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Describing Function Analysis of Dahl Model Friction

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Abstract— **Mechanical systems subject to friction hysteresis exhibit amplitude dependent frequency responses. This paper derives the describing function (DF) of the Dahl friction model subject to a sinusoidal input. This DF is useful for analysis and modeling of systems subject to friction hysteresis from frequency response data.**

I. INTRODUCTION

The use of frequency response data of mechanical systems for analysis, modeling, and controller design has been common practice for decades. While frequency response techniques are most useful for linear systems, the DF (amplitude dependent frequency response at the excitation frequency) can be used for some systems with nonlinear components [9], [10]. The Fourier series representation of the periodic output of the nonlinearity subject to a sinusoidal input as a function of the frequency and input amplitude enables researchers to relate analytical DF results to models typically examined in the time domain.

There are many models of hysteretic friction, [2], [3], [4], [5], [8]. The Dahl friction model [6] is one of the most popular because of its simplicity and its ability to capture much of the behavior of hysteretic friction observed in practice. The Dahl model lacks the nonlocal memory property of friction, but the exact Fourier series representation of other models does not currently exist in the literature [2]. The objective of this paper is to derive an analytical expression for the DF of the Dahl friction model for use in identifying and understanding friction in the frequency domain.

This paper is organized as follows. Section II introduces the Dahl model and derives its Fourier series. Section III illustrates the Fourier components modeling the linear Dahl model, and it shows the DFs for several different combinations of the parameters of the linear Dahl model. Section IV provides concluding remarks.

II. FOURIER SERIES OF THE DAHL MODEL

To derive the DF of the Dahl friction model, we determine the Fourier series represenation

$$F_{hys}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t) \qquad (1)$$

where F_{hys} is the output of Dahl dynamic model subject to a sinusoidal input. The Dahl model consists of the following

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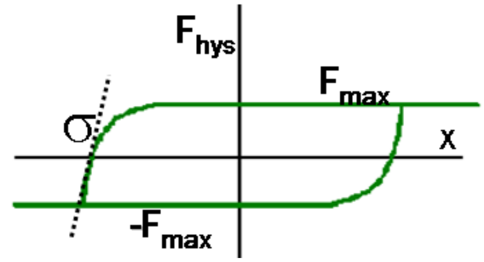

Fig. 1. Linear Dahl model in the displacement domain with F_{max} and σ defined pictorially.

nonlinear first order differential equation in the displacement variable x

$$\frac{dF_{hys}}{dx} = \sigma \left(1 - \frac{F_{hys}}{F_{max}} \mathrm{sgn}(\dot{x})\right)^{\alpha} \qquad (2)$$

where F_{max} is the maximum value of the friction. This paper employs a frequently used simplification of this model where the exponent $\alpha = 1$. This simplification is called the linear Dahl model. For the linear Dahl model, the solution is exponential with respect to position, and σ represents the slope of the friction with respect to position at the zero crossing, as shown in Figure 1.

A. Time Domain Solution of the Dahl Model

This derivation of the Fourier coefficients starts with the output of the Dahl model as a function of time. Integrating the differential equation of the Dahl model with respect to x in one direction gives the hysteresis force as a function of x. Substituting $x(t)$ for x gives the solution to the Dahl model as a function of time as long as the sign of the velocity does not change.

$$\begin{aligned} F_{hys}(t) &= \mathrm{sgn}(\dot{x}(t)) \Bigg(-F_0 + \\ &\quad + (F_{max} + F_0)\left(1 - e^{-\frac{\sigma}{F_{max}}|x(t)-x_0|}\right)\Bigg) \end{aligned} \qquad (3)$$

where x_0 is the initial position and F_0 is the hysteresis force at that position.

For the purposes of determining the Fourier series, we assume the position $x(t)$ is given by

$$x(t) = A\cos(\omega t) \qquad (4)$$

The variable x_0 represents the position at the time of direction reversals. During steady state oscillations

$$x_0 = -\mathrm{sgn}(\dot{x}(t))A = \begin{cases} -A & \dot{x} > 0 \\ A & \dot{x} \le 0 \end{cases} \qquad (5)$$

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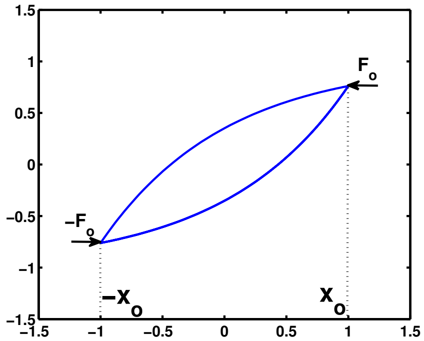

