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A Unified Extension of The Robust Two-Stage Kalman Filter and Its Application to Functional Filtering

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Abstract— **This paper extends previous work on robust two-stage Kalman filter (RTSKF) for systems with unknown inputs affecting both the system state and the output. By making use of an augmented known input model, an augmented unknown input model, an unknown input error dynamics model, and the previously proposed RTSKF, a unified extension of the RTSKF is further proposed to enhance the unknown input filtering performance. Through the global optimality analysis technique, the conditions under which the unknown input filter and the system state estimator of the RTSKF can both achieve the globally optimal filtering performances are provided. An application of this new RTSKF to functional filtering problem is addressed.**

I. INTRODUCTION

Unknown inputs filtering (UIF) has played a significant role in many applications, e.g., bias compensation [1], [2], geophysical and environmental applications [3], fault detection and isolation problems [4], and functional filtering [5]. Except for the first above-mentioned application, the UIF problem is often solved by assuming that no prior information about the unknown input is available, which is the main concern of this paper.

A general approach to solve the state estimation for unknown inputs that have arbitrary statistics is to apply unknown inputs decoupled state estimation, in which four major approaches have been used in the literature: the first is unbiased minimum-variance estimation (UMVE) [3], [4], [6], [7], [8], [9], the second is the equivalent system description (ESD) method [10], [11], the third is designed based on state estimation techniques for descriptor systems [12], and the last is joint input and state estimation (JISE) [13], [14], [15], [16]. For the description of the above approaches and papers, see [17] for further information.

In this paper, we continue the previous research [13], [16], [17] and further consider a unified extension of the previously proposed RTSKF [13] to solve general unknown input filtering problems that the unknown input may also affect the output. It is shown that using an augmented known input model, an augmented unknown input model, an unknown input error dynamics model, and the RTSKF, a unified extension of the RTSKF is proposed to enhance the unknown input filtering performance. The issue of global optimality concerning the unknown input and system state estimators of this RTSKF is also explored. Through this new RTSKF, a refined version of the previously proposed unified two-stage functional filter (UTSFF) [5] is further proposed to enhance the functional filtering performance.

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II. STATEMENT OF THE PROBLEM

Consider the discrete-time stochastic linear time-varying system with unknown inputs in the form

$$x_{k+1} = A_k x_k + B_k u_k + F_k d_k + w_k \quad (1)$$
$$y_k = H_k x_k + G_k d_k + v_k \quad (2)$$

where $x_k \in R^n$ is the system state, $u_k \in R^m$ is the known input, $d_k \in R^q$ is the unknown input, and $y_k \in R^p$ is the output. w_k and v_k are uncorrelated white noises sequences of zero-mean and covariance matrices $Q_k \geq 0$ and $R_k > 0$, respectively. The initial state x_0 is with unbiased mean $\hat{x}_0$ and covariance matrix P_0^x and is independent of w_k and v_k.

The problem of interest in this paper is to design an optimal state estimator of x_k, denoted by $x_{k|k}$, such that $tr(P_{k|k}^x) = tr(E[e_k e_k'])$ is minimized under the unbiasedness condition $E[e_k] = 0$, where $e_k = x_k - x_{k|k}$ and $'$ denotes transpose. The filter considered in this paper is given as the following RTSKF filter:

$$x_{k|k} = \bar{x}_{k|k} + V_k d_{k|k}$$
$$P_{k|k}^x = P_{k|k}^{\bar{x}} + V_k P_{k|k}^d V_k'$$

where $\bar{x}_{k|k}$, $d_{k|k}$, $P_{k|k}^{\bar{x}}$, $P_{k|k}^d$, and V_k are to be determined; this extends the original filter [13] to more general systems with direct feedthrough of the unknown input to the output.

The main aims of this paper are 1) to derive a unified extension of the RTSKF in [13] to solve the more general unknown input filtering problem, 2) to explore the optimality issue of the obtained new RTSKF, and 3) to enhance the filtering performance of the two-stage functional filtering problem [5] by using the proposed RTSKF.

III. A UNIFIED EXTENSION OF THE RTSKF

First, we give an optimal estimate of the unknown input as follows (see [16] for details):

$$\hat{d}_k = M_k(y_k - H_k \bar{x}_{k|k-1}) \quad (3)$$
$$P_k^d = M_k C_k M_k' \quad (4)$$

where

$$M_k = \begin{bmatrix} \Phi_k & 0_q \end{bmatrix} (S_k' C_k^{-1} S_k)^+ S_k' C_k^{-1} \quad (5)$$
$$S_k = \begin{bmatrix} G_k & H_k \tilde{F}_{k-1} \end{bmatrix}, \quad \tilde{F}_k = F_k(I - \Phi_k) \quad (6)$$
$$C_k = H_k P_{k|k-1}^{\bar{x}} H_k' + R_k, \quad \Phi_k = G_k^+ G_k \quad (7)$$

in which M^+ is the Moore-Penrose pseudo-inverse of M and $\bar{x}_{k|k-1}$ and $P^{\bar{x}}_{k|k-1}$ will be defined later. With the above optimal unknown input estimator, we augment the known input u_k as follows:

$$u_k \to \bar{u}_k = \begin{bmatrix} u'_k & \hat{d}'_k \end{bmatrix}'. \quad (8)$$

Thus, the system model (1) can be represented as follows:

$$x_{k+1} = A_k x_k + \bar{B}_k \bar{u}_k + F_k \tilde{d}_k + w_k \quad (9)$$

where $\bar{B}_k = \begin{bmatrix} B_k & F_k \end{bmatrix}$ and $\tilde{d}_k = d_k - \hat{d}_k$.

