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Observer Design for Polynomial Systems with Bounded Disturbances

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Abstract— **Computational methods of filter and observer design are presented for a class of polynomial systems with $\mathcal{L}_2$-bounded disturbance via convex optimization. A measurement and estimated state dependent polynomial filter gain stabilizes the origin of the error dynamics in an invariant set. In addition to the stability of the error dynamics, a polynomial observer gain guarantees a stability of the origin of the closed-loop system in another invariant set for a given polynomial dependent estimated state feedback law. To compute the filter and observer gains and the invariant sets, matrix sum of squares relaxation and semidefinite programming are effectively applied. Numerical examples illustrate the design methods of the paper.**

I. Introduction

The availability of all the state for direct measurement is a rare occasion in practical feedback control systems. In most cases, there is a true need for a reliable estimation of unmeasurable state variables, especially when they are used for the synthesis of model-based controllers or for process monitoring purposes. For these purposes, a state observer is usually employed, in order to accurately reconstruct the unmeasurable state variables. There are several approaches in observer design for nonlinear systems. To discuss our results, we should mention three approaches in the literature [1], [2], [3], [4], [5], [6]. The first approach is based on a canonical form with linear techniques to design the observer [1] but this approach necessitates conservative conditions. The second approach is based on a decomposition into a linear part and a vector of nonlinear function for a class of nonlinear systems [2], [3], [4], [7], [8]. Observer gains have been designed for Lipschitz nonlinear systems [2], [4] and for a specific class of sector-bounded nonlinear systems [7], [8]. In particular, computational methods by using semidefinite programming (SDP) are discussed [4], [8]. This approach gives better performance than the first approach. However, almost observer gains are restricted to constant matrices. The third approach is high-gain observer [5], [6] which is based on output feedback stabilization to overcome a peaking phenomenon by saturating the control when a global stabilizable state feedback controller is given. From a viewpoint of output feedback stabilization, the third approach is quite different from other two approaches. Global stabilization of nonlinear systems is sometimes hard by state feedback control.

Recently, sum of squares (SOS) relaxation [9], [10] using SDP has been applied to state feedback controller design [11], [12], [13], [14] for polynomial control systems. One of the advantages to adopt such controller design is that local stabilization is liable to be achieved even if it is difficult to design a global stabilizable controller. A performance index is introduced to the controller design [13] and stabilization of the systems with bounded disturbances are discussed [11], [14]. Then invariant sets with respect to the closed-loop systems can be obtained. Observer design for polynomial systems using SOS relaxation has been proposed for global stabilization of the error dynamics [12]. Since problems of global stabilization of the error dynamics are hard to solve, we have proposed a convex formulation of locally stabilization of the error dynamics and closed-loop system with observer for given estimated state feedback law [15]. However, in [15], the observer gains have been limited to constant matrices , and class of the disturbance has not been specified. These issues may restrict the performances of observer.

In this paper, we propose an observer design method for polynomial systems by using SOS relaxation and SDP without any assumptions about nonlinear terms of the error dynamics except polynomially. To improve performance of observer design, we make the filter and observer gains depend on measurement and estimated state in polynomially. Then local stabilization of the error dynamics is studied on the basis of Lyapunov's stability theorem and invariance principal. Since unbounded disturbances may unstable the system locally, we consider bounded disturbances by $\mathcal{L}_2$ gain. In the paper, firstly, an observer design without control input, that is, a state estimator, will be discussed. Secondly, another observer design will be discussed for a given estimated state feedback law. It is assumed that the feedback law stabilizes the closed-loop system in an invariant set if we apply the law as state feedback control not as estimated state feedback. In each design, invariant sets of the error dynamics as well as the closed-loop system are obtained.

The paper is organized as follows. Section II describes observer problems we are interested in. A class of polynomial systems are clarified. Section III presents the observer design without control for polynomial systems, the filter design. The error dynamics is locally stabilized. A class of $\mathcal{L}_2$-disturbance are defined. Section IV presents the observer design with control for polynomial systems. The closed-loop system as well as the error dynamics is locally stabilized. A class of estimated state feedback law are described. Section V illustrates examples. Lastly, section VI concludes with remarks.

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Notation: The notation used is standard. For a vector $x \in \mathbb{R}^n$, $\mathbb{R}^{p\times q}$ means the real matrix of size $p \times q$, $\mathbb{S}^m$ means the symmetric matrix of size $m \times m$, $\mathbb{R}[x]$ means the ring of real polynomials of x, and $\Sigma[x]$ means the SOS polynomials of x. $X \succ (\succeq) 0$ means that a matrix $X \in \mathbb{S}^m$ is positive (semi)definite. $\mathbb{R}[x]^{p\times q}$ means the ring of real matrix polynomials of size $p\times q$, and $\Sigma[x]^m$ means the matrix SOS polynomials of size $m\times m$. For a vector $x \in \mathbb{R}^n$, $\|x\|_2 = (x^T x)^{\frac{1}{2}}$, and $\|x\|_{\mathcal{L}_2} = \left(\int_0^\infty \|x\|_2^2 dt\right)^{\frac{1}{2}}$.