Fig. 2. F_0 and $-F_0$ on a hysteresis loop where $F_{max}=1$, $\sigma=1$, and $A=1$

If F_0 is the friction at $x(t)=A$, the point where $\dot{x}(t)$ changes from positive to negative, then $-F_0$ is the friction value when $x(t)=-a$, the point where $\dot{x}(t)$ changes from negative to positive during steady state oscillations. Figure 2 shows F_0 and $-F_0$ on a hysteresis loop where $F_{max}=1$, $\sigma=1$, and $A=1$.

Taking $x_0=-A$ with corresponding friction value $-F_0$ in Eqn. 3, the following relation holds when $x(t)=A$

$$F_0=-F_0+(F_{max}+F_0)\left(1-e^{-2\frac{\sigma}{F_{max}}A}\right) \tag{6}$$

At this point it is convenient to define the constant β to simplify notation

$$\beta \triangleq \frac{\sigma A}{F_{max}} \tag{7}$$

Solving for F_0 gives

$$F_0=\frac{F_{max}\left(1-e^{-2\beta}\right)}{\left(1+e^{-2\beta}\right)}=F_{max}\tanh(\beta) \tag{8}$$

Substituting back into the Dahl model Eqn. 3

$$\begin{aligned} F_{hys}(t) \quad = \quad & \operatorname{sgn}(\dot{x}(t))\Bigg(-F_{max}+F_{max}(1+\tanh(\beta))* \\ & *e^{-\frac{\sigma}{F_{max}}|x(t)+A\operatorname{sgn}(\dot{x}(t))|}\Bigg) \end{aligned} \tag{9}$$

When the velocity $\dot{x}(t)$ is positive, $x(t)+A\operatorname{sgn}(\dot{x}(t))>0$ the solution is

$$F_{hys}(t)=F_{max}-F_{max}(1+\tanh(\beta))e^{-\frac{\sigma}{F_{max}}(x(t)+A)} \tag{10}$$

When the velocity $\dot{x}(t)$ is negative, $x(t)+A\operatorname{sgn}(\dot{x}(t))<0$ then

$$F_{hys}(t)=-F_{max}+F_{max}(1+\tanh(\beta))e^{\frac{\sigma}{F_{max}}(x(t)-A)} \tag{11}$$

B. Cosine Terms

The Fourier coefficients for the cosine and sine are

$$a_0 \quad = \quad \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F_{hys}(t)dt \tag{12}$$

$$a_k \quad = \quad \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F_{hys}(t)\cos(k\omega t)dt \tag{13}$$

$$b_k \quad = \quad \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} F_{hys}(t)\sin(k\omega t)dt \tag{14}$$

where $T=\frac{2\pi}{\omega}$ and k is an integer greater than zero. This section presents the derivation of the cosine coefficients a_k.

The symmetry of $F_{hys}(t)$ over one period implies that $a_0=0$. Determining the other cosine coefficients, a_k, begins with the substitution of F_{hys} into Eqn. 13. For clarity the integral is split into the periods based on the change in the sign of the velocity allowing direct substitution of Eqn. 10 and 11.

$$\begin{aligned} a_k = \frac{2F_{max}}{T}\Bigg[&\int_{-\frac{T}{2}}^{0}\left(1-(1+\tanh(\beta))e^{-\frac{\sigma}{F_{max}}(x(t)+A)}\right)* \\ &*\cos(k\omega t)dt+ \\ &+\int_{0}^{\frac{T}{2}}\left(-1+(1+\tanh(\beta))e^{\frac{\sigma}{F_{max}}(x(t)-A)}\right)* \\ &*\cos(k\omega t)dt\Bigg] \end{aligned} \tag{15}$$

The constant terms cancel, leaving

$$\begin{aligned} a_k = &\frac{2F_{max}}{T}\Bigg[\int_{0}^{\frac{T}{2}}(1+\tanh(\beta))e^{-\beta}e^{\frac{\sigma}{F_{max}}x(t)}\cos(k\omega t)dt+ \\ &-\int_{-\frac{T}{2}}^{0}(1+\tanh(\beta))e^{-\beta}e^{-\frac{\sigma}{F_{max}}x(t)}\cos(k\omega t)dt\Bigg] \end{aligned} \tag{16}$$