Second, we augment the unknown input d_k as follows:

$$d_k \to d^a_k = \begin{bmatrix} d'_k & \tilde{d}'_k \end{bmatrix}'. \quad (10)$$

Thus, the system model (9) and (2) can be represented, respectively, as follows:

$$\begin{aligned} x_{k+1} &= A_k x_k + \bar{B}_k \bar{u}_k + F^a_k d^a_k + w_k \quad (11)\\ y_k &= H_k x_k + \bar{G}_k d^a_k + v_k \quad (12) \end{aligned}$$

where $F^a_k = \begin{bmatrix} 0 & F_k \end{bmatrix}$ and $\bar{G}_k = \begin{bmatrix} G_k & 0 \end{bmatrix}$.

Third, recall from [16] that the expectation of the error dynamics of the unknown input filter $\hat{d}_k$ is given as follows:

$$E[\tilde{d}_k] = (I - \Phi_k) d_k. \quad (13)$$

Hence, using (6), (10), (13), and the following notation:

$$\bar{d}_k = \begin{bmatrix} d'_k & d'_k \end{bmatrix}' \quad (14)$$

(11) and (12) can be represented alternatively as follows:

$$\begin{aligned} x_{k+1} &= A_k x_k + \bar{B}_k \bar{u}_k + \bar{F}_k \bar{d}_k + w_k \quad (15)\\ y_k &= H_k x_k + \bar{G}_k \bar{d}_k + v_k \quad (16) \end{aligned}$$

where $\bar{F}_k = \begin{bmatrix} 0_{n\times q} & \tilde{F}_k \end{bmatrix}$.

Finally, considering that the unknown input also enters into the measurement equation, and then applying the RTSKF [13] to the system (15)-(16), we obtain

$$\begin{aligned} x_{k|k} &= \bar{x}_{k|k} + V_k d_{k|k} \quad (17)\\ P^x_{k|k} &= P^{\bar{x}}_{k|k} + V_k P^d_{k|k} V'_k \quad (18) \end{aligned}$$

where $\bar{x}_{k|k}$ is given by

$$\begin{aligned} \bar{x}_{k|k-1} &= A_{k-1} x_{k-1|k-1} + B_{k-1} u_{k-1} + F_{k-1} \hat{d}_{k-1} \quad (19)\\ \bar{x}_{k|k} &= \bar{x}_{k|k-1} + K^{\bar{x}}_k (y_k - H_k \bar{x}_{k|k-1}) \quad (20)\\ P^{\bar{x}}_{k|k-1} &= A_{k-1} P^x_{k-1|k-1} A'_{k-1} + \bar{Q}_{k-1} \quad (21)\\ K^{\bar{x}}_k &= P^{\bar{x}}_{k|k-1} H'_k C^{-1}_k \quad (22)\\ P^{\bar{x}}_{k|k} &= (I - K^{\bar{x}}_k H_k) P^{\bar{x}}_{k|k-1} \quad (23)\\ \bar{Q}_k &= Q_k + F_k P^{dx}_k A'_k + A_k P^{xd}_k F'_k + F_k P^d_k F'_k \quad (24)\\ P^{xd}_k &= (P^{dx}_k)' = V_k P^d_{k|k} \begin{bmatrix} \Phi_k & 0 \end{bmatrix}' \quad (25) \end{aligned}$$

$d_{k|k}$ is given by

$$\begin{aligned} d_{k|k} &= K^d_k (y_k - H_k \bar{x}_{k|k-1}) \quad (26)\\ K^d_k &= P^d_{k|k} S'_k C^{-1}_k \quad (27)\\ P^d_{k|k} &= (S'_k C^{-1}_k S_k)^+ \quad (28) \end{aligned}$$

and

$$V_k = \begin{bmatrix} -K^{\bar{x}}_k G_k & (I - K^{\bar{x}}_k H_k) \tilde{F}_{k-1} \end{bmatrix} \quad (29)$$

with the following initial conditions:

$$\bar{x}_{0|-1} = \hat{x}_0, \quad P^{\bar{x}}_{0|-1} = P^x_0. \quad (30)$$

The optimal unknown input filter $\hat{d}_k$ (3)-(4) can be expressed alternatively in the form of the filter $d_{k|k}$ as follows:

$$\begin{aligned} \hat{d}_k &= \begin{bmatrix} \Phi_k & 0 \end{bmatrix} d_{k|k} \quad (31)\\ P^d_k &= \begin{bmatrix} \Phi_k & 0 \end{bmatrix} P^d_{k|k} \begin{bmatrix} \Phi_k & 0 \end{bmatrix}'. \quad (32) \end{aligned}$$

