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Improved Controller Design for Switching Fuzzy Model-based Control

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Abstract— **This paper presents improved LMI (linear matrix inequality) controller design condition for switching fuzzy model-based control using slack variable approach based on the inverse use of the elimination lemma. In our previous papers, we derived controller design conditions for augmented switching fuzzy model which consists of a switching fuzzy model and a stable linear system. However, in the papers, we have to determine the stable linear system in advance. In this paper, by employing slack variable approach based on the inverse use of the elimination lemma, we derive LMI controller design conditions for the switching fuzzy model without determining the stable linear system in advance. A design example illustrates the utility of this approach.**

I. INTRODUCTION

In general, nonlinear controls require special and rather involved knowledge [1]. It is not easy to utilize nonlinear control theories for practical engineers. On the other hand, Takagi-Sugeno (T-S) fuzzy model-based control which has been rapidly developed in recent years [2]–[6] is simple and natural. By employing the T-S fuzzy model [7], which utilizes local linear system description for each rule, we can devise a control methodology to fully take advantages of linear control theory.

However, the complexity of a system makes the number of rules of a fuzzy model exponentially increase. The curse of the number of rules makes controller design difficult. To decrease the number of rules which fire simultaneously, we proposed a switching fuzzy model [8]. The switching fuzzy model is constructed by dividing the state space and by finding the sector which can cover the nonlinear dynamics [9], [10]. Moreover, we derived controller design conditions based on the switching Lyapunov function by introducing the augmented system which consists of the switching fuzzy model and a stable linear system. The stable linear system is an additional system in order to derive controller design condition. However, we have to determine the stable linear system in advance. Therefore, it is likely that how to determine the stable system affects stability analysis results and controller designs.

In this paper, we derive improved LMI controller design condition for switching fuzzy model-based control. We employ slack variable approach [13] based on the inverse use of the elimination lemma [12]. By utilizing the approach, a system matrix of the additional stable system can be converted into a LMI variable. This means that we do not have to determine the stable system in advance. A design example illustrates the utility of this approach.

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II. PRELIMINARY RESULTS

In this section, we explain switching fuzzy model construction and controller design.

A. Switching Fuzzy Model Construction [9], [10]

Consider the following nonlinear function.

$$y = f(\boldsymbol{x}) = f(x_1, x_2, \cdots, x_n) \tag{1}$$

where $f(\mathbf{0}) = 0$. In this subsection, we show how to convert the nonlinear function into the swtiching fuzzy model.

To begin with, determine the dividing planes. We assume that the dividing planes contain the origin. dividing planes are represented by the following linear equations.

$$\lambda_{\gamma 1} x_1 + \lambda_{\gamma 2} x_2 + \cdots + \lambda_{\gamma n} x_n = \boldsymbol{\Lambda}_\gamma \boldsymbol{x} = 0 \tag{2}$$

where $\gamma = 1, 2, \cdots, \Gamma$ and Γ is the number of dividing planes. One dividing plane divides the state space into the following two regions.

$$\overline{S}_\gamma = \{x | \boldsymbol{\Lambda}_\gamma \boldsymbol{x} \geq 0\}, \quad \underline{S}_\gamma = \{x | \boldsymbol{\Lambda}_\gamma \boldsymbol{x} \leq 0\}$$

The state space is divided into Q regions by Γ dividing planes. Note that $Q = 2^\Gamma$ is not necessarily satisfied. One region constructed by dividing planes is defined as follows:

$$\begin{aligned} R_q = \{x | \boldsymbol{\Lambda}_1 \boldsymbol{x} \geq 0, \boldsymbol{\Lambda}_2 \boldsymbol{x} \leq 0, \boldsymbol{\Lambda}_3 \boldsymbol{x} \leq 0, \\ \boldsymbol{\Lambda}_4 \boldsymbol{x} \geq 0, \cdots, \boldsymbol{\Lambda}_\Gamma \boldsymbol{x} \geq 0\} \end{aligned} \tag{3}$$

We represent the region as follows:

$$\begin{aligned} & R_q(s_1, s_2, s_3, s_4, \cdots, s_\Gamma) \\ & s_1 = 1, s_4, \cdots, s_\Gamma = 1, s_2, s_3 = 0 \end{aligned}$$

or

$$R_q(1, 0, 0, 1, \cdots, 1)$$

where

$$\begin{cases} s_\gamma = 1 & \boldsymbol{\Lambda}_\gamma \boldsymbol{x} \geq 0 \\ s_\gamma = 0 & \boldsymbol{\Lambda}_\gamma \boldsymbol{x} \leq 0 \end{cases}$$

Next we calculate the continuity matrix $\boldsymbol{K}_q \in \mathbb{R}^{(2\Gamma + n) \times n}$ as follows:

$$\begin{aligned} \boldsymbol{K}_q = \Big[\eta_{q11} \boldsymbol{\Lambda}_1^T \ \eta_{q12} \boldsymbol{\Lambda}_2^T \ \cdots \ \eta_{q1\Gamma} \boldsymbol{\Lambda}_\Gamma^T \\ -\eta_{q21} \boldsymbol{\Lambda}_1^T \ -\eta_{q22} \boldsymbol{\Lambda}_2^T \ \cdots \ -\eta_{q2\Gamma} \boldsymbol{\Lambda}_\Gamma^T \ \boldsymbol{I}_n \Big]^T \end{aligned}$$

where

$$\eta_{q1\gamma}=\begin{cases}1 & R_q\in\overline{S}_\gamma\\ 0 & R_q\notin\overline{S}_\gamma\end{cases}\quad \eta_{q2\gamma}=\begin{cases}1 & R_q\in\underline{S}_\gamma\\ 0 & R_q\notin\underline{S}_\gamma\end{cases}$$
$$\overline{S}_\gamma=\{\boldsymbol{x}|\boldsymbol{\Lambda}_\gamma\boldsymbol{x}\geq 0\},\quad \underline{S}_\gamma=\{\boldsymbol{x}|\boldsymbol{\Lambda}_\gamma\boldsymbol{x}\leq 0\}.$$