To further simplify the notation we define the constant

$$\gamma \triangleq \frac{2F_{max}}{\pi}(1+\tanh(\beta))e^{-\beta} \tag{17}$$

Eqn. 16 becomes

$$\begin{aligned} a_k = \frac{\omega\gamma}{2}\Bigg[&\int_{0}^{\frac{T}{2}} e^{\frac{\sigma}{F_{max}}x}\cos(k\omega t)dt+ \\ &-\int_{-\frac{T}{2}}^{0} e^{-\frac{\sigma}{F_{max}}x}\cos(k\omega t)dt\Bigg] \end{aligned} \tag{18}$$

The limits of integration are further subdivided at $\frac{T}{4}$ and $-\frac{T}{4}$ to properly account for the zero crossings of $\cos(\omega t)$. Substituting in $x(t)=A\cos(\omega t)$ and using the definition of β gives.

$$\begin{aligned} a_k \quad = \quad \frac{\omega\gamma}{2}\Bigg[&\int_{0}^{\frac{T}{4}} e^{\beta\cos(\omega t)}\cos(k\omega t)dt+ \\ &+\int_{\frac{T}{4}}^{\frac{T}{2}} e^{\beta\cos(\omega t)}\cos(k\omega t)dt+ \\ &-\int_{-\frac{T}{2}}^{-\frac{T}{4}} e^{-\beta\cos(\omega t)}\cos(k\omega t)dt+ \\ &-\int_{-\frac{T}{4}}^{0} e^{-\beta\cos(\omega t)}\cos(k\omega t)dt\Bigg] \end{aligned} \tag{19}$$

Substitution of $u=t+\frac{T}{2}$ in the third and fourth integrals above and noting that $\frac{T\omega}{2}=\pi$ and $\cos(\zeta-\pi)=-\cos(\zeta)$, gives

$$a_k = \frac{\omega\gamma}{2}\left[\int_0^{\frac{T}{4}} e^{\beta\cos(\omega t)}\cos(k\omega t)dt + \right.$$
$$+\int_{\frac{T}{4}}^{\frac{T}{2}} e^{\beta\cos(\omega t)}\cos(k\omega t)dt +$$
$$-\int_0^{\frac{T}{4}} e^{\beta\cos(\omega u)}\cos(\omega ku - k\pi)du +$$
$$\left. -\int_{\frac{T}{4}}^{\frac{T}{2}} e^{\beta\cos(\omega u)}\cos(\omega ku - k\pi)du\right] \quad (20)$$

Eqn. 20 simplifies to

$$a_k = \begin{cases} 0 & k = 0,2,4,\ldots \\ \omega\gamma\int_0^{\frac{T}{2}} e^{\beta\cos(\omega t)}\cos(k\omega t)dt & k = 1,3,5,\ldots \end{cases} \quad (21)$$

The derivation of a_k for even k is complete.

The derivation for odd values of k requires further work prior to integration. The derivation proceeds with a change of variables using $u = \omega t$ and $du = \omega dt$ to arrive at

$$a_k = \gamma\int_0^{\pi} e^{\beta\cos(u)}\cos(ku)du \quad (22)$$

There is no closed form solution for the integrals above. To arrive at an expression for a_k we apply the Taylor Series expansion for the exponential to Eqn. 22 to obtain

$$a_k = \gamma\int_0^{\pi}\sum_{n=0}^{\infty}\frac{(\beta\cos(u))^n}{n!}\cos(ku)du \quad (23)$$

We expand $cos(ku)$ using the following identity from [7]

$$\cos(ku) = \cos(u)\left[1 + \sum_{j=1}^{\frac{k+1}{2}-1}\left(\frac{(-1)^j}{(2j)!}\sin^{2j}(u) * \right.\right.$$
$$\left.\left. * \prod_{i=1}^{j}(k^2 - (2i-1)^2)\right)\right] \quad (24)$$