Remark 1: Using (17)-(18), (25), and (31)-(32), Eqs. (19) and (21) can be further simplified, respectively, as follows:

$$\begin{aligned} \bar{x}_{k|k-1} &= A_{k-1} \bar{x}_{k-1|k-1} + B_{k-1} u_{k-1}\\ &\quad + \breve{U}_{k-1} d_{k-1|k-1} \quad (33)\\ P^{\bar{x}}_{k|k-1} &= A_{k-1} P^{\bar{x}}_{k-1|k-1} A'_{k-1} + \breve{U}_{k-1} P^d_{k-1|k-1} \breve{U}'_{k-1}\\ &\quad + Q_{k-1} \quad (34) \end{aligned}$$

where

$$\breve{U}_k = A_k V_k + \breve{F}_k, \quad \breve{F}_k = F_k \begin{bmatrix} \Phi_k & 0 \end{bmatrix}.$$

Remark 2: For the case that $0 < rank[G_k] \le q$, one can verify that the above RTSKF will be equivalent to the filter presented in [16] as well as the ARTSKF given in [17].

Remark 3: For the special case that matrix $G_k = 0$, one obtains

$$\begin{aligned} &\Phi_k = 0, \ \breve{F}_k = 0, \ \tilde{F}_k = F_k, \ S_k = \begin{bmatrix} 0 & H_k F_{k-1} \end{bmatrix},\\ &\breve{U}_k = A_k V_k, \quad V_k = \begin{bmatrix} 0 & (I - K^{\bar{x}}_k H_k) F_{k-1} \end{bmatrix} \end{aligned}$$

which renders (33) and (34) to be simplified as follows:

$$\begin{aligned} \bar{x}_{k|k-1} &= A_{k-1} x_{k-1|k-1} + B_{k-1} u_{k-1}\\ P^{\bar{x}}_{k|k-1} &= A_{k-1} P^x_{k-1|k-1} A'_{k-1} + Q_{k-1} \end{aligned}$$

then one can easily verify that this specific RTSKF is equivalent to the original one presented in [13]. Note that in this special case, the optimal unknown input filter $\hat{d}_{k-1}$ can be obtained in the form of the filter $d_{k|k}$ as follows [14]:

$$\begin{aligned} \hat{d}_{k-1} &= \begin{bmatrix} 0 & I_q \end{bmatrix} d_{k|k}\\ P^d_{k-1} &= \begin{bmatrix} 0 & I_q \end{bmatrix} P^d_{k|k} \begin{bmatrix} 0 & I_q \end{bmatrix}' \end{aligned}$$

assuming that $rank[H_k F_{k-1}] = rank[F_{k-1}] = q$.

Remark 4: For the special case that matrix G_k is of full column rank, i.e., $rank[G_k] = q$, one obtains

$$\begin{aligned} &\Phi_k = I, \ \breve{F}_k = \begin{bmatrix} F_k & 0 \end{bmatrix}, \ \tilde{F}_k = 0,\\ &S_k = \begin{bmatrix} G_k & 0 \end{bmatrix}, \quad V_k = \begin{bmatrix} -K^{\bar{x}}_k G_k & 0 \end{bmatrix}. \end{aligned}$$

Then, one can easily verify that this specific RTSKF is equivalent to the filter presented in [15].

Remark 5: The above RTSKF can be tailored in order to be applicable for the following system:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + F^1_k d^1_k + F^2_k d^2_k + w_k \quad (35)\\ y_k &= H_k x_k + G^1_k d^1_k + v_k \quad (36) \end{aligned}$$

where the matrix G_k^1 is of full column rank. This is achieved by using the similar approach in deriving (15)-(16) to obtain an alternative to the system (35)-(36) as follows:

$$x_{k+1} = A_k x_k + \bar{B}_k \bar{u}_k + \bar{F}_k \bar{d}_k + w_k \quad (37)$$
$$y_k = H_k x_k + \bar{G}_k \bar{d}_k + v_k \quad (38)$$

where $\bar{B}_k = \begin{bmatrix} B_k & F_k^1 \end{bmatrix}$, $\bar{F}_k = \begin{bmatrix} 0 & F_k^2 \end{bmatrix}$, $\bar{G}_k = \begin{bmatrix} G_k^1 & 0 \end{bmatrix}$, $\bar{u}_k = \begin{bmatrix} u_k' & (\hat{d}_k^1)' \end{bmatrix}'$, $\bar{d}_k = \begin{bmatrix} (d_k^1)' & (d_k^2)' \end{bmatrix}'$. And then, applying the RTSKF (17)-(32) to the system (37)-(38), we obtain the dedicate result. Note that in this specific RTSKF, the system parameters Φ_k, S_k, and V_k are given as

$$\Phi_k = I, \qquad S_k = \begin{bmatrix} G_k^1 & H_k F_{k-1}^2 \end{bmatrix}$$
$$V_k = \begin{bmatrix} -K_k^{\bar{x}} G_k^1 & (I - K_k^{\bar{x}} H_k) F_{k-1}^2 \end{bmatrix}.$$

IV. On the Global Optimality of the Unknown Input Estimator

In this section, we show the global optimality of the unknown input estimator of the new proposed RTSKF by verifying that it satisfies the following requirements:

$$E[d_k - \hat{d}_k] = 0, \quad min\ E[||d_k - \hat{d}_k||^2]$$

within the general framework for obtaining the linear minimum-variance unbiased (LMVU) estimate of d_k as addressed in [18]. The derivation basically follows those of [18] and [14].