The continuity matrix $\boldsymbol{K}_q$ satisfies the following condition on region boundaries [11].

$$\boldsymbol{K}_{q_1}\boldsymbol{x}=\boldsymbol{K}_{q_2}\boldsymbol{x},\quad \boldsymbol{x}\in R_{q_1}\cap R_{q_2}. \tag{4}$$

By using $\boldsymbol{K}_q$ and solving the following conditions, we construct the tight sector which can cover the nonlinear function (1).

$$\begin{aligned}&\underset{\boldsymbol{a}_1,\boldsymbol{a}_2}{\text{minimize}}\sum_{q=1}^{Q}|Y_{q1}(\boldsymbol{a}_1)-Y_{q2}(\boldsymbol{a}_2)| \\ &\quad\text{subject to}\\ &\qquad y_{q1}(\boldsymbol{a}_1,\boldsymbol{x})-f(\boldsymbol{x})\geq 0,\quad \boldsymbol{x}\in R_q,\ \forall q\\ &\qquad f(\boldsymbol{x})-y_{q2}(\boldsymbol{a}_2,\boldsymbol{x})\geq 0,\quad \boldsymbol{x}\in R_q,\ \forall q\end{aligned} \tag{5}$$

where

$$\begin{aligned}&\boldsymbol{a}_i=[\,a_{i1}\ a_{i2}\ \cdots\ a_{i(2\Gamma+n)}\,]\\ &Y_{qi}(\boldsymbol{a}_i)=\boldsymbol{a}_i\boldsymbol{K}_q\boldsymbol{D}_q\\ &\boldsymbol{D}_q=\int\cdots\int\int_{R_q}\boldsymbol{x}\ dx_1dx_2\cdots dx_n\\ &y_{qi}(\boldsymbol{a}_i,\boldsymbol{x})=\boldsymbol{a}_i\boldsymbol{K}_q\boldsymbol{x}\\ &|x_1|\leq d_1,|x_2|\leq d_2,\cdots,|x_n|\leq d_n\end{aligned}$$

The sector in qth region is represented by the following two linear models.

$$y_{q1}(\boldsymbol{x})=\boldsymbol{a}_1\boldsymbol{K}_q\boldsymbol{x} \tag{6}$$
$$y_{q2}(\boldsymbol{x})=\boldsymbol{a}_2\boldsymbol{K}_q\boldsymbol{x} \tag{7}$$

By using (6) and (7), the switching fuzzy model can be constructed as follows:

$$y=\sum_{q=1}^{Q}\sum_{i=1}^{2}v_q(\boldsymbol{x})h_{qi}(\boldsymbol{x})\boldsymbol{a}_i\boldsymbol{K}_q\boldsymbol{x} \tag{8}$$

where

$$v_q(\boldsymbol{x})=\begin{cases}1, & \boldsymbol{x}\in R_q,\\ 0, & \boldsymbol{x}\notin R_q.\end{cases} \tag{9}$$

The membership functions are represented by the following equations.

$$h_{q1}(\boldsymbol{x})=\frac{f(\boldsymbol{x})-y_{q2}(\boldsymbol{x})}{y_{q1}(\boldsymbol{x})-y_{q2}(\boldsymbol{x})} \tag{10}$$
$$h_{q2}(\boldsymbol{x})=\frac{y_{q1}(\boldsymbol{x})-f(\boldsymbol{x})}{y_{q1}(\boldsymbol{x})-y_{q2}(\boldsymbol{x})} \tag{11}$$

where $h_{q1}(\boldsymbol{x})\geq 0$, $h_{q2}(\boldsymbol{x})\geq 0$ and $h_{q1}(\boldsymbol{x})+h_{q2}(\boldsymbol{x})=1$.

B. Construction of Dynamic Switching Fuzzy Model

This subsection shows the switching fuzzy model construction for the following dynamic state equation.

$$\begin{aligned}\dot{\boldsymbol{x}}(t)=&\begin{bmatrix}f_1(\boldsymbol{x}(t))\\ f_2(\boldsymbol{x}(t))\\ \vdots\\ f_n(\boldsymbol{x}(t))\end{bmatrix}\\ &+\begin{bmatrix}g_{11}(\boldsymbol{x}(t)) & g_{21}(\boldsymbol{x}(t)) & \cdots & g_{m1}(\boldsymbol{x}(t))\\ g_{12}(\boldsymbol{x}(t)) & g_{22}(\boldsymbol{x}(t)) & \cdots & g_{m2}(\boldsymbol{x}(t))\\ \vdots & & & \\ g_{1n}(\boldsymbol{x}(t)) & g_{2n}(\boldsymbol{x}(t)) & \cdots & g_{mn}(\boldsymbol{x}(t))\end{bmatrix}\boldsymbol{u}(t)\end{aligned} \tag{12}$$

where $\boldsymbol{x}(t)=[x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^T$ is the state vector, $\boldsymbol{u}(t)=[u_1(t)\ u_2(t)\cdots\ u_m(t)]^T$ is the input vector. $f_j(\boldsymbol{x}(t))$ and $g_{ej}(\boldsymbol{x}(t))$ are scalar linear or nonlinear functions, respectively. $j=1,\ 2,\ \cdots,\ n$, $e=1,\ 2,\ \cdots,\ m$. By applying the switching fuzzy model construction method described in Section II-A to each $f_j(\boldsymbol{x}(t))$, we can obtain the following switching fuzzy model.