Substituting this into Eqn. 23 and expanding gives

$$a_k = \gamma\int_0^{\pi}\sum_{n=0}^{\infty}\frac{\beta^n}{n!}\cos^{n+1}(u)du +$$
$$-\gamma\int_0^{\pi}\sum_{n=0}^{\infty}\frac{\beta^n}{n!}\cos^{n+1}(u)\frac{k^2-1^2}{2!}\sin^2(u)du +$$
$$+\gamma\int_0^{\pi}\sum_{n=0}^{\infty}\frac{\beta^n}{n!}\cos^{n+1}(u) *$$
$$* \frac{(k^2-1^2)(k^2-3^2)}{4!}\sin^4(u)du +$$
$$-\gamma\int_0^{\pi}\sum_{n=0}^{\infty}\frac{\beta^n}{n!}\cos^{n+1}(u) *$$
$$* \frac{(k^2-1^2)(k^2-3^2)(k^2-5^2)}{6!}\sin^6(u)du + \ldots \quad (25)$$

We now employ the following relations from [7]

$$\int\cos^{2l}u\ du = \frac{\sin u}{2l}\left[\cos^{2l-1}u + \right.$$
$$+\sum_{k=1}^{t-1}\frac{(2l-1)(2l-3)\ldots(2l-2k+1)}{2^k(l-1)(l-2)\ldots(l-k)} *$$
$$\left. *\cos^{2l-2k-1}u\right] + \frac{(2l-1)!!}{2^l l!}u \quad (26)$$

and

$$\int\cos^{2l+1}u\ du = \frac{\sin u}{2l+1}\left[\cos^{2l}u + \right.$$
$$+\sum_{k=0}^{t-1}\frac{2^{k+1}(l-1)\ldots(l-k)}{(2l-1)\ldots(2l-2k-1)} *$$
$$\left. *\cos^{2l-2k-2}u\right] \quad (27)$$

where

$$x!! \triangleq \begin{cases} 1 & \text{if } x = -1,\ 0, \text{ or } 1 \\ x(x-2)!! & \text{if } x \geq 2 \end{cases} \quad (28)$$

Since $\sin(\pi) = \sin(0) = 0$, Eqn. 26 becomes

$$\int_0^{\pi}\cos^{2l}(u)du = \frac{\pi(2l-1)!!}{2^l l!} \quad (29)$$

Similarly, Eqn. 27 becomes

$$\int_0^{\pi}\cos^{2l+1}(u)du = 0 \quad (30)$$

The terms involving $\cos^p u\ \sin^{2n} u$ can be evaluated using the following relation from [7]

$$\int\cos^p u\ \sin^{2n} u\ du = -\frac{\cos^{p+1}u}{2n+p}\left[\sin^{2n-1}u + \right.$$
$$\left. +\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)\ldots(2n-2k+1)\sin^{2n-2k-1}u}{(2n+p-2)(2n+p-4)\ldots(2n+p-2k)}\right] +$$
$$+\frac{(2n-1)!!}{(2n+p)(2n+p-2)\ldots(p+2)}\int\cos^p u\ du \quad (31)$$

and using $\sin(\pi) = \sin(0) = 0$ to obtain

$$\int_0^{\pi}\cos^p(u)\sin^{2n}(u)du =$$
$$= \frac{(2n-1)!!}{(2n+p)(2n+p-2)\ldots(p+2)}\int\cos^p(u)du \quad (32)$$

Substituting these relations into Eqn. 25 gives

$$
\begin{aligned}
a_k \;=\;& \gamma \sum_{n=1,3,5,\ldots}^{\infty} \frac{\pi\beta^n n!!}{n! 2^{\frac{n+1}{2}} (\frac{n+1}{2})!} + \\
& -\gamma \sum_{n=1,3,5,\ldots}^{\infty} \left(\frac{\beta^n}{n!} \frac{k^2-1^2}{2!} (2(1)-1)!! * \right.\\
& \left. * \prod_{l=0}^{1-1} \frac{1}{n+1+2(1)-2l} \frac{n!!}{2^{\frac{n+1}{2}} (\frac{n+1}{2})!} \right) + \\
& +\gamma \sum_{n=1,3,5,\ldots}^{\infty} \left(\frac{\beta^n}{n!} \frac{(k^2-1^2)(k^2-3^2)}{4!} (2(2)-1)!! * \right.\\
& \left. * \prod_{l=0}^{2-1} \frac{1}{n+1+2(2)-2l} \frac{n!!}{2^{\frac{n+1}{2}} (\frac{n+1}{2})!} \right) + \\
& -\gamma \sum_{n=1,3,5,\ldots}^{\infty} \left(\frac{\beta^n}{n!} \frac{(k^2-1^2)(k^2-3^2)(k^2-5^2)}{6!} * \right.\\
& *(2(3)-1)!! * \\
& \left. * \prod_{l=0}^{3-1} \frac{1}{n+1+2(2)-2l} \frac{n!!}{2^{\frac{n+1}{2}} (\frac{n+1}{2})!} \right) + \ldots \qquad (33)
\end{aligned}
$$