First, we note that the most general estimate of d_k can be written in the form (see [14] for details):

$$\hat{d}_k = M_k \tilde{y}_k + \sum_{i=0}^{k-1} D_i \tilde{y}_i + E_0 \hat{x}_0, \quad (39)$$

where $\tilde{y}_k = y_k - H_k \bar{x}_{k|k-1}$. A necessary and sufficient condition for (39) to be an unbiased estimator of d_k is given in the following lemma.

Lemma 1: The estimator (39) is unbiased if and only if $E_0 = 0$ and M_k and D_i $(0 \le i \le k-1)$ satisfy the following conditions:

$$M_k \begin{bmatrix} G_k & H_k \tilde{F}_{k-1} \end{bmatrix} = \begin{bmatrix} I & -D_{k-1} G_{k-1} \end{bmatrix},$$
$$D_{i+1} H_{i+1} \tilde{F}_i = -D_i G_i \ \ (0 \le i \le k-2). \quad (40)$$

Proof: Using (1), (2), and (19), we obtain

$$\tilde{y}_k = H_k (A_{k-1} e_{k-1} + w_{k-1} + F_{k-1} \tilde{d}_{k-1}) + v_k + G_k d_k. \quad (41)$$

Using (6), (13), and the facts: $E[e_i] = 0$, $E[w_i] = 0$, and $E[v_{i+1}] = 0$ for $i < k$, and substituting (41) into (39), we obtain

$$E[\hat{d}_k] = M_k G_k d_k + (M_k H_k \tilde{F}_{k-1} + D_{k-1} G_{k-1}) d_{k-1} + \sum_{i=0}^{k-2} (D_{i+1} H_{i+1} \tilde{F}_i + D_i G_i) d_i + E_0 \hat{x}_0$$

from which we obtain that the estimator (39) is unbiased if and only if $E_0 = 0$ and M_k and D_i satisfy (40).

Next, to proceed to show the conditions under which the unbiased estimator (39) can achieve the minimum mean square error (MMSE), we have the following lemma.

Lemma 2: For every $i < k$ and every D_i satisfying $D_i S_i = 0$, $E[\tilde{y}_k (D_i \tilde{y}_i)'] = 0$ and $E[d_k (D_i \tilde{y}_i)'] = 0$.

Proof: Defining notations $\bar{y}_i$ and $\bar{e}_i$ as follows:

$$\bar{y}_i = H_i \bar{e}_i + v_i, \quad \bar{e}_i = A_{i-1} e_{i-1} + w_{i-1} + F_{i-1} \tilde{d}_{i-1} \quad (42)$$

we have

$$E[\bar{e}_i \bar{e}_i'] = P_{i|i-1}^{\bar{x}}, \quad E[\bar{e}_i v_i'] = 0. \quad (43)$$

Using (1), (17), (20), (26), (29), and (42), we obtain the error dynamics e_i as follows:

$$e_i = (I - L_i H_i) \bar{e}_i - L_i v_i \quad (44)$$

where $L_i = K_i^{\bar{x}} + V_i K_i^d$. Using (44) in (42) yields

$$\bar{e}_{i+1} = A_i (I - L_i H_i) \bar{e}_i - A_i L_i v_i + w_i + F_i \tilde{d}_i \quad (45)$$

from which and using (43), we obtain

$$E[\bar{e}_i \bar{e}_j'] = A_{i-1} (I - L_{i-1} H_{i-1}) E[\bar{e}_{i-1} \bar{e}_j'] \quad (46)$$
$$E[\bar{e}_i v_j'] = A_{i-1} (I - L_{i-1} H_{i-1}) E[\bar{e}_{i-1} v_j'] - A_{i-1} L_{i-1} E[v_{i-1} v_j'] \quad (47)$$

where $i > j$. Then, we note that

$$E[d_k (D_i \tilde{y}_i)'] = d_k d_i' (D_i G_i)' = 0$$
$$E[\tilde{y}_k (D_i \tilde{y}_i)'] = H_k E[\bar{e}_k \bar{y}_i'] D_i'. \quad (48)$$

Using (7), (22), (27), (42), (43), and (45)-(47), we obtain

$$\begin{aligned} E[\bar{e}_k \bar{y}_i'] &= \bar{A}(k-1, i)(E[\bar{e}_{i+1} \bar{e}_i'] H_i' + E[\bar{e}_{i+1} v_i']) \\ &= \bar{A}(k-1, i)(\bar{A}_i P_{i|i-1}^{\bar{x}} H_i' - A_i L_i R_i) \\ &= \bar{A}(k-1, i) A_i (P_{i|i-1}^{\bar{x}} H_i' - K_i^{\bar{x}} C_i - V_i K_i^d C_i) \\ &= -\bar{A}(k-1, i) A_i V_i P_{i|i}^d S_i' \end{aligned} \quad (49)$$

where $\bar{A}_i = A_i (I - L_i H_i)$ and

$$\bar{A}(m, n) = \bar{A}_m \times \cdots \times \bar{A}_{n+1}, \quad m > n, \quad \bar{A}(m, m) = I.$$

Using (49) in (48) yields that $E[\tilde{y}_k (D_i \tilde{y}_i)'] = 0$ if all $D_i (i < k)$ satisfying $D_i S_i = 0$.