$$\begin{aligned}\dot{\boldsymbol{x}}(t)=&\begin{bmatrix}f_1(\boldsymbol{x}(t))\\ f_2(\boldsymbol{x}(t))\\ \vdots\\ f_n(\boldsymbol{x}(t))\end{bmatrix}\\ &+\begin{bmatrix}g_{11}(\boldsymbol{x}(t)) & g_{21}(\boldsymbol{x}(t)) & \cdots & g_{m1}(\boldsymbol{x}(t))\\ g_{12}(\boldsymbol{x}(t)) & g_{22}(\boldsymbol{x}(t)) & \cdots & g_{m2}(\boldsymbol{x}(t))\\ \vdots & & & \\ g_{1n}(\boldsymbol{x}(t)) & g_{2n}(\boldsymbol{x}(t)) & \cdots & g_{mn}(\boldsymbol{x}(t))\end{bmatrix}\boldsymbol{u}(t)\\ =&\sum_{q=1}^{Q}v_q(\boldsymbol{x}(t))\begin{bmatrix}\sum_{i=1}^{\rho_1}h_{qi1}(\boldsymbol{x}(t))\boldsymbol{a}_{i1}\boldsymbol{K}_q\boldsymbol{x}(t)\\ \quad+\sum_{i=1}^{\sigma_1}w_{qi1}(\boldsymbol{x}(t))\boldsymbol{b}_{qi1}\boldsymbol{u}(t)\\ \sum_{i=1}^{\rho_2}h_{qi2}(\boldsymbol{x}(t))\boldsymbol{a}_{i2}\boldsymbol{K}_q\boldsymbol{x}(t)\\ \quad+\sum_{i=1}^{\sigma_2}w_{qi2}(\boldsymbol{x}(t))\boldsymbol{b}_{qi2}\boldsymbol{u}(t)\\ \vdots\\ \sum_{i=1}^{\rho_n}h_{qin}(\boldsymbol{x}(t))\boldsymbol{a}_{in}\boldsymbol{K}_q\boldsymbol{x}(t)\\ \quad+\sum_{i=1}^{\sigma_n}w_{qin}(\boldsymbol{x}(t))\boldsymbol{b}_{qin}\boldsymbol{u}(t)\end{bmatrix}\\ =&\sum_{q=1}^{Q}\sum_{i_1=1}^{\rho_1}\sum_{i_2=1}^{\rho_2}\cdots\sum_{i_n=1}^{\rho_n}\\ &\sum_{i_{(n+1)}=1}^{\sigma_1}\sum_{i_{(n+2)}=1}^{\sigma_2}\cdots\sum_{i_{(2n)}=1}^{\sigma_n}v_q(\boldsymbol{x}(t))h_{qi_1 1}(\boldsymbol{x}(t))\\ &\times h_{qi_2 2}(\boldsymbol{x}(t))\times\cdots\times h_{qi_n n}(\boldsymbol{x}(t))w_{qi_{(n+1)}1}(\boldsymbol{x}(t))\\ &\times w_{qi_{(n+2)}2}(\boldsymbol{x}(t))\times\cdots\times w_{qi_{(2n)}n}(\boldsymbol{x}(t))\\ &\times\left\{\begin{bmatrix}\boldsymbol{a}_{i_1 1}\\ \boldsymbol{a}_{i_2 2}\\ \vdots\\ \boldsymbol{a}_{i_n n}\end{bmatrix}\boldsymbol{K}_q\boldsymbol{x}(t)+\begin{bmatrix}\boldsymbol{b}_{qi_{(n+1)}1}\\ \boldsymbol{b}_{qi_{(n+2)}2}\\ \vdots\\ \boldsymbol{b}_{qi_{(2n)}n}\end{bmatrix}\boldsymbol{u}(t)\right\}\\ =&\sum_{q=1}^{Q}\sum_{i=1}^{r}v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t))\left(\boldsymbol{A}_{qi}\boldsymbol{x}(t)+\boldsymbol{B}_{qi}\boldsymbol{u}(t)\right)\end{aligned} \tag{13}$$

where

$$\rho_j = \begin{cases} 1, & f_j \text{ is linear} \\ 2, & f_j \text{ is nonlinear} \end{cases}$$

$$\beta_{ej} = \begin{cases} 1, & g_{ej} \text{ is constant} \\ 2, & g_{ej} \text{ is a function with respect to } \boldsymbol{x}(t) \end{cases}$$

$$\sigma_j = \prod_{e=1}^{m} \beta_{ej}$$

$$r = \rho_1 \times \rho_2 \times \cdots \times \rho_n \times \sigma_1 \times \sigma_2 \times \cdots \times \sigma_n$$

$$\boldsymbol{a}_{ij} = [a_{i1j}\ a_{i2j}\ \cdots\ a_{inj}],$$

$$\boldsymbol{b}_{qi_{(n+1)}1} = [b_{q1i_{(n+1)}1}\ b_{q2i_{(n+1)}1}\ \cdots\ b_{qmi_{(n+1)}1}]$$

$$\boldsymbol{b}_{qi_{(n+2)}2} = [b_{q1i_{(n+2)}2}\ b_{q2i_{(n+2)}2}\ \cdots\ b_{qmi_{(n+2)}2}]$$

$$\vdots$$

$$\boldsymbol{b}_{qi_{(2n)}n} = [b_{q1i_{(2n)}n}\ b_{q2i_{(2n)}n}\ \cdots\ b_{qmi_{(2n)}n}]$$

$$b_{qeij} = \begin{cases} \max\limits_{\boldsymbol{x}(t)\in R_q} g_{ej}(\boldsymbol{x}(t)), \\ \min\limits_{\boldsymbol{x}(t)\in R_q} g_{ej}(\boldsymbol{x}(t)), \end{cases}$$

$$\begin{aligned} &\sum_{i=1}^{r} \hat{h}_{qi}(\boldsymbol{x}(t)) \\ &:= \sum_{i_1=1}^{\rho_1}\sum_{i_2=1}^{\rho_2}\cdots\sum_{i_n=1}^{\rho_n}\sum_{i_{(n+1)}=1}^{\sigma_1}\sum_{i_{(n+2)}=1}^{\sigma_2}\cdots\sum_{i_{(2n)}=1}^{\sigma_n} \\ &\quad \times h_{qi_1 1}(\boldsymbol{x}(t)) h_{qi_2 2}(\boldsymbol{x}(t)) \times \cdots \times h_{qi_n n}(\boldsymbol{x}(t)) \\ &\quad \times w_{qi_{(n+1)}1}(\boldsymbol{x}(t)) w_{qi_{(n+2)}2}(\boldsymbol{x}(t)) \\ &\quad \times \cdots \times w_{qi_{(2n)}n}(\boldsymbol{x}(t)) \end{aligned}$$

r is the number of rules of the fuzzy model in each region.