II. Filtering and Observation

We first describe a filtering problem. Consider the following polynomial system:

$$\dot{x} = f(x) + B_w(x)w \tag{1}$$

$$y = Cx \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^r$ is the measurement, $w \in \mathcal{W}_{\mathcal{L}_2}(\beta)$ is the disturbance,

$$\mathcal{W}_{\mathcal{L}_2}(\beta) = \left\{ \ w(t) \in \mathbb{R}^{m_w} \ \mid \ \|w\|_{\mathcal{L}_2}^2 < \beta \ \right\} \tag{3}$$

$f(x) \in \mathbb{R}[x]^n$ with $f(0) = 0$, $B_w(x) \in \mathbb{R}[x]^{n\times m_w}$, and $C \in \mathbb{R}^{r\times n}$. A filter to estimate x from y will have the form

$$\dot{\hat{x}} = f(\hat{x}) + L(y, \hat{x})(y - C\hat{x}) \tag{4}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimated state, $L(y, \hat{x}) \in \mathbb{R}[y, \hat{x}]^{n\times r}$ is the filter gain matrix polynomial of y and $\hat{x}$ to be designed to ensure convergence of $\hat{x}$ to x. $L(y, \hat{x})$ is detailed with $\sum_{\pi\in\mathcal{F}} y^\rho \hat{x}^\sigma L_\pi$ where $y^\rho = y_1^{\rho_1} y_2^{\rho_2} \cdots y_r^{\rho_r}$ and $\hat{x}^\sigma = \hat{x}_1^{\sigma_1} \hat{x}_2^{\sigma_2} \cdots \hat{x}_n^{\sigma_n}$, $\pi = \{\rho, \sigma\}$, $L_\pi \in \mathbb{R}^{n\times r}$, and $\mathcal{F}$ is the set of indexes on monomials in y and $\hat{x}$, respectively. For example, in the case where the maximum degree of $L(y, \hat{x})$ is 2, $r = 1$, and $n = 2$, $\mathcal{F}$ is $\{000, 100, 010, 001, 200, 020, 002, 110, 011, 101\}$, $L(y, \hat{x})$ is

$$\begin{aligned} &L_{(000)} + yL_{(100)} + \hat{x}_1 L_{(010)} + \hat{x}_2 L_{(001)} \\ &+y^2 L_{(200)} + \hat{x}_1^2 L_{(020)} + \hat{x}_2^2 L_{(002)} \\ &+y\hat{x}_1 L_{(110)} + \hat{x}_1\hat{x}_2 L_{(011)} + y\hat{x}_2 L_{(101)}. \end{aligned}$$

To study the convergence and performance of this filter, we look at the dynamics of the estimation error defined by $e = x - \hat{x}$. The resulting error dynamics is

$$\dot{e} = f(x) - f(\hat{x}) - L(y, \hat{x})Ce + B_w(x)w. \tag{5}$$

The purpose here is to design a filter gain $L(y, \hat{x})$ to converge the estimation error e to zero. In general, it has been difficult to design the gain by constructive methods except a few special cases on the nonlinear terms of the right hand of (5) [5], [7]. We will discuss the filtering problem in section III under no assumptions about the nonlinear terms except polynomially.

Next we also describe an observer problem for the polynomial systems. Consider the system (1) with control inputs such as

$$\dot{x} = f(x) + B(x)u + B_w(x)w \tag{6}$$

where $u \in \mathbb{R}^{m_u}$ is the control input and other signals follow in (1). Since full measurement is not available, there is no choice than using state estimators. An observer to estimate x from y, $\hat{x}$ and u will have the form

$$\dot{\hat{x}} = f(\hat{x}) + B(\hat{x})u + L(y, \hat{x})(y - C\hat{x}) \tag{7}$$

and (2) where $L(y, \hat{x})$ is an observer gain to be designed to ensure convergence of $\hat{x}$ to x under some control inputs. To construct a closed-loop system, we need a given estimated state feedback such as

$$u = k(\hat{x}) \tag{8}$$

where $k(\hat{x}) \in \mathbb{R}[\hat{x}]$. We assume that the state feedback $u = k(x)$ stabilize the equilibrium point $x = 0$ of the system (6) and a positively invariant set with respect to the closed-loop system, $\dot{x} = f(x) + B(x)k(x)$, is known. The error dynamics is

$$\begin{aligned} \dot{e} = f(x) - f(\hat{x}) &+ (B(x) - B(\hat{x}))k(\hat{x}) \\ &- L(y, \hat{x})Ce + B_w(x)w. \end{aligned} \tag{9}$$

One of the purposes here is to design an observer gain $L(y, \hat{x})$ to converge the estimation error to zero with a given estimated state feedback. It has been difficult to design the observer gain for the same reason as the filter design. In addition to the purpose of regulating the error dynamics, another purpose will naturally appear. That is, the closed-loop system from (6), (2), (7), and (8) must be stabilized because a given estimated state feedback does not always stabilize the closed-loop system with an observer that is able to regulate the estimation error. This purpose is important from practical viewpoint of observer design. We will discuss the observer design problem in section IV.