Finally, we have the following theorem.

Theorem 1: Let $rank[G_k] = q$ and $\hat{d}_k$ given by (39) be unbiased, then the mean square error $\sigma_k^2 = E[\|d_k - \hat{d}_k\|_2^2]$ achieves a minimum when $D_0 = D_1 = \cdots = D_{k-1} = 0$.

Proof: Using $\tilde{F}_k = 0$, Lemmas 1 and 2, and the proof of [14,Theorem 10], the theorem is proved.

V. On the Global Optimality of the State Estimator

In this section, we show the global optimality of the state estimator of the new proposed RTSKF by verifying that it satisfies the following requirements:

$$E[x_k - x_{k|k}] = 0, \quad min\ E[||x_k - x_{k|k}||^2]$$

within the general framework for obtaining the LMVU estimate of x_k as addressed in [18]. The derivation basically follows that of [18].

As illustrated in [18], the most general estimate of x_k can be written in the form

$$x_{k|k} = \bar{x}_{k|k-1} + L_k \tilde{y}_k + \sum_{i=0}^{k-1} C_i \tilde{y}_i. \quad (50)$$

A necessary and sufficient condition for (50) to be an unbiased estimator of x_k is given in the following lemma.

Lemma 3: The estimator (50) is unbiased if and only if L_k and $C_i (0 \le i \le k-1)$ satisfy the following conditions:

$$\begin{aligned} L_k G_k = 0, \quad (I - L_k H_k)\tilde{F}_{k-1} = C_{k-1} G_{k-1}, \\ C_{i+1} H_{i+1} \tilde{F}_i = -C_i G_i \ \ (0 \le i \le k-2). \end{aligned} \quad (51)$$

Proof: Using the fact that $E[e_i] = 0$ for $i < k$ and Eqs. (6), (13), and (41), we obtain

$$E[\tilde{y}_k] = H_k \tilde{F}_{k-1} d_{k-1} + G_k d_k. \quad (52)$$

Using (1), (6), (19), (50), and (52), we obtain

$$\begin{aligned} E[e_k] = A_{k-1} E[e_{k-1}] - L_k G_k d_k + [(I - L_k H_k)\tilde{F}_{k-1} \\ -C_{k-1} G_{k-1}] d_{k-1} - \sum_{i=0}^{k-2} (C_{i+1} H_{i+1} \tilde{F}_i + C_i G_i) d_i \end{aligned}$$

from which it can be checked that the estimator (50) is unbiased if and only if L_k and $C_i (0 \le i \le k-1)$ satisfy the conditions in (51).

Next, to proceed to show the conditions under which the unbiased estimator (50) can achieve the MMSE, we have the following lemma.

Lemma 4: For every $i < k$ and every C_i satisfying $C_i S_i = 0$, $E[\bar{e}_k (C_i \tilde{y}_i)'] = 0$.

Proof: Using (41), (42), and (49), we obtain

$$\begin{aligned} E[\bar{e}_k (C_i \tilde{y}_i)'] = E[\bar{e}_k (C_i \bar{y}_i + C_i G_i d_i)'] \\ = \quad -\bar{A}(k-1, i) A_i V_i P^d_{i|i} S'_i C'_i + E[\bar{e}_k d'_i] G'_i C'_i \end{aligned}$$

which equals to zero if all $C_i (i < k)$ satisfying $C_i S_i = 0$.

Now, we are in place to show some sufficient conditions under which the unbiased estimator (50) can achieve the MMSE.

Theorem 2: If the gain matrices C_i of the unbiased estimator (50) are further constrained as follows:

$$C_i G_i = 0, \quad 0 \le i \le k-1 \quad (53)$$

then the mean square error $\bar{\sigma}_k^2 = E[\|x_k - x_{k|k}\|_2^2]$ achieves a minimum when $C_0 = C_1 = \cdots = C_{k-1} = 0$.

Proof: Using (6) and (53) in (51) yields

$$L_k G_k = 0, \quad (I - L_k H_k)\tilde{F}_{k-1} = 0, \quad C_i S_i = 0. \quad (54)$$

Then, we note that the error dynamics e_k is given as follows:

$$e_k = (I - L_k H_k)\bar{e}_k - L_k v_k - \sum_{i=0}^{k-1} C_i \tilde{y}_i. \quad (55)$$

Thus, using (54), (55), and Lemma 4 and following the same approach given in the proof of [18,Theorem 3], the theorem is proved.

Theorem 3: For $rank[G_i] = q$ $(0 \le i \le k-1)$ and $x_{k|k}$ given by (50) with conditions in (51), the mean square error $\bar{\sigma}_k^2 = E[\|x_k - x_{k|k}\|_2^2]$ achieves a minimum when $C_0 = C_1 = \cdots = C_{k-1} = 0$.