C. Switching Fuzzy Controller Design

To stabilize the switching fuzzy model (13), we employ augmented system approach [9]. Consider the following stable linear system.

$$\dot{\hat{\boldsymbol{x}}}(t) = -\alpha \boldsymbol{I}_{2\Gamma}\hat{\boldsymbol{x}}(t) \tag{14}$$

where α is a positive constant, $\boldsymbol{I}_{2\Gamma}$ is an identity matrix, $\hat{\boldsymbol{x}}(t) = [\hat{x}_1(t)\ \hat{x}_2(t)\ \cdots\ \hat{x}_{2\Gamma}(t)]^T$ is a state vector for the linear system (14). By adding the stable linear system (14) to the switching fuzzy model (13), we construct the following augmented system.

$$\dot{\tilde{\boldsymbol{x}}}(t) = \sum_{q=1}^{Q}\sum_{i=1}^{r} v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t))\left(\tilde{\boldsymbol{A}}_{qi}\tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{B}}_{qi}\boldsymbol{u}(t)\right) \tag{15}$$

where

$$\begin{aligned} \tilde{\boldsymbol{x}}(t) &= [\boldsymbol{x}^T(t)\ \ \hat{\boldsymbol{x}}^T(t)]^T \\ &= [x_1(t)\ \cdots\ x_n(t)\ \hat{x}_1(t)\ \cdots\ \hat{x}_{2\Gamma}(t)]^T \end{aligned}$$

$$\tilde{\boldsymbol{A}}_{qi} = \begin{bmatrix} \boldsymbol{A}_{qi} & \boldsymbol{0} \\ \boldsymbol{0} & -\alpha\boldsymbol{I}_{2\Gamma} \end{bmatrix},\ \tilde{\boldsymbol{B}}_{qi} = \begin{bmatrix} \boldsymbol{B}_{qi} \\ \boldsymbol{0} \end{bmatrix}$$

By employing the so-called parallel distributed compensation (PDC) [2], [3], the switching fuzzy controller is represented as

$$\boldsymbol{u}(t) = -\sum_{q=1}^{Q}\sum_{i=1}^{r} v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t))\boldsymbol{F}_{qi}\boldsymbol{E}_q\tilde{\boldsymbol{x}}(t) \tag{16}$$

where $\boldsymbol{F}_{qi} \in R^{m\times(2\Gamma+n)}$ is a feedback gain and

$$\boldsymbol{E}_q = [\boldsymbol{K}_q\ \boldsymbol{K}_q^{\perp}] \tag{17}$$

$\boldsymbol{K}_q^{\perp}$ is the orthogonal complement of $\boldsymbol{K}_q$. Note that $\boldsymbol{E}_q$ becomes a nonsingular matrix because of the property of the orthogonal complement. The feedback gain $\boldsymbol{F}_{qi}$ can be determined by solving controller design conditions (18), (19) and (20) in Theorem 1.

Theorem 1: [9], [10] If there exist positive definite matrix $\boldsymbol{X} \in R^{(2\Gamma+n)\times(2\Gamma+n)}$ and $\boldsymbol{M}_{qi} \in R^{m\times(2\Gamma+n)}$ satisfying (18), (19) and (20) and the initial state is $\tilde{\boldsymbol{x}}(0) = [\boldsymbol{x}^T(0)\ \boldsymbol{0}^T]^T$, then the augmented system (15) can be stabilized by the switching fuzzy controller (16).

$$\boldsymbol{X} > \boldsymbol{0}, \tag{18}$$

$$\begin{aligned} &\boldsymbol{E}_q\tilde{\boldsymbol{A}}_{qi}\boldsymbol{E}_q^{-1}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{E}_q^{-T}\tilde{\boldsymbol{A}}_{qi}^T\boldsymbol{E}_q^T \\ &\quad -\boldsymbol{E}_q\tilde{\boldsymbol{B}}_{qi}\boldsymbol{M}_{qi} - \boldsymbol{M}_{qi}^T\tilde{\boldsymbol{B}}_{qi}^T\boldsymbol{E}_q^T < \boldsymbol{0},\ \forall i, q, \end{aligned} \tag{19}$$

$$\begin{aligned} &\boldsymbol{E}_q\tilde{\boldsymbol{A}}_{qi}\boldsymbol{E}_q^{-1}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{E}_q^{-T}\tilde{\boldsymbol{A}}_{qi}^T\boldsymbol{E}_q^T \\ &\quad +\boldsymbol{E}_q\tilde{\boldsymbol{A}}_{qj}\boldsymbol{E}_q^{-1}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{E}_q^{-T}\tilde{\boldsymbol{A}}_{qj}^T\boldsymbol{E}_q^T \\ &\quad -\boldsymbol{E}_q\tilde{\boldsymbol{B}}_{qi}\boldsymbol{M}_{qj} - \boldsymbol{M}_{qj}^T\tilde{\boldsymbol{B}}_{qi}^T\boldsymbol{E}_q^T \\ &\quad -\boldsymbol{E}_q\tilde{\boldsymbol{B}}_{qj}\boldsymbol{M}_{qi} - \boldsymbol{M}_{qi}^T\tilde{\boldsymbol{B}}_{qj}^T\boldsymbol{E}_q^T < \boldsymbol{0},\ \forall i, q, i < j, \end{aligned} \tag{20}$$

where $\boldsymbol{F}_{qi} = \boldsymbol{M}_{qi}\boldsymbol{X}^{-1}$.

III. Main Result

In our previous approach, we utilized simple and stable linear system (14) in order to derive LMI controller design condition. In this section, we derive controller design condition without determining stable system in advance by employing slack variable approach [13] based on the inverse use of elimination lemma [12].