To simplify the notation again we define the term

$$
\Xi \stackrel{\triangle}{=} \gamma \sum_{n=1,3,5,\ldots}^{\infty} \frac{\pi\beta^n n!!}{n! 2^{\frac{n+1}{2}} (\frac{n+1}{2})!} \qquad (34)
$$

and rewrite Eqn. 33 as

$$
\begin{aligned}
a_k \;=\;& \Xi - \Xi \frac{k^2-1^2}{2!} (2(1)-1)!! * \\
& * \prod_{l=0}^{1-1} \frac{1}{n+1+2(1)-2l} + \\
& + \Xi \frac{(k^2-1^2)(k^2-3^2)}{4!} (2(2)-1)!! * \\
& * \prod_{l=0}^{2-1} \frac{1}{n+1+2(2)-2l} + \\
& - \Xi \frac{(k^2-1^2)(k^2-3^2)(k^2-5^2)}{6!} (2(3)-1)!! * \\
& * \prod_{l=0}^{3-1} \frac{1}{n+1+2(2)-2l} + \ldots \qquad (35) \\
a_k \;=\;& \Xi \left(1 + \sum_{j=1}^{\frac{k+1}{2}} \frac{(-1)^j (2j-1)!!}{(2j)!} * \right.\\
& \left. * \prod_{l=0}^{j-1} \frac{1}{n+1+2j-2l} \prod_{l=1}^{j} (k^2-(2l-1)^2) \right) \qquad (36)
\end{aligned}
$$

for $k = 1, 3, 5, \ldots$.

C. Sine Terms

The derivation of the sine coefficients, b_k, is similar to the derivation of the cosine coefficients with the substitution of F_{hys} into the equation for the Fourier sine components. The limits of integration are again split according to the the sign of the velocity $(\dot{x})$. Equations 10 and 11 are substituted directly into Eqn. 14.

$$
\begin{aligned}
b_k \;=\;& \frac{2}{T} \int_{-\frac{T}{2}}^{0} \left(F_{max} - F_{max}(1+\tanh(\beta)) e^{-\frac{\sigma}{F_{max}}(x+A)} \right) * \\
& * \sin(k\omega t) dt + \\
& + \frac{2}{T} \int_{0}^{\frac{T}{2}} \left(-F_{max} + F_{max}(1+\tanh(\beta)) e^{\frac{\sigma}{F_{max}}(x-A)} \right) * \\
& * \sin(k\omega t) dt \qquad (37)
\end{aligned}
$$

Rearranging, separating the limits of integration at $\frac{-T}{4}$ and $\frac{T}{4}$ to account for the zero crossings of $cos(\omega t)$, applying the definition of β and integrating the constant terms gives.

$$
\begin{aligned}
b_k \;=\;& \frac{F_{max}}{k\pi} (2\cos(k\pi) - 2) + \\
& + \frac{\Psi}{2} \int_{0}^{\frac{T}{4}} e^{\beta \cos(\omega t)} \sin(k\omega t) dt + \\
& + \frac{\Psi}{2} \int_{\frac{T}{4}}^{\frac{T}{2}} e^{\beta \cos(\omega t)} \sin(k\omega t) dt + \\
& - \frac{\Psi}{2} \int_{-\frac{T}{2}}^{-\frac{T}{4}} e^{-\beta \cos(\omega t)} \sin(k\omega t) dt + \\
& - \frac{\Psi}{2} \int_{-\frac{T}{4}}^{0} e^{-\beta \cos(\omega t)} \sin(k\omega t) dt \qquad (38)
\end{aligned}
$$

where

$$
\Psi \stackrel{\triangle}{=} \frac{4F_{max}(1+\tanh(\beta)e^{-\beta})}{T} \qquad (39)
$$