Proof: For $rank[G_i] = q$, one has $\tilde{F}_i = 0$. Then, using (51), one obtains $C_i G_i = 0$. Thus, the theorem follows immediately from Theorem 2.

VI. APPLICATION TO FUNCTIONAL FILTERING

The functional filtering problem considered in [5] was revisited. Consider the following discrete-time linear system:

$$x_{k+1} = A_k x_k + B_k u_k + w_k \quad (56)$$

$$y_k = C_k x_k + \eta_k \quad (57)$$

$$z_k = L_k x_k \quad (58)$$

where $x_k \in R^n$ is the system state vector, $u_k \in R^p$ is the input vector, $y_k \in R^m$ is the measurement vector, $z_k \in R^r$ is the vector to be estimated, and $r \le n$. The functional filtering problem is to estimate z_k from the measurements $\{y_t\}$, where $0 \le t \le k$. A unified filter structure, named as the UTSFF, was proposed in [5] to solve the aforementioned functional filtering problem. The UTSFF serves as a unified filter structure to represent the OROF, SOROF, RTSKF, and TSFF (see [5] for details).

The basic idea of obtaining the UTSFF for the system (56)-(58) is first to transfer (56) and (57) into the following

$$d_{k+1} = \tilde{F}_k d_k + \tilde{A}_k z_k + \tilde{B}_k u_k + \tilde{L}_{k+1} w_k \quad (59)$$

$$z_{k+1} = \bar{F}_k d_k + \bar{A}_k z_k + \bar{B}_k u_k + L_{k+1} w_k \quad (60)$$

$$y_k = C_k H_k d_k + C_k G_k z_k + \eta_k \quad (61)$$

where

$$\tilde{F}_k = \tilde{L}_{k+1} A_k H_k, \ \ \tilde{A}_k = \tilde{L}_{k+1} A_k G_k, \ \ \tilde{B}_k = \tilde{L}_{k+1} B_k$$
$$\bar{F}_k = L_{k+1} A_k H_k, \ \ \bar{A}_k = L_{k+1} A_k G_k, \ \ \bar{B}_k = L_{k+1} B_k$$

via the following generalized state transformation:

$$\begin{bmatrix} z_k \\ d_k \end{bmatrix} = \begin{bmatrix} L_k \\ \tilde{L}_k \end{bmatrix} x_k$$

where d_k is an auxiliary state and $\tilde{L}_k$ is a design parameter such that

$$rank \begin{bmatrix} L'_k & \tilde{L}'_k \end{bmatrix}' = n$$

and through the following system state reconstruction:

$$x_k = G_k z_k + H_k d_k, \quad \begin{bmatrix} G_k & H_k \end{bmatrix} = \begin{bmatrix} L_k \\ \tilde{L}_k \end{bmatrix}^+.$$

Then, the derivation of the aforementioned filters depends mainly on the existence of the following rank condition:

$$rank \begin{bmatrix} L_k \\ C_k \end{bmatrix} = n. \quad (62)$$

A. Rank Condition (62) Holds

Assuming that the rank condition (62) holds, and one can find suitable matrices Γ_k and $\hat{D}_k$, where $rank \begin{bmatrix} \Gamma'_k & \hat{D}'_k \end{bmatrix}' = dim(y_k)$, such that

$$\begin{bmatrix} \bar{y}_k^1 \\ \bar{y}_k^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ \bar{D}_k & \bar{C}_k \end{bmatrix} \begin{bmatrix} d_k \\ z_k \end{bmatrix} + \begin{bmatrix} \Gamma_k \\ \hat{D}_k \end{bmatrix} \eta_k \quad (63)$$

where $\bar{y}_k^1 = \Gamma_k y_k$, $\bar{y}_k^2 = \hat{D}_k y_k$, and

$$\bar{D}_k = \hat{D}_k C_k H_k, \quad \bar{C}_k = \hat{D}_k C_k G_k, \quad \Gamma_k \eta_k = 0$$

then, applying the GTSKF [20] to the system [(59), (60), and (63)], one obtains the following OROF:

$$\begin{aligned} z_{k|k-1} &= \bar{A}_{k-1} z_{k-1|k-1} + \bar{B}_{k-1} u_{k-1} + \bar{F}_{k-1} \bar{y}_{k-1}^1 & (64) \\ z_{k|k} &= z_{k|k-1} + K_k^z (y_k^z - S_k^z z_{k|k-1}) & (65) \\ P_{k|k-1}^z &= \bar{A}_{k-1} P_{k-1|k-1}^z \bar{A}'_{k-1} + L_k Q_{k-1} L'_k & (66) \\ K_k^z &= P_{k|k-1}^z (S_k^z)' \left(S_k^z P_{k|k-1}^z (S_k^z)' + R_k^z \right)^+ & (67) \\ P_{k|k}^z &= (I - K_k^z S_k^z) P_{k|k-1}^z & (68) \end{aligned}$$

where

$$y_k^z = \begin{bmatrix} \bar{y}_k^1 - d_{k|k-1} \\ \bar{y}_k^2 - \bar{D}_k \bar{y}_k^1 \end{bmatrix}, \quad S_k^z = \begin{bmatrix} \tilde{K}_k^z \\ \bar{C}_k \end{bmatrix} \quad (69)$$