Recall the switching fuzzy model (13).

$$\dot{\boldsymbol{x}}(t) = \sum_{q=1}^{Q}\sum_{i=1}^{r} v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t))\left(\boldsymbol{A}_{qi}\boldsymbol{x}(t) + \boldsymbol{B}_{qi}\boldsymbol{u}(t)\right) \tag{21}$$

For the above system, we consider the following additional nonlinear system instead of Eq. (14).

$$\dot{\hat{\boldsymbol{x}}}(t) = \sum_{q=1}^{Q}\sum_{i=1}^{r} v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t))\boldsymbol{C}_{qi}\hat{\boldsymbol{x}}(t) \tag{22}$$

where $\boldsymbol{C}_{qi}$ is a matrix variable which is determined with feedback gains of the fuzzy controller. Important property of this system is $\hat{\boldsymbol{x}}(t) = \boldsymbol{0}$ when $\hat{\boldsymbol{x}}(0) = \boldsymbol{0}$ even if $v_q(\boldsymbol{x}(t))$ and $\hat{h}_{qi}(\boldsymbol{x}(t))$ are not zero. By adding (22) to (21), we can obtain the following augmented system.

$$\begin{aligned} \begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\hat{\boldsymbol{x}}}(t) \end{bmatrix} &= \sum_{q=1}^{Q}\sum_{i=1}^{r} v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t)) \\ &\quad \times \left(\begin{bmatrix} \boldsymbol{A}_{qi} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}_{qi} \end{bmatrix}\begin{bmatrix} \boldsymbol{x}(t) \\ \hat{\boldsymbol{x}}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_{qi} \\ \boldsymbol{0} \end{bmatrix}\boldsymbol{u}(t)\right) \end{aligned} \tag{23}$$

To stabilize the augmented system (23), we employ the following switching fuzzy controller.

$$\boldsymbol{u}(t)=-\sum_{q=1}^{Q}\sum_{i=1}^{r}v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t))\left[\boldsymbol{F}_{1qi}\ \ \boldsymbol{F}_{2qi}\right]\left[\begin{array}{c}\boldsymbol{x}(t)\\ \hat{\boldsymbol{x}}(t)\end{array}\right] \tag{24}$$

where $\boldsymbol{F}_{1qi}$ and $\boldsymbol{F}_{2qi}$ are feedback gains. By substituting (24) into (23), we can obtain the following switching fuzzy control system.

$$\begin{aligned}
&\left[\begin{array}{c}\dot{\boldsymbol{x}}(t)\\ \dot{\hat{\boldsymbol{x}}}(t)\end{array}\right]\\
&=\sum_{q=1}^{Q}\sum_{i=1}^{r}\sum_{j=1}^{r}v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t))\hat{h}_{qj}(\boldsymbol{x}(t))\\
&\quad\times\left(\left[\begin{array}{cc}\boldsymbol{A}_{qi} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{C}_{qi}\end{array}\right]-\left[\begin{array}{c}\boldsymbol{B}_{qi}\\ \boldsymbol{0}\end{array}\right]\left[\boldsymbol{F}_{1qj}\ \ \boldsymbol{F}_{2qj}\right]\right)\left[\begin{array}{c}\boldsymbol{x}(t)\\ \hat{\boldsymbol{x}}(t)\end{array}\right]\\
&=\sum_{q=1}^{Q}\sum_{i=1}^{r}\sum_{j=1}^{r}v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t))\hat{h}_{qj}(\boldsymbol{x}(t))\\
&\quad\times\left(\left[\begin{array}{cc}\boldsymbol{A}_{qi} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{array}\right]\right.\\
&\qquad\left.-\left[\begin{array}{cc}\boldsymbol{B}_{qi} & \boldsymbol{0}\\ \boldsymbol{0} & -\boldsymbol{I}_{2\Gamma}\end{array}\right]\left[\begin{array}{cc}\boldsymbol{F}_{1qj} & \boldsymbol{F}_{2qj}\\ \boldsymbol{0} & \boldsymbol{C}_{qj}\end{array}\right]\right)\left[\begin{array}{c}\boldsymbol{x}(t)\\ \hat{\boldsymbol{x}}(t)\end{array}\right]\\
&=\sum_{q=1}^{Q}\sum_{i=1}^{r}\sum_{j=1}^{r}v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t))\hat{h}_{qj}(\boldsymbol{x}(t))\\
&\qquad\times\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qj}\right)\tilde{\boldsymbol{x}}(t).
\end{aligned} \tag{25}$$

where

$$\breve{\boldsymbol{A}}_{qi}=\left[\begin{array}{cc}\boldsymbol{A}_{qi} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{array}\right],\ \breve{\boldsymbol{B}}_{qi}=\left[\begin{array}{cc}\boldsymbol{B}_{qi} & \boldsymbol{0}\\ \boldsymbol{0} & -\boldsymbol{I}_{2\Gamma}\end{array}\right],$$
$$\breve{\boldsymbol{F}}_{qj}=\left[\begin{array}{cc}\boldsymbol{F}_{1qj} & \boldsymbol{F}_{2qj}\\ \boldsymbol{0} & \boldsymbol{C}_{qj}\end{array}\right]$$

The feedback gain $\breve{\boldsymbol{F}}_{qi}$ can be determined by solving controller design conditions (26), (27) and (28) in Theorem 2. Note that (26), (27) and (28) are represented in terms of LMIs. Hence we can effectively determine the feedback gain by computer software like MATLAB.