Defining

$$
\Upsilon_k = \begin{cases} \frac{-4F_{max}}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & k = 0, 2, 4, \ldots \end{cases} \qquad (40)
$$

at this point allows Eqn. 38 to simplify to

$$
\begin{aligned}
b_k \;=\;& \Upsilon_k + \frac{\Psi}{2} \int_{0}^{\frac{T}{4}} e^{\beta \cos(\omega t)} \sin(k\omega t) dt + \\
& + \frac{\Psi}{2} \int_{\frac{T}{4}}^{\frac{T}{2}} e^{\beta \cos(\omega t)} \sin(k\omega t) dt + \\
& - \frac{\Psi}{2} \int_{-\frac{T}{2}}^{-\frac{T}{4}} e^{-\beta \cos(\omega t)} \sin(k\omega t) dt + \\
& - \frac{\Psi}{2} \int_{-\frac{T}{4}}^{0} e^{-\beta \cos(\omega t)} \sin(k\omega t) dt \qquad (41)
\end{aligned}
$$

Again, substitution of $u = t + \frac{T}{2}$ in the third and fourth integrals above and noting that $\frac{T\omega}{2} = \pi$ and $\cos(\zeta - \pi) = -\cos(\zeta)$, gives

$$
\begin{aligned}
b_k \;=\;& \Upsilon_k + \frac{\Psi}{2} \int_{0}^{\frac{T}{4}} e^{\beta \cos(\omega t)} \sin(k\omega t) dt + \\
& + \frac{\Psi}{2} \int_{\frac{T}{4}}^{\frac{T}{2}} e^{\beta \cos(\omega t)} \sin(k\omega t) dt + \\
& - \frac{\Psi}{2} \int_{0}^{\frac{T}{4}} e^{\beta \cos(\omega u)} \sin(\omega k u - k\pi) du + \\
& - \frac{\Psi}{2} \int_{\frac{T}{4}}^{\frac{T}{2}} e^{\beta \cos(\omega u)} \sin(\omega k u - k\pi) du \qquad (42)
\end{aligned}
$$

Since $\sin(\omega k u - k\pi) = \sin(\omega k u)$ for $k = 0, 2, 4, \ldots$, $b_k = 0$ for even k.

The Fourier coefficients with $k = 1, 3, 5, \ldots$ simplify to

$$b_k = \Upsilon_k + \Psi \int_0^{\frac{T}{2}} e^{\beta \cos(\omega t)} \sin(k\omega t) dt \quad (43)$$

Similar to the prior section, applying the change of variables $u = \omega t$ and $du = \omega dt$ and expanding the exponential in a Taylor series gives

$$b_k = \Upsilon + \frac{\Psi}{\omega} \int_0^{\pi} \sum_{n=0}^{\infty} \frac{(\beta \cos(u))^n}{n!} \sin(ku) du \quad (44)$$

Substituting the following relation from [7] for odd k

$$\begin{aligned}
&\int \cos^n(u) \sin(ku) du = \\
&= (-1)^{\frac{k+1}{2}} \left[\frac{\cos^{n+1}(u)}{n+1} + \right. \\
&+ \sum_{j=1}^{\frac{k-1}{2}} \left((-1)^j \cos^{2j+n+1}(u) * \right. \\
&\left.\left. * \frac{(k^2-1^2)(k^2-3^2)\ldots(k^2-(2j-1)^2)}{(2j)!(2j+n+1)} \right) \right]
\end{aligned} \quad (45)$$

into Eqn. 44 gives

$$\begin{aligned}
b_k \;\; = \;\; & \Upsilon_k + \frac{\Psi}{\omega} \sum_{n=0}^{\infty} \frac{\beta^n}{n!} (-1)^{\frac{k+1}{2}} \left[\frac{\cos^{n+1}(u)}{n+1} + \right. \\
& + \sum_{j=1}^{\frac{k-1}{2}} \left((-1)^j \cos^{2j+n+1}(u) * \right. \\
& \left.\left. * \frac{(k^2-1^2)(k^2-3^2)\ldots(k^2-(2j-1)^2)}{(2j)!(2j+n+1)} \right) \right]
\end{aligned} \quad (46)$$