III. Filter Design for Polynomial Systems

In this section, the error dynamics (5) is stabilized to estimate the state of the system (1). At first, we define two sets $\mathcal{X}_S$ of states and $\mathcal{X}_E$ of estimation errors:

$$\begin{aligned} \mathcal{X}_S &= \{ \ x \in \mathbb{R}^n \mid g_S(x) \geq 0, \ \ g_S(x) \in \mathbb{R}[x] \ \} \\ \mathcal{X}_E &= \{ \ e \in \mathbb{R}^n \mid g_E(e) \geq 0, \ \ g_E(e) \in \mathbb{R}[e] \ \} \end{aligned}$$

where $g_S(x)$ has the form $1 - x^T S_X x$ with $S_X \succ 0$ and $g_E(x)$ has the form $1 - e^T S_E e$ with $S_E \succ 0$. We consider the trajectories of x and e on these given sets.

A candidate Lyapunov function of the error dynamics is $V(e) = e^T S e$ where $S \in \mathbb{S}^n$. If the trajectories of e beginning from every point in

$$\mathcal{E}_e(S, \alpha) = \{ \ e \in \mathbb{R}^n \mid V(e) \leq \alpha \ \} \tag{10}$$

will remain in $\mathcal{E}_e(S, \alpha+\beta)$ where $\alpha > 0$, then $\mathcal{E}_e(S, \alpha+\beta)$ is an invariant set with respect to the error dynamics under a bounded disturbance $w \in \mathcal{W}_{\mathcal{L}_2}(\beta)$. Negativity of $\dot{V}(e) - \|w\|_2^2$ on $\mathcal{E}_e(S, \alpha + \beta)$ must be guaranteed to make the invariant set positive. Since we consider the error dynamics on $\mathcal{X}_E$, $\mathcal{E}_e(S, \alpha+\beta)$ should be inside of $\mathcal{X}_E$:

$$\mathcal{E}_e(S, \alpha+\beta) \subseteq \mathcal{X}_E. \tag{11}$$

Here we make two assumptions on initial point and the sets $\mathcal{X}_S$ and $\mathcal{X}_E$ for simplicity.

Assumption 1: $\hat{x}(0) = 0$ and $e(0) \in \mathcal{E}_e(S, \alpha)$.

Assumption 2: $\mathcal{X}_S$ and $\mathcal{X}_E$ satisfy $\mathcal{X}_S \subseteq \tilde{\mathcal{X}}_E$ where

$$\tilde{\mathcal{X}}_E = \{\ x \in \mathbb{R}^n \mid \mathcal{X}_E \text{ with } \hat{x} = 0\ \}.$$

From Assumption 1, $x(0)$ must satisfy $x(0)^T S x(0) \leq \alpha$. $\tilde{\mathcal{X}}_E$ is introduced into Assumption 2 for comparing the two sets $\mathcal{X}_S$ and $\mathcal{X}_E$. $\mathcal{X}_S \subseteq \tilde{\mathcal{X}}_E$ is a necessary condition of that $\mathcal{E}_x(S, \alpha) = \{\ x \in \mathbb{R}^n \mid x^T S x \leq \alpha\ \}$ is an admissible set to be able to put initial states $x(0)$.

Under these assumptions, we need a relation that the admissible set of $x(0)$ is a subset of $\mathcal{X}_S$ at least. If the following relation:

$$\mathcal{E}_x(S, \alpha + \beta) \subseteq \mathcal{X}_S \tag{12}$$

holds, then the admissible set becomes a subset of $\mathcal{X}_S$ because of $\mathcal{E}_x(S, \alpha) \subset \mathcal{E}_x(S, \alpha + \beta)$. From (12) and $\mathcal{X}_S \subseteq \tilde{\mathcal{X}}_E$ in Assumption 2, we have

$$\mathcal{E}_x(S, \alpha + \beta) \subset \tilde{\mathcal{X}}_E. \tag{13}$$

The above relation satisfies (11) numerically. That is, if (11) and (13) are reduced to LMI conditions, the LMI condition from (13) satisfies that from (11). Thus, if we take into account of (12), then we do not need (11) any more. The following theorem gives an answer to the filtering problem.

Theorem 1: Under Assumption 1 and 2, consider the system (1) and (2), and the filter (4). If there exist $S (> 0)$, $H(y, \hat{x}) \in \mathbb{R}[y, \hat{x}]^{n \times r}$, $\alpha (> 0)$, $s_{10}(x, \hat{x})$, $s_{11}(x, \hat{x})$, $s_{12}(x, \hat{x})$, $s_{21}(x, \hat{x})$, $s_{22}(x, \hat{x}) \in \Sigma[x, \hat{x}]$, $s_{30}(x)$, $s_{31}(x) \in \Sigma[x]$, $S_{20}(x, \hat{x}) \in \Sigma[x, \hat{x}]^{n+m_w}$ satisfying

$$\begin{aligned} V(e) - s_{11}(x, \hat{x}) g_S(x) - s_{12}(x, \hat{x}) g_E(e) \\ = s_{10}(x, \hat{x}) \quad \forall (x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n \end{aligned} \tag{14}$$