Theorem 2: If there exist positive definite matrix $\boldsymbol{X}\in R^{(2\Gamma+n)\times(2\Gamma+n)}$ and block upper triangular matrices $\boldsymbol{V}\in R^{(2\Gamma+n)\times(2\Gamma+n)}$ and $\breve{\boldsymbol{M}}_{qi}\in R^{m\times(2\Gamma+n)}$ satisfying (26), (27) and (28) and the initial state is $\tilde{\boldsymbol{x}}(0)=[\boldsymbol{x}^T(0)\ \boldsymbol{0}^T]^T$, then the augmented system (23) can be stabilized by the switching fuzzy controller (24).

$$\boldsymbol{X}>\boldsymbol{0}, \tag{26}$$

$$\left[\begin{array}{cc}\left(\begin{array}{c}\breve{\boldsymbol{A}}_{qi}\boldsymbol{V}+\boldsymbol{V}^T\breve{\boldsymbol{A}}_{qi}^T\\ -\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{M}}_{qi}-\breve{\boldsymbol{M}}_{qi}^T\breve{\boldsymbol{B}}_{qi}^T\end{array}\right) & *\\ \left(\begin{array}{c}\varepsilon\left(\breve{\boldsymbol{A}}_{qi}\boldsymbol{V}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{M}}_{qi}\right)^T\\ -\boldsymbol{V}+\boldsymbol{E}_q^{-1}\boldsymbol{X}\boldsymbol{E}_q^{-T}\end{array}\right) & -\varepsilon\boldsymbol{V}-\varepsilon\boldsymbol{V}^T\end{array}\right]<\boldsymbol{0} \tag{27}$$
$$\forall i,q$$

$$\left[\begin{array}{cc}\left(\begin{array}{c}\breve{\boldsymbol{A}}_{qi}\boldsymbol{V}+\boldsymbol{V}^T\breve{\boldsymbol{A}}_{qi}^T\\ \breve{\boldsymbol{A}}_{qj}\boldsymbol{V}+\boldsymbol{V}^T\breve{\boldsymbol{A}}_{qj}^T\\ -\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{M}}_{qj}-\breve{\boldsymbol{M}}_{qj}^T\breve{\boldsymbol{B}}_{qi}^T\\ -\breve{\boldsymbol{B}}_{qj}\breve{\boldsymbol{M}}_{qi}-\breve{\boldsymbol{M}}_{qi}^T\breve{\boldsymbol{B}}_{qj}^T\end{array}\right) & *\\ \left(\begin{array}{c}\varepsilon\left(\begin{array}{c}\breve{\boldsymbol{A}}_{qi}\boldsymbol{V}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{M}}_{qj}\\ \breve{\boldsymbol{A}}_{qj}\boldsymbol{V}-\breve{\boldsymbol{B}}_{qj}\breve{\boldsymbol{M}}_{qi}\end{array}\right)^T\\ -\boldsymbol{V}+\boldsymbol{E}_q^{-1}\boldsymbol{X}\boldsymbol{E}_q^{-T}\end{array}\right) & -\varepsilon\boldsymbol{V}-\varepsilon\boldsymbol{V}^T\end{array}\right]<\boldsymbol{0} \tag{28}$$
$$\forall i,q,i<j$$

where

$$\begin{aligned}
&\boldsymbol{V}=\left[\begin{array}{cc}\boldsymbol{V}_1 & \boldsymbol{V}_2\\ \boldsymbol{0} & \boldsymbol{V}_3\end{array}\right]\\
&\boldsymbol{V}_1\in R^{n\times n},\boldsymbol{V}_2\in R^{n\times 2\Gamma},\boldsymbol{V}_3\in R^{2\Gamma\times 2\Gamma}\\
&\breve{\boldsymbol{M}}_{qi}=\left[\begin{array}{cc}\boldsymbol{M}_{1qi} & \boldsymbol{M}_{2qi}\\ \boldsymbol{0} & \boldsymbol{M}_{3qi}\end{array}\right]\\
&\boldsymbol{M}_{1qi}\in R^{m\times n},\boldsymbol{M}_{2qi}\in R^{m\times 2\Gamma},\boldsymbol{M}_{3qi}\in R^{2\Gamma\times 2\Gamma}
\end{aligned}$$

ε is a line-search parameter [13]. The symbol $*$ denotes the transposed matrices for symmetric positions. $\breve{\boldsymbol{F}}_{qi}=\breve{\boldsymbol{M}}_{qi}\boldsymbol{V}^{-1}$.

[proof]

We consider the following switching Lyapunov function.

$$V(\tilde{\boldsymbol{x}}(t))=\begin{cases}\tilde{\boldsymbol{x}}^T(t)\boldsymbol{E}_1^T\boldsymbol{P}\boldsymbol{E}_1\tilde{\boldsymbol{x}}(t) & \boldsymbol{x}(t)\in R_1\\ \tilde{\boldsymbol{x}}^T(t)\boldsymbol{E}_2^T\boldsymbol{P}\boldsymbol{E}_2\tilde{\boldsymbol{x}}(t) & \boldsymbol{x}(t)\in R_2\\ \quad\vdots & \\ \tilde{\boldsymbol{x}}^T(t)\boldsymbol{E}_Q^T\boldsymbol{P}\boldsymbol{E}_Q\tilde{\boldsymbol{x}}(t) & \boldsymbol{x}(t)\in R_Q\end{cases} \tag{29}$$

To prove the theorem, we have to show $V(\tilde{\boldsymbol{x}}(t))>0$, $\dot{V}(\tilde{\boldsymbol{x}}(t))<0$ in each region and the continuity of $V(\tilde{\boldsymbol{x}}(t))$ on the region boundaries. $V(\tilde{\boldsymbol{x}}(t))>0$ and the continuity of the Lyapunov function on the region boundaries are explained in [9]. We show only $\dot{V}(\tilde{\boldsymbol{x}}(t))<0$. This proof focuses on the switching Lyapunov function in the qth region. The same technique can be applied to the other regions. The time derivative of (29) along the trajectories of the system is represented as follows:

$$\dot{V}(\tilde{\boldsymbol{x}}(t))=\dot{\tilde{\boldsymbol{x}}}^T(t)\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q\tilde{\boldsymbol{x}}(t)+\tilde{\boldsymbol{x}}^T(t)\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q\dot{\tilde{\boldsymbol{x}}}(t) \tag{30}$$