Applying the following relations for integer j and n

$$\cos^{2j+n+1}(\pi) - \cos^{2j+n+1}(0) = \cos^{n+1}(\pi) - \cos^{n+1}(0) \quad (47)$$

and

$$\cos^{n+1}(\pi) - \cos^{n+1}(0) = \begin{cases} -2 & \text{if } n = 0, 2, 4, \ldots \\ 0 & \text{if } n = 1, 3, 5, \ldots \end{cases} \quad (48)$$

to simplify Eqn. 46 gives

$$\begin{aligned}
b_k \;\; = \;\; & \Upsilon_k + \frac{2\Psi}{\omega} \sum_{n=0,2,4,\ldots}^{\infty} \frac{\beta^n}{n!} (-1)^{\frac{k-1}{2}} \left[\frac{1}{n+1} + \right. \\
& \left. + \sum_{j=1}^{\frac{k-1}{2}} \frac{(-1)^j (k^2-1^2)\ldots(k^2-(2j-1)^2)}{(2j)!(2j+n+1)} \right] \quad (49) \\
= \;\; & \Upsilon_k + \frac{2\Psi}{\omega} \sum_{n=0,2,4,\ldots}^{\infty} \frac{\beta^n}{n!} (-1)^{\frac{k-1}{2}} \left[\frac{1}{n+1} + \right. \\
& \left. + \sum_{j=1}^{\frac{k-1}{2}} \frac{(-1)^j \prod_{m=1}^{j} (k^2-(2m-1)^2)}{(2j)!(2j+n+1)} \right] \quad (50)
\end{aligned}$$

for $k = 1, 3, 5, \ldots$.

Note that ω and T drop out of the expressions for a_k and b_k. That is, there is no frequency or time dependence; there is only amplitude dependence. This make sense because

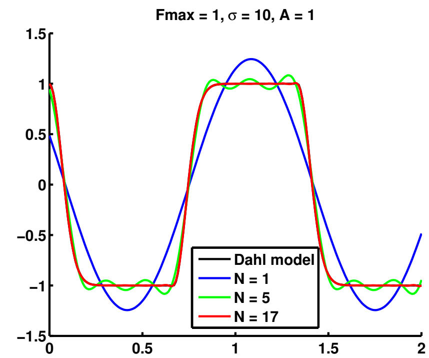

Fig. 3. Dahl model and Fourier series approximations for $F_{max} = 1$, $\sigma = 10$, and $A = 1$.

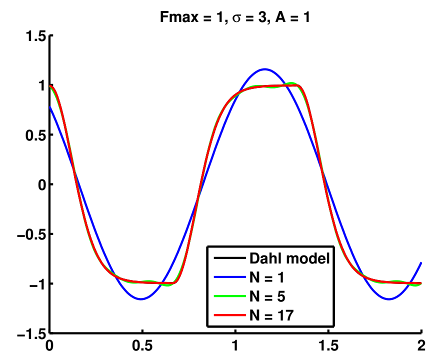

Fig. 4. Dahl model and Fourier series approximations for $F_{max} = 1$, $\sigma = 3$, and $A = 1$.

$F_{hys}(t)$ in Eqn. 10 and 11 depend algebraically on $x(t)$. In the following section, we are able to plot the friction hysteresis response independent of the frequency.

III. EXAMPLES

This section shows examples using the Fourier series of the Dahl model. Figures 3 - 5 show the accuracy of the Fourier series approximation where $F_{max} = 1$, $A = 1$ and $\sigma = 0.1$, 1, and 3 and 10.

Figure 6 shows examples of the DF of the Dahl model for several cases where the maximum force output is $F_{max} = 1$. We consider the cases where $\sigma = 0.1$, 0.32, 1, 3, and 10. The magnitude is given by

$$M = \frac{(a_1^2 + b_1^2)^{\frac{1}{2}}}{A} \quad (51)$$

and the phase is given by

$$\phi = \arg(a_1 - jb_1) \quad (52)$$

where j is the imaginary unit.

Figure 6 shows that for low values of the input amplitude such that $\sigma A << F_{max}$ the gain of the DF is equal to σ. When $\sigma A = F_{max}$, the gain of the DF is approximately 0.8σ and the phase distortion is about 20^o. The gain drops to 0.5σ when $\sigma A \approx 2.2 F_{max}$. The phase distortion is 45^o when $\sigma A \approx 2.7 F_{max}$. For $\sigma A >> F_{max}$ the gain of the DF is F_{max}/A and the phase approaches 90^o.

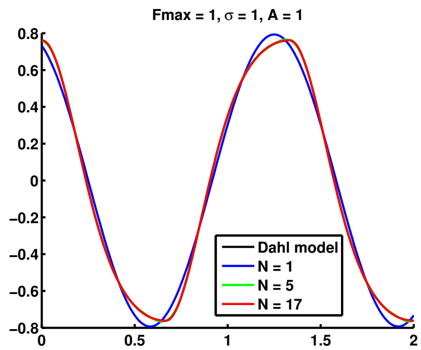

Fig. 5. Dahl model and Fourier series approximations for $F_{max} = 1$, $\sigma = 1$, and $A = 1$.

