer design for nonlinear systems," *International Journal of Systems Sciences*, vol. **37**, no. 15, pp. 1097–1109, 2006.

[13] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[14] J. P. Gauthier, H. Hammouri, and S. Othman, "A simple observer for nonlinear systems: Application to bioreactors," *IEEE Transactions on Automatic Control*, vol. **37**, no. 6, pp. 875–880, June 1992.