$$\begin{bmatrix} \mathcal{F}_2^{11}(x, \hat{x}) & (*)^T \\ B_w(x)^T S e & -I \end{bmatrix} = -S_{20}(x, \hat{x}) \quad \forall (x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n \tag{15}$$

$$s_{31}(x) g_S(x) - (\alpha + \beta - V(x)) = s_{30}(x) \quad \forall x \in \mathbb{R}^n \tag{16}$$

where $\mathcal{F}_2^{11}(x, \hat{x})$ and $h(x, \hat{x})$ are

$$h(x, \hat{x}) + s_{21}(x, \hat{x}) g_S(x) + s_{22}(x, \hat{x}) g_E(e)$$

and

$$2e^T \{S(f(x) - f(\hat{x})) - H(y, \hat{x}) C e\},$$

respectively, then a filter gain $L(y, \hat{x})$ is $S^{-1} H(y, \hat{x})$, and trajectories of the estimation error beginning from $\mathcal{E}_e(S, \alpha)$ remain in $\mathcal{E}_e(S, \alpha + \beta)$ and converge to zero.

Proof: If (14) holds, then $V(e) > 0$ for all $x \in \mathcal{X}_S$ and $e \in \mathcal{X}_E$ and thus $V(e)$ could be a candidate Lyapunov function of the error dynamics (5). Using Lemma 1, we can write that the derivative of $V(e)$ along (5) is

$$\begin{aligned} \dot{V}(e) &= h(x, \hat{x}) + 2e^T S B_w(x) w \\ &\leq h(x, \hat{x}) + w^T w + e^T S B_w(x) B_w(x)^T S e. \end{aligned}$$

If (15) holds, then $\dot{V}(e(\tau)) \leq \|w(\tau)\|_2^2$ for all $x \in \mathcal{X}_S$ and $e \in \mathcal{X}_E$. Integrating both side of the above inequality from $\tau = 0$ to t, we have a relation: $e(t)^T S e(t) \leq e(0)^T S e(0) + \|w(t)\|_{\mathcal{L}_2}^2$. From Assumption 1 and $w \in \mathcal{W}_{\mathcal{L}_2}(\beta)$, $e(t)^T S e(t) \leq \alpha + \beta$, $e(t) \in \mathcal{E}_e(S, \alpha + \beta)$. That is, $\mathcal{E}_e(S, \alpha + \beta)$ is a positively invariant set of the error dynamics, and thus the trajectories beginning from $\mathcal{E}_e(S, \alpha)$ remain in $\mathcal{E}_e(S, \alpha + \beta)$ and converge to zero. To be sure that $\mathcal{X}_S$ includes $x(0)$ and $\mathcal{X}_E$ includes $\mathcal{E}_e(S, \alpha + \beta)$, we need (16) that means (12). ■

Remark 1: To find a feasible solution of Theorem 1, the SOS techniques [9], [16], [17] are available for reducing the problem to SDP. See the appendix.

Remark 2: Although we restrict the state of the system (1) by $\mathcal{X}_S$, such an arbitrary restriction itself does *not* guarantee that trajectories of x will remain in $\mathcal{X}_S$. If a trajectory comes out from $\mathcal{X}_S$, then there is a possibility the error will not converges to zero even if the error is inside of $\mathcal{E}_e(S, \alpha + \beta)$ at the present time.

IV. Observer Design for Polynomial Systems

The closed-loop system from (6), (2), (7), and (8), as well as the error dynamics (9), is stabilized in this section by choosing an observer gain. We define a set of states in the closed-loop system and the error dynamics:

$$\mathcal{X}_F = \{\ (x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n \mid g_F(x, \hat{x}) \geq 0\ \}$$

where $g_F(x, \hat{x})$ has the form $1 + \beta - x^T P x - e^T S_E e$ with $P > 0$ and $S_E > 0$. Here we make an assumption on P and $k(\cdot)$.

Assumption 3: $\mathcal{X}_S = \{ x \in \mathbb{R}^n \mid x^T P x \leq 1 + \beta \}$ is a positively invariant set with respect to the closed-loop system *without* observer,

$$\dot{x} = f(x) + B(x) k(x) + B_w(x) w,\ w \in \mathcal{W}_{\mathcal{L}_2}(\beta).$$

That is, the trajectories beginning from $\{ x \in \mathbb{R}^n \mid x^T P x \leq 1 \}$ remain in $\mathcal{X}_S$ and converge to the origin.

Remark 3: A state feedback gain $k(x)$ satisfying Assumption 3 has been available in [18], [13], [14].

A candidate Lyapunov function of the closed-loop system is

$$U(x, \hat{x}) = (\alpha + \beta) x^T P x + e^T S e,\ \ \alpha > 0 \tag{17}$$

where $S \in \mathbb{S}^n$. We also make an assumption on initial point.