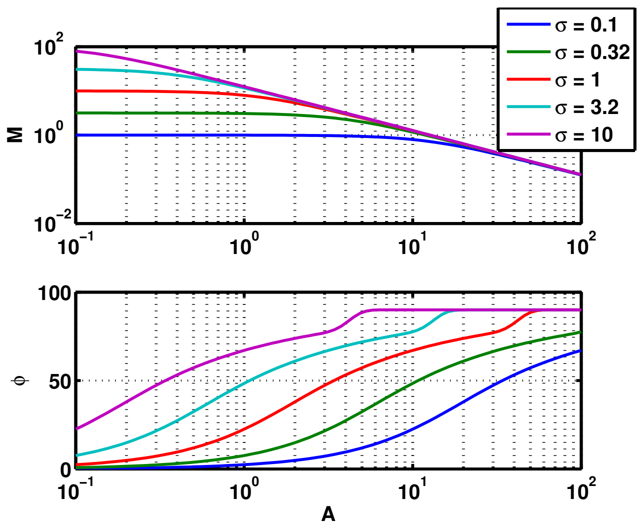

Fig. 6. DF magnitude and phase for $F_{max} = 1$ $\sigma = 0.1$, 0.32, 1.0, 3.2, and 10, and A varying from 0.1 to 100.

We also simulate the frequency response of a mass-spring-damper system with hysteretic friction of the form

$$m\ddot{x} + b\dot{x} + kx + F_h(A) = u \tag{53}$$

where A is the amplitude of the output $x = Ae^{j(\omega t+\phi)}$, the input $u = u_0 e^{j\omega t}$, and the friction hysteresis $F_{hys}(A) = [a_1(A) - jb_1(A)]e^{j(\omega t+\phi)}$. Numerically solving for A and ϕ gives the frequency response.

Figure 7 shows the plot of 51 frequency responses for u_0 varying from 10^{-4} to 10^1 with logarithmic spacing. The parameters are $m = 0.3$, $b = 0.24$, $k = 0.3$, $F_{max} = 0.1$, and $\sigma = 30$. The frequency response exhibits behavior similar to that reported in literature for actual systems [1], [2], [11]. In particular the DC gain rapidly changes from a low value to a high value when u_0 changes between 0.1 and 0.2. The resonance changes from a high frequency/low damping resonance to a low frequency/high damping resonance. The transition occurs over a small range of u_0. The damping of the high frequency resonance increases as u_0 approaches a value near 0.1 in this case, before the resonance frequency suddenly shifts.

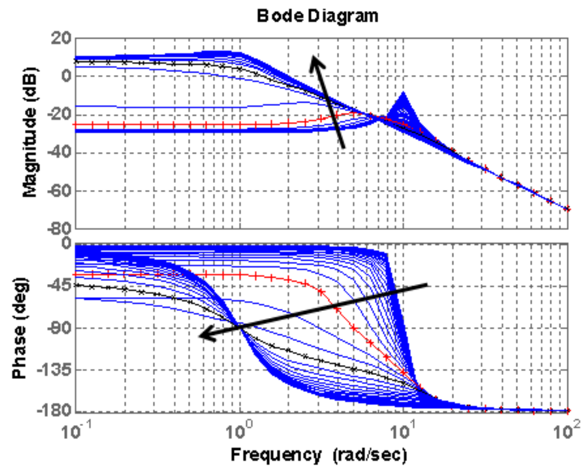

Fig. 7. Frequency response of a mass-spring-damper system with hysteretic friction model where $m = 0.3$, $b = 0.24$, $k = 0.3$, $F_{max} = 0.1$, and $\sigma = 30$. The red line with + marks denotes $u_0 = 0.1$, and the black line with x marks denotes $u_0 = 0.2$.

IV. CONCLUSIONS

This paper derived the coefficients for the Dahl model of friction for a sinusoid input. The coefficients for $k = 1$ can be used to determine the DF of the Dahl model. Examples of the Fourier coefficient fits for several parameter combinations of the Dahl model and input excitations were given. Plots of the DF of a mass-spring-damper system using this DF to represent the nonlinear hysteresis exhibits characteristics of experimental systems reported in the literature. We believe that this derivation will prove useful to practitioners employing frequency response data for analyzing and modeling systems subject to friction hysteresis.

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