By substituting (25) into (30),

$$\begin{aligned}
\dot{V}(\tilde{\boldsymbol{x}}(t))\ &=\sum_{q=1}^{Q}\sum_{i=1}^{r}\sum_{j=1}^{r}v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t))\hat{h}_{qj}(\boldsymbol{x}(t))\\
&\quad\times\tilde{\boldsymbol{x}}^T(t)\left[\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qj}\right)^T\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q\right.\\
&\qquad\left.+\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qj}\right)\right]\tilde{\boldsymbol{x}}(t)\\
&=\sum_{q=1}^{Q}\sum_{i=1}^{r}v_q(\boldsymbol{x}(t))\hat{h}_{qi}^2(\boldsymbol{x}(t))\\
&\quad\times\tilde{\boldsymbol{x}}^T(t)\left[\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qi}\right)^T\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q\right.\\
&\qquad\left.+\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qi}\right)\right]\tilde{\boldsymbol{x}}(t)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{q=1}^{Q}\sum_{i=1}^{r}\sum_{i<j} v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t))\hat{h}_{qj}(\boldsymbol{x}(t)) \\
&\quad \times\tilde{\boldsymbol{x}}^T(t)\left[\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qj}\right)^T\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q\right. \\
&\qquad +\left(\breve{\boldsymbol{A}}_{qj}-\breve{\boldsymbol{B}}_{qj}\breve{\boldsymbol{F}}_{qi}\right)^T\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q \\
&\qquad +\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qj}\right) \\
&\qquad \left.+\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q\left(\breve{\boldsymbol{A}}_{qj}-\breve{\boldsymbol{B}}_{qj}\breve{\boldsymbol{F}}_{qi}\right)\right]\tilde{\boldsymbol{x}}(t)
\end{aligned}
$$

Therefore, $\dot{V}(\tilde{\boldsymbol{x}}(t)) < 0$ at $\tilde{\boldsymbol{x}}(t) \neq \mathbf{0}$ when the following conditions are satisfied.

$$
\begin{aligned}
&\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qi}\right)^T\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q \\
&\quad +\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qi}\right) < \mathbf{0},\ \forall q, i \qquad (31)\\
&\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qj}\right)^T\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q \\
&\quad +\left(\breve{\boldsymbol{A}}_{qj}-\breve{\boldsymbol{B}}_{qj}\breve{\boldsymbol{F}}_{qi}\right)^T\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q \\
&\quad +\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qj}\right) \\
&\quad +\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q\left(\breve{\boldsymbol{A}}_{qj}-\breve{\boldsymbol{B}}_{qj}\breve{\boldsymbol{F}}_{qi}\right) < \mathbf{0},\ \forall q, i,\ i<j \qquad (32)
\end{aligned}
$$

We focus on (31). By multiplying $\boldsymbol{E}_q^{-1}\boldsymbol{X}\boldsymbol{E}_q^{-T} = (\boldsymbol{E}_q^T\boldsymbol{P}\boldsymbol{E}_q)^{-1}$ on the left and right side, we can obtain the following inequality.

$$
\begin{aligned}
&\boldsymbol{E}_q^{-1}\boldsymbol{X}\boldsymbol{E}_q^{-T}\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qi}\right)^T \\
&\quad +\left(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qi}\right)\boldsymbol{E}_q^{-1}\boldsymbol{X}\boldsymbol{E}_q^{-T} < \mathbf{0} \qquad (33)
\end{aligned}
$$

By applying the inverse of elimination lemma [12] to (33) with $-2\varepsilon\boldsymbol{X} < \mathbf{0}$ where ε is a positive scalar variable, (33) is converted into the following form.

$$
\begin{aligned}
&\begin{bmatrix} \mathbf{0} & \boldsymbol{E}_q^{-1}\boldsymbol{X}\boldsymbol{E}_q^{-T} \\ \boldsymbol{E}_q^{-1}\boldsymbol{X}\boldsymbol{E}_q^{-T} & \mathbf{0} \end{bmatrix} \\
&+\begin{bmatrix} \breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qi} \\ -\boldsymbol{I} \end{bmatrix}\boldsymbol{V}\left[\boldsymbol{I}\ \varepsilon\boldsymbol{I}\right] \\
&+\begin{bmatrix} \boldsymbol{I} \\ \varepsilon\boldsymbol{I} \end{bmatrix}\boldsymbol{V}^T\left[(\breve{\boldsymbol{A}}_{qi}-\breve{\boldsymbol{B}}_{qi}\breve{\boldsymbol{F}}_{qi})^T\ \ -\boldsymbol{I}\right] < \mathbf{0}
\end{aligned}
$$

where $\boldsymbol{V}$ is a block upper triangular matrix. By defining $\breve{\boldsymbol{M}}_{qi} = \breve{\boldsymbol{F}}_{qi}\boldsymbol{V}$ and $\breve{\boldsymbol{M}}_{qi}$ is a block upper triangular matrix, we can obtain (27). Note that $\breve{\boldsymbol{F}}_{qi} = \breve{\boldsymbol{M}}_{qi}\boldsymbol{V}^{-1}$ becomes a block upper triangular matrix if $\breve{\boldsymbol{M}}_{qi}$ and $\boldsymbol{V}^{-1}$ are block upper triangular matrices. By applying same procedure to (32), we can obtain (28).

IV. Design Example

Consider the following nonlinear system [9], [10].

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} \\
&= \begin{bmatrix} x_2(t) \\ -x_1^3(t)-x_2^3(t)+5x_1^2(t)x_2(t)+5x_1(t)x_2^2(t) \\ -3x_1(t)x_2(t)-x_1(t)-x_2(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ -0.7+x_1(t)x_2(t) \end{bmatrix}u(t),
\end{aligned}
$$

$$
-d \le x_1 \le d,\ -d \le x_2 \le d.
$$

We select the following four dividing planes.

$$
\begin{aligned}
&x_2(t) = [0\ 1]\boldsymbol{x}(t) = \boldsymbol{\Lambda}_1\boldsymbol{x}(t) = 0, \\
&x_1(t)-x_2(t) = [1\ -1]\boldsymbol{x}(t) = \boldsymbol{\Lambda}_2\boldsymbol{x}(t) = 0, \\
&x_1(t) = [1\ 0]\boldsymbol{x}(t) = \boldsymbol{\Lambda}_3\boldsymbol{x}(t) = 0, \\
&x_1(t)+x_2(t) = [1\ 1]\boldsymbol{x}(t) = \boldsymbol{\Lambda}_4\boldsymbol{x}(t) = 0
\end{aligned}
$$