Assumption 4: $(x(0), \hat{x}(0)) \in \mathcal{X}_\Omega(\alpha)$ where

$$\mathcal{X}_\Omega(\alpha) = \{\ (x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n \mid U(x, \hat{x}) \leq \beta(\alpha + \beta) + \alpha\ \}.$$

If the trajectories $(x, \hat{x})$ beginning from every point in $\mathcal{X}_\Omega(\alpha)$ will remain in

$$\mathcal{X}_\Omega(\alpha + \beta) = \{\ (x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n \mid U(x, \hat{x}) \leq (1 + \beta)(\alpha + \beta)\ \}$$

then $\mathcal{X}_\Omega(\alpha + \beta)$ could be an invariant set with respect to the closed-loop system under bounded disturbances $w \in \mathcal{W}_{\mathcal{L}_2}(\beta)$. Negativity of $\dot{U}(x, \hat{x}) - \|w\|_2^2$ on $\mathcal{X}_\Omega$ must be guaranteed to be the invariant set positive.

On the other hand, we readopt $V(e) = e^T S e$ as a candidate Lyapunov function for the error dynamics (9). From the similar discussion in section III, we will guarantee that

$\dot{V}(e) - \|w\|_2^2$ is negative on $\mathcal{E}_e(S, \alpha+\beta)$ under Assumption 1 and 2. Let $g_S(x)$ and $g_E(e)$ be $1+\beta - x^T P x$ and $1 - e^T S_E e$. Then the following theorem gives an answer to the observer problem.

Theorem 2: Under Assumption 1, 2, 3, and 4, consider the system (6) and (2), the estimated state feedback (8) and the observer (7). If there exist $S(>0)$, $H(y,\hat{x}) \in \mathbb{R}[y,\hat{x}]^{n\times r}$, $\alpha(>0)$, $s_{10}(x,\hat{x})$, $s_{11}(x,\hat{x})$, $s_{12}(x,\hat{x})$, $s_{21}(x,\hat{x})$, $s_{22}(x,\hat{x})$, $s_{40}(x,\hat{x})$, $s_{41}(x,\hat{x})$, $s_{51}(x,\hat{x}) \in \sum[x,\hat{x}]$, $s_{30}(x)$, $s_{31}(x) \in \sum[x]$, $S_{20}(x,\hat{x})$, $S_{50}(x,\hat{x}) \in \sum[x,\hat{x}]^{n+m_w}$ satisfying (14), (16),

$$\begin{bmatrix} \bar{\mathcal{F}}_2^{11}(x,\hat{x}) & (*)^T \\ B_w(x)^T S e & -I \end{bmatrix} = -S_{20}(x,\hat{x}) \quad \forall (x,\hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n \tag{18}$$

$$U(x,\hat{x}) - s_{41}(x,\hat{x}) g_F(x,\hat{x}) = s_{40}(x,\hat{x}) \quad \forall (x,\hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n \tag{19}$$

$$\begin{bmatrix} \bar{\mathcal{F}}_5^{11}(x,\hat{x}) & (*)^T \\ B_w(x)^T((\alpha+\beta)Px + Se) & -I \end{bmatrix} = -S_{50}(x,\hat{x}) \quad \forall (x,\hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n \tag{20}$$

where $\bar{\mathcal{F}}_2^{11}(x,\hat{x})$, $h_K(x,\hat{x})$, $\bar{\mathcal{F}}_5^{11}(x,\hat{x})$, and $h_F(x,\hat{x})$ are

$$\begin{aligned} &h_K(x,\hat{x}) + s_{21}(x,\hat{x}) g_S(x) + s_{22}(x,\hat{x}) g_E(e), \\ &h(x,\hat{x}) + 2e^T S(B(x) - B(\hat{x}))k(\hat{x}), \\ &h_F(x,\hat{x}) + s_{51}(x,\hat{x}) g_F(x) \end{aligned}$$

and

$$2\begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} (\alpha+\beta)P(f(x) + B(x)k(\hat{x})) \\ \begin{pmatrix} S(f(x) - f(\hat{x})) \\ +S(B(x) - B(\hat{x})) - H(y,\hat{x})Ce \end{pmatrix} \end{bmatrix},$$

respectively, then an observer gain $L(y,\hat{x})$ is $S^{-1}H(y,\hat{x})$, and trajectories of the estimation error beginning from $\mathcal{E}_e(S,\alpha)$ remain in $\mathcal{E}_e(S,\alpha+\beta)$ and converge to zero, and trajectories of the closed-loop system beginning from $\mathcal{X}_\Omega(\alpha)$ remain in $\mathcal{X}_\Omega(\alpha+\beta)$ and converge to zero.