$$R_k^z = \begin{bmatrix} P_{k|k-1}^d - \tilde{K}_k^z P_{k|k-1}^z (\tilde{K}_k^z)' & 0 \\ 0 & \hat{D}_k R_k \hat{D}'_k \end{bmatrix} \quad (70)$$

in which

$$d_{k|k-1} = \tilde{A}_{k-1} z_{k-1|k-1} + \tilde{F}_{k-1} \bar{y}_{k-1}^1 + \tilde{B}_{k-1} u_{k-1} \quad (71)$$
$$P_{k|k-1}^d = \tilde{A}_{k-1} P_{k-1|k-1}^z \tilde{A}'_{k-1} + \tilde{L}_k Q_{k-1} \tilde{L}'_k \quad (72)$$
$$\tilde{K}_k^z = (\bar{A}_{k-1} P_{k-1|k-1}^z \tilde{A}'_{k-1} + L_k Q_{k-1} \tilde{L}'_k)(P_{k|k-1}^d)^+. \quad (73)$$

Note that the above OROF (64)-(73) is an alternative to that given in [5,(50)-(59)]. Moreover, using $d_{k|k-1} = 0$, $P_{k|k-1}^d = 0$, $\bar{D}_k = 0$, and $\Gamma_k \eta_k \neq 0$ in the above OROF and replacing (66) with the following

$$\begin{aligned} P_{k|k-1}^z &= \bar{A}_{k-1} P_{k-1|k-1}^z \bar{A}'_{k-1} + L_k Q_{k-1} L'_k \\ &\quad + \bar{F}_{k-1} \Gamma_{k-1} R_{k-1} \Gamma'_{k-1} \bar{F}'_{k-1} \end{aligned}$$

one obtains the SOROF in [5,(62)-(65)].

In the following, we show that (65) and (68) can be further simplified by using the two-stage measurement update equations, i.e., [20,(73)-(80)], respectively, as follows:

$$\begin{aligned} z_{k|k} &= \tilde{z}_{k|k} + \bar{K}_k^z (\bar{y}_k^2 - \bar{D}_k \bar{y}_k^1 - \bar{C}_k \tilde{z}_{k|k}) & (74) \\ P_{k|k}^z &= (I - \bar{K}_k^z \bar{C}_k) \tilde{P}_{k|k}^z & (75) \end{aligned}$$

where

$$\begin{aligned} \tilde{z}_{k|k} &= z_{k|k-1} + \tilde{K}_k^z (\bar{y}_k^1 - d_{k|k-1}) & (76) \\ \bar{K}_k^z &= \tilde{P}_{k|k}^z \bar{C}'_k (\bar{C}_k \tilde{P}_{k|k}^z \bar{C}'_k + \hat{D}_k R_k \hat{D}'_k)^{-1} & (77) \\ \tilde{P}_{k|k}^z &= P_{k|k-1}^z - \tilde{K}_k^z P_{k|k-1}^d (\tilde{K}_k^z)'. & (78) \end{aligned}$$

B. Rank Condition (62) Does Not Hold

Assuming that the rank condition (62) does not hold, then the first equation of (63) is discarded by using $\Gamma_k = 0$ and $\hat{D}_k = I_m$. Thus, (63) reduces to

$$\bar{y}_k^2 = \bar{C}_k z_k + \bar{D}_k d_k + \hat{D}_k \eta_k. \quad (79)$$

Applying the new proposed RTSKF (17)-(32) to the system [(60) and (79)], one obtains

$$\begin{aligned} z_{k|k} &= \bar{z}_{k|k} + V_k d_{k|k} & (80) \\ P_{k|k}^z &= P_{k|k}^{\bar{z}} + V_k P_{k|k}^d V'_k & (81) \end{aligned}$$

where $\bar{z}_{k|k}$ is given by

$$\bar{z}_{k|k-1} = \bar{A}_{k-1} z_{k-1|k-1} + \bar{B}_{k-1} u_{k-1} + \bar{u}_{k-1} \quad (82)$$
$$\bar{z}_{k|k} = \bar{z}_{k|k-1} + K_k^{\bar{z}} (\bar{y}_k^2 - \bar{C}_k \bar{z}_{k|k-1}) \quad (83)$$
$$P_{k|k-1}^{\bar{z}} = \bar{A}_{k-1} P_{k-1|k-1}^z \bar{A}'_{k-1} + L_k Q_{k-1} L'_k + P_{k-1}^{\bar{u}} \quad (84)$$
$$K_k^{\bar{z}} = P_{k|k-1}^{\bar{z}} \bar{C}'_k \Omega_k^{-1}, \ \Omega_k = \bar{C}_k P_{k|k-1}^{\bar{z}} \bar{C}'_k + \hat{D}_k R_k \hat{D}'_k \quad (85)$$
$$P_{k|k}^{\bar{z}} = (I - K_k^{\bar{z}} \bar{C}_k) P_{k|k-1}^{\bar{z}} \quad (86)$$
$$\bar{u}_k = \breve{F}_k d_{k|k}, \ \breve{F}_k = \bar{F}_k \begin{bmatrix} \Phi_k & 0 \end{bmatrix}, \ \Phi_k = \bar{D}_k^+ \bar{D}_k \quad (87)$$
$$P_k^{\bar{u}} = \breve{F}_k P_{k|k}^d V'_k \bar{A}'_k + \bar{A}_k V_k P_{k|k}^d \breve{F}'_k + \breve{F}_k P_{k|k}^d \breve{F}'_k \quad (88)$$