The state space is divided into the following eight regions by these dividing planes.

$$
\begin{aligned}
&R_1(1,1,1,1), \quad R_2(1,0,1,1) \\
&R_3(1,0,0,1), \quad R_4(1,0,0,0) \\
&R_5(0,0,0,0), \quad R_6(0,1,0,0) \\
&R_7(0,1,1,0), \quad R_8(0,1,1,1)
\end{aligned}
$$

The continuity matrices are obtained as follows:

$$
\begin{aligned}
\boldsymbol{K}_1 &= \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \\
\boldsymbol{K}_2 &= \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}^T \\
\boldsymbol{K}_3 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}^T \\
\boldsymbol{K}_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}^T \\
\boldsymbol{K}_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}^T \\
\boldsymbol{K}_6 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}^T \\
\boldsymbol{K}_7 &= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}^T \\
\boldsymbol{K}_8 &= \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T
\end{aligned}
$$

When $d = 0.836$, by applying the switching fuzzy model construction method described in Section II-A, the switching fuzzy model (13) is constructed as

$$
\dot{\boldsymbol{x}}(t)=\sum_{q=1}^{8}\sum_{i=1}^{4} v_q(\boldsymbol{x}(t))\hat{h}_{qi}(\boldsymbol{x}(t))\left(\boldsymbol{A}_{qi}\boldsymbol{x}(t)+\boldsymbol{B}_{qi}\boldsymbol{u}(t)\right) \quad (34)
$$

where

$$\boldsymbol{A}_{q1} = \boldsymbol{A}_{q3} = \begin{bmatrix} 0 & 1 \\ \multicolumn{2}{c}{\boldsymbol{a}_1 \boldsymbol{K}_q} \end{bmatrix}$$

$$\boldsymbol{A}_{q2} = \boldsymbol{A}_{q4} = \begin{bmatrix} 0 & 1 \\ \multicolumn{2}{c}{\boldsymbol{a}_2 \boldsymbol{K}_q} \end{bmatrix}$$

$$\boldsymbol{a}_1 = [1.900\ 0.101\ 2.220\ -1.221\ 1.299\\ \quad 0.0314\ 2.590\ 0.129\ 0.076\ -0.670]$$

$$\boldsymbol{a}_2 = [-3.731\ 0.991\ -4.48\ 1.701\ -4.425\\ \quad 0.241\ -3.265\ 0.884\ 0.0880\ 0.09]$$

$$\boldsymbol{B}_{11} = \boldsymbol{B}_{12} = \boldsymbol{B}_{21} = \boldsymbol{B}_{22}$$

$$= \boldsymbol{B}_{51} = \boldsymbol{B}_{52} = \boldsymbol{B}_{61} = \boldsymbol{B}_{72} = \begin{bmatrix} 0 \\ -1.301 \end{bmatrix}$$

$$\boldsymbol{B}_{13} = \boldsymbol{B}_{14} = \boldsymbol{B}_{23} = \boldsymbol{B}_{24}$$

$$= \boldsymbol{B}_{53} = \boldsymbol{B}_{54} = \boldsymbol{B}_{63} = \boldsymbol{B}_{74} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\boldsymbol{B}_{31} = \boldsymbol{B}_{32} = \boldsymbol{B}_{41} = \boldsymbol{B}_{42}$$

$$= \boldsymbol{B}_{71} = \boldsymbol{B}_{72} = \boldsymbol{B}_{81} = \boldsymbol{B}_{82} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\boldsymbol{B}_{33} = \boldsymbol{B}_{34} = \boldsymbol{B}_{43} = \boldsymbol{B}_{44}$$

$$= \boldsymbol{B}_{73} = \boldsymbol{B}_{74} = \boldsymbol{B}_{83} = \boldsymbol{B}_{84} = \begin{bmatrix} 0 \\ -2.699 \end{bmatrix}$$

The membership functions $\hat{h}_{qi}(\boldsymbol{x}(t))$ are omitted due to lack of space. By constructing the switching fuzzy control system (25) and solving Theorem 2 with $\varepsilon = 1$, the switching fuzzy controller (24) is designed. Figures 1 and 2 show the control result and the time trajectory of the switching Lyapunov function, where initial states are $\boldsymbol{x}(0) = [0.1 \quad -0.6]^T$. The designed controller can stabilize the nonlinear system (34) and the switching Lyapunov function continuously and monotonously decrease.

V. Conclusions

In this paper, we have derived improved LMI controller design condition for switching fuzzy model-based control by employing slack variable approach based on the inverse use of the elimination lemma. By utilizing the approach, we have shown that a system matrix of the additional stable system can be converted into a LMI variable. A design example has illustrated the utility of this approach.

Our future work is to apply this approach to real complicated systems.

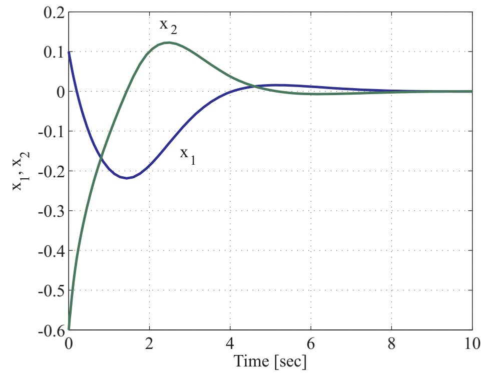

Fig. 1. Control result.

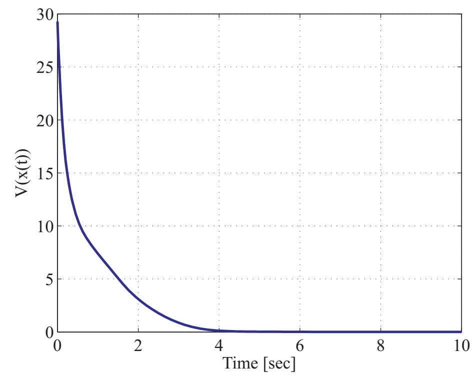

Fig. 2. Lyapunov function.

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