Proof: It is required to show that both the error dynamics and the closed-loop system have positively invariant set, respectively. In a similar way in the proof of Theorem 1, $\mathcal{E}_e(S,\alpha+\beta)$ is a positively invariant set of the error dynamics (9) from (14), (16), and (18). Note that (16) means (11). On the other hand, if (19) holds, then $U(e) > 0$ for all $(x,\hat{x}) \in \mathcal{X}_F$ and thus $U(e)$ could be a candidate Lyapunov function of the closed-loop system. Using Lemma 1, the derivative of $U(x,\hat{x})$ along the closed-loop system is

$$\begin{aligned} \dot{U}(x,\hat{x}) &= h_F(x,\hat{x}) + 2[(\alpha+\beta)x^T P + e^T S]B_w(x)w \\ &\le h_F(x,\hat{x}) + w^T w \\ &\quad +[(\alpha+\beta)x^T P + e^T S]B_w(x)B_w(x)^T[(*)]^T. \end{aligned}$$

If (20) holds, then $\dot{U}(x,\hat{x}) \le \|w\|_2^2$ for all $(x,\hat{x}) \in \mathcal{X}_F$. From Assumption 4 and $w \in \mathcal{W}_{\mathcal{L}_2}(\beta)$, we can write that $U(x,\hat{x}) \le \beta(\alpha+\beta) + \alpha + \beta = (1+\beta)(\alpha+\beta)$. That is, $\mathcal{X}_\Omega(\alpha+\beta)$ is a positively invariant set of the closed-loop system, and thus the trajectories beginning from $\mathcal{X}_\Omega(\alpha)$ remain in $\mathcal{X}_\Omega(\alpha+\beta)$ and converge to zero. The remainder is to show that $\mathcal{X}_F$ includes $\mathcal{X}_\Omega(\alpha+\beta)$. From the relation (11), we can write that

$$\begin{aligned} &\left(x^T P x + e^T S e/(\alpha+\beta)\right) - \left(x^T P x + e^T S_E e\right) \\ = \;& e^T\left(S/(\alpha+\beta) - S_E\right)e \ge 0. \end{aligned}$$

This relation means $\mathcal{X}_\Omega(\alpha+\beta) \subseteq \mathcal{X}_F$. ∎

Remark 4: An admissible region to be able to put the initial state $x(0)$ to converge both the error dynamics and the closed-loop systems is the intersection of the two ellipsoids $\mathcal{E}(S,\alpha)$ and $\mathcal{X}_\Omega(\alpha)$, that is, $x(0)$ must satisfy $V(x(0)) \le \alpha$ and $U(x(0),0) \le \beta(\alpha+\beta) + \alpha$.

Remark 5: In case the error is zero, we may expect the estimated state feedback recovers the performance as the same level as the full measurement feedback. If the error is occurred, the invariant set $\mathcal{X}_\Omega(\alpha+\beta)$ becomes smaller than $\mathcal{X}_S$ of the invariant set of the full measurement feedback.

V. Numerical Examples

We show two examples to illustrate the results of the previous sections. The examples are computed by using Matlab, YALMIP [19] and SeDuMi [20].

Example 1: The Van der Pol oscillator, the polynomial system (1) and (2) with

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 + x_2 - x_1^2 x_2 \end{bmatrix}, \quad B_w(x) = \begin{bmatrix} 0 \\ 0.5x_1x_2 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

illustrates the filter design presented in Theorem 1. Both two matrices S_X in the set $\mathcal{X}_S$ and S_E in the sets $\mathcal{X}_E$ are chosen as $\mathrm{diag}\begin{bmatrix} 1 & 1 \end{bmatrix}/10^2$, and β is given by 10. The highest degree of $L(y,\hat{x})$ is fixed at 2.

The resulting variables of a feasible solution are

$$\alpha = 48.1173, \quad S = \begin{bmatrix} 13.6473 & 1.1770 \\ 1.1770 & 0.7097 \end{bmatrix}$$

and

$$\begin{aligned} L(y,\hat{x}) = &\begin{bmatrix} -0.9041 \\ 87.7033 \end{bmatrix} + \begin{bmatrix} 0.0000 \\ -0.0001 \end{bmatrix} y + \begin{bmatrix} 0.0000 \\ 0.0001 \end{bmatrix} \hat{x}_1 \\ &+ \begin{bmatrix} 0.0000 \\ 0.0004 \end{bmatrix} \hat{x}_2 + \begin{bmatrix} 1.3759 \\ 24.9577 \end{bmatrix} y^2 + \begin{bmatrix} -0.0197 \\ 7.5759 \end{bmatrix} \hat{x}_1^2 \\ &+ \begin{bmatrix} 0.2344 \\ 4.6782 \end{bmatrix} \hat{x}_2^2 + \begin{bmatrix} -0.5009 \\ 5.3669 \end{bmatrix} y\hat{x}_1 + \begin{bmatrix} 0.0640 \\ 0.2783 \end{bmatrix} \hat{x}_1\hat{x}_2 \\ &+ \begin{bmatrix} -0.6079 \\ 4.5145 \end{bmatrix} y\hat{x}_2. \end{aligned}$$

A trajectory of the estimated state and the invariant set of the error dynamics are shown in Fig. 1. In some cases where highest degree of $L(y,\hat{x})$ is smaller than 2 for y and $\hat{x}$, problems could be infeasible or coefficient matrices of $L(y,\hat{x})$ tend to be very large.

Example 2: The polynomial system (6) and (2) with

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 - x_2 + x_2^3/10 \end{bmatrix}, \quad B(x) = \begin{bmatrix} 0 \\ 1 + x_1/20 \end{bmatrix}$$
$$B_w(x) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \quad C = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$$

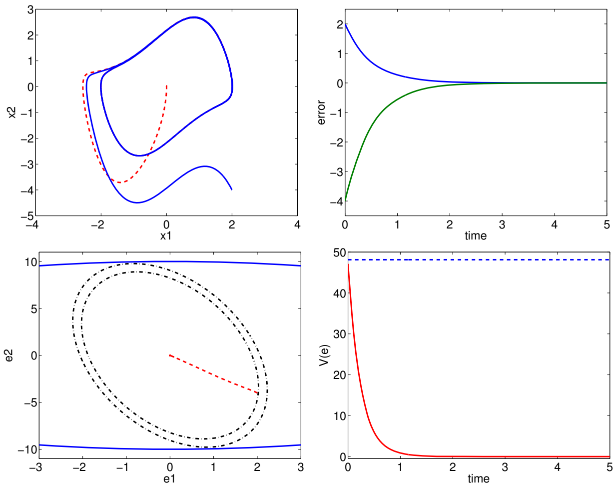

Fig. 1. Trajectory of estimated state and error, and invariant set: In *upper left*, solid curve shows the state trajectory of the oscillator from $x(0) = [2 \ \ -4]^T$, and dashed curve shows the estimated state trajectory from $\hat{x}(0) = [0 \ \ 0]^T$. In *upper right*, time history of the error $e(t)$ is shown. In *lower left*, dashed curve shows the error trajectory from $e(0)$, solid ellipsoid shows $\mathcal{X}_E$, outer chain ellipsoid shows $\mathcal{E}_e(S, \alpha + \beta)$, and inner chain ellipsoid shows $\mathcal{E}_e(S, \alpha)$. In *lower right*, solid curve shows time history of $V(e(t))$, and dashed line shows α.

illustrates the observer design presented in Theorem 2. A estimated state feedback gain $k(\hat{x})$ is given by

$$\begin{aligned}&-0.012139x_1 - 0.042499x_2 - 0.0015438x_2^2\\ &\quad +0.0016002x_1^2 + 0.0072906x_1x_2 - 0.022431x_1^3\\ &\quad -0.13984x_1^2x_2 - 0.014513x_1x_2^2 - 0.080659x_2^3.\end{aligned}$$

Both two matrices S_X and S_E are given by

$$\begin{bmatrix} 0.0316 & 0.0035 \\ 0.0035 & 0.0316 \end{bmatrix}$$

and β is given by 1. The highest degree of $L(y, \hat{x})$ is fixed at 2.

The resulting variables of a feasible solution are

$$\alpha = 4.0967, \quad S = \begin{bmatrix} 0.8362 & 0.8479 \\ 0.8479 & 1.0356 \end{bmatrix}$$

and

$$\begin{aligned}L(y, \hat{x}) &= \begin{bmatrix} -196.9951 \\ 245.9035 \end{bmatrix} + \begin{bmatrix} -0.1948 \\ 0.1656 \end{bmatrix} y + \begin{bmatrix} 0.0288 \\ -0.0384 \end{bmatrix} \hat{x}_1\\ &+ \begin{bmatrix} -0.1634 \\ 0.1394 \end{bmatrix} \hat{x}_2 + \begin{bmatrix} -214.7819 \\ 266.2865 \end{bmatrix} y^2 + \begin{bmatrix} -206.5736 \\ 261.4746 \end{bmatrix} \hat{x}_1^2\\ &+ \begin{bmatrix} -211.0425 \\ 265.1378 \end{bmatrix} \hat{x}_2^2 + \begin{bmatrix} -73.2845 \\ 85.6688 \end{bmatrix} y\hat{x}_1 + \begin{bmatrix} 59.0844 \\ -66.1435 \end{bmatrix} \hat{x}_1\hat{x}_2\\ &+ \begin{bmatrix} -124.5001 \\ 155.7589 \end{bmatrix} y\hat{x}_2.\end{aligned}$$

A trajectory of the error dynamics and the state of the closed-loop system are shown in Fig. 2.

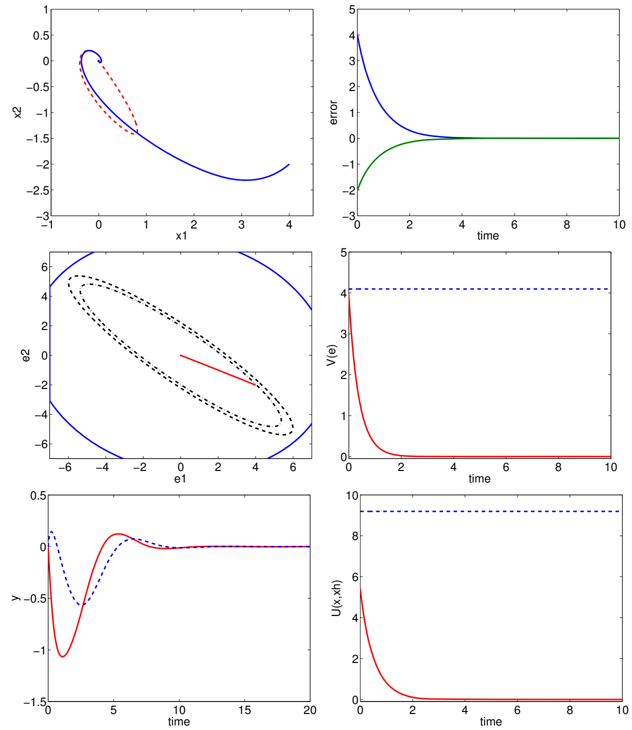

Fig. 2. Trajectories of estimated state and error, and invariant sets: In *upper left*, solid curve shows the state trajectory of the system from $x(0) = [4 \ \ -2]^T$, and dashed curve shows the estimated state trajectory from $\hat{x}(0) = [0 \ \ 0]^T$. In *upper right*, time history of error $e(t)$ is shown. In *middle left*, dashed curve shows the error trajectory from $e(0)$, solid ellipsoid shows $\mathcal{X}_E$, outer chain ellipsoid shows $\mathcal{E}_e(S, \alpha + \beta)$, and inner chain ellipsoid shows $\mathcal{E}_e(S, \alpha)$. In *middle right*, solid curve shows time history of $V(e(t))$, and dashed line shows α. In *lower left*, solid curve shows time history of y, and dashed curve also shows time history of y in the case of full measurement feedback. In *lower right*, solid curve shows time history of $U(x(t), \hat{x}(t))$, and dashed line shows $\beta(\alpha + \beta) + \alpha$.

VI. Conclusions

The filter and observer design were proposed for the polynomial systems with $\mathcal{L}_2$-bounded disturbance using the sum of squares relaxations. The origin of the error dynamics was locally stabilized by a measurement and state estimate dependent polynomial filter/observer gain, and positively invariant sets were obtained. The origin of the closed-loop system was also locally stabilized assuming that a given estimated state feedback law for the bounded disturbance is able to stabilize an another closed-loop system when the law was applied to as state feedback not estimated state feedback, and a positively invariant set of the observer-based closed-loop system was also obtained. The both design problems were reduced to the convex conditions.

Appendix

Lemma 1: For any $\eta(x, \hat{x}) \in \Sigma[x, \hat{x}]$,

$$\begin{aligned}2\zeta(x, \hat{x})B_w(x)w &\le \eta(x, \hat{x})w^T w\\ &\quad + \eta(x, \hat{x})^{-1}\zeta(x, \hat{x})^T B_w(x)B_w(x)^T \zeta(x, \hat{x})\end{aligned}$$

holds for all $(x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n$ and $w \in \mathbb{R}^{m_w}$.

Background of the SOS techniques

Let us consider a symmetric matrix polynomial $F(x) = \sum_{\alpha\in\mathcal{F}} F_\alpha x^\alpha \in \mathbb{S}[x]^m$, where $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $\mathcal{F} = \{\alpha \in \mathbb{R}^n \big| \sum_{i=1}^n \alpha_i \le 2N\}$. For some polynomial $G(x) \in \mathbb{R}[x]^{q\times m}$, $F(x) = G(x)^T G(x)$ is a matrix SOS polynomial [17], or $F(x) \in \Sigma[x]^m$, where $G(x) \in \mathbb{R}[x]^{k\times m}$.

Lemma 2 ([17]): $F(x) \in \mathbb{S}[x]^m$ is a matrix SOS polynomial if and only if there exists $Q \in \mathbb{S}_+^{md_{[N]}}$ satisfying

$$F(x) = (I_m \otimes z_{[N]})^T Q (I_m \otimes z_{[N]}) \quad \forall x \in \mathbb{R}^n, \tag{21}$$

where $z_{[N]} = z_{[N]}(x) \in \mathbb{R}^{d_{[N]}}$ is a monomial vector of x whose highest degree is N.

Lemma 3 ([17]): There exists $Q \in \mathbb{S}_+^{md_{[N]}}$ satisfying (21) if and only if there also exists it satisfying

$$< (I_m \otimes A_\alpha), Q >_m = F_\alpha \quad \forall \alpha \in \mathcal{F}, \tag{22}$$

where $A_\alpha \in \mathbb{S}^{d_{[N]}}$ satisfies $z_{[N]} z_{[N]}^T = \sum_{\alpha\in\mathcal{F}} A_\alpha x^\alpha$.

The relationship between the above facts is as follows:

$$\begin{aligned} F(x) \in \mathbb{S}_+[x]^m \quad &\Leftarrow \quad F(x) \in \Sigma[x]^m \\ &\Leftrightarrow \quad \exists Q \in \mathbb{S}_+^{md_{[N]}} \text{ s.t. } (21) \\ &\Leftrightarrow \quad \exists Q \in \mathbb{S}_+^{md_{[N]}} \text{ s.t. } (22) \end{aligned}$$

If $F(x)$ is SOS, then $F(x)$ is positive semidefinite for all $x \in \mathbb{R}^n$. All real matrix polynomials having the representation $F(x) = G(x)^T G(x)$ is SOS, which is equivalently written with a matrix quadratic form (21). Finding Q in (21) is the certification of $F(x)$ being a matrix SOS polynomial. The certification is reduced to solving the SDP (22).

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