$d_{k|k}$ is given by

$$\begin{aligned} d_{k|k} &= K_k^d (\bar{y}_k^2 - \bar{C}_k \bar{z}_{k|k-1}) & (89) \\ K_k^d &= P_{k|k}^d S'_k \Omega_k^{-1} & (90) \\ P_{k|k}^d &= (S'_k \Omega_k^{-1} S_k)^+ & (91) \end{aligned}$$

and

$$\begin{aligned} V_k &= \begin{bmatrix} -K_k^{\bar{z}} \bar{D}_k & (I - K_k^{\bar{z}} \bar{C}_k) \tilde{F}_{k-1} \end{bmatrix} & (92) \\ S_k &= \begin{bmatrix} \bar{D}_k & \bar{C}_k \tilde{F}_{k-1} \end{bmatrix}, \quad \tilde{F}_k = \bar{F}_k (I - \Phi_k). & (93) \end{aligned}$$

Finally, combining the OROF [(64), (66), (71)-(73), and (74)-(78)] and the RTSKF [(80)-(93)] into a unified result, a refined version of the previous proposed UTSFF [5] is given as follows:

$$\begin{aligned} z_{k|k} &= \bar{z}_{k|k} + V_k \bar{d}_{k|k} \\ P_{k|k}^z &= P_{k|k}^{\bar{z}} + V_k P_{k|k}^{\bar{d}} V'_k \end{aligned}$$

where $\bar{z}_{k|k}$ is the main functional estimate given by

$$\bar{z}_{k|k} = \bar{z}_{k|k-1} + K_k^{\bar{z}} (\bar{y}_k^2 - \bar{D}_k \bar{y}_k^1 - \bar{C}_k \bar{z}_{k|k-1})$$

$\bar{z}_{k|k-1}$, $P_{k|k-1}^{\bar{z}}$, $K_k^{\bar{z}}$, and $P_{k|k}^{\bar{z}}$ are given by (82), (84), (85), and (86), respectively, $\bar{d}_{k|k}$ is the complementary functional estimate given by

$$\bar{d}_{k|k} = (I - K_k^{\bar{d}} S_k) \bar{d}_{k|k-1} + K_k^{\bar{d}} (\bar{y}_k^2 - \bar{D}_k \bar{y}_k^1 - \bar{C}_k \bar{z}_{k|k-1})$$
$$K_k^{\bar{d}} = P_{k|k}^{\bar{d}} S'_k \Omega_k^{-1}, \quad P_{k|k}^{\bar{d}} = \left((P_{k|k-1}^{\bar{d}})^+ + S'_k \Omega_k^{-1} S_k \right)^+$$

V_k and S_k are given as follows:

$$\begin{aligned} V_k &= \begin{bmatrix} -K_k^{\bar{z}} N_k & (I - K_k^{\bar{z}} \bar{C}_k) U_k \end{bmatrix} \\ S_k &= \begin{bmatrix} N_k & \bar{C}_k U_k \end{bmatrix} \end{aligned}$$

TABLE I

PERFORMANCES OF THE OROF, OUFF, TSFF[1], DUFF, AND RTSKF

Case	rmse	OROF	OUFF	TSFF[1]	DUFF	RTSKF
1	x^3	1.3359	1.4753	7.1880	5.7086	1.4792
	x^4	0.8349	0.9607	1.0233	0.9611	0.9607
2	x^3	NA	1.6858	4.5831	4.8125	1.6853
	x^4	NA	0.9276	1.0347	0.9436	0.9269
3	x^3	NA	5.0166	4.5831	4.6075	5.0170
	x^4	NA	1.0320	1.0347	1.0316	1.0320
4	x^3	NA	NA	7.2644	7.2644	7.2644
	x^4	NA	NA	1.1505	1.1505	1.1505

and with the following computing structure:

$$\Theta_k = \left(\tilde{L}_k, \hat{D}_k, \Gamma_k, \bar{u}_{k-1}, P^{\bar{u}}_{k-1}\right)$$
$$\Xi_k = \left(N_k, U_k, \bar{d}_{k|k-1}, P^{\bar{d}}_{k|k-1}\right).$$

Remark 6: The OROF [(64), (66), (71)-(73), and (74)-(78)] can be reformulated as the above refined UTSFF by using the following computing structure:

$$\Theta_k = \left(\tilde{L}_k, \hat{D}_k, \Gamma_k, \bar{F}_{k-1}\bar{y}^1_{k-1}, 0\right)$$
$$\Xi_k = \left(\phi, \tilde{K}^z_k, \bar{y}^1_k - d_{k|k-1}, -P^d_{k|k-1}\right).$$

VII. AN ILLUSTRATIVE EXAMPLE

To illustrate the usefulness of the new proposed RTSKF, in this section, we shall evaluate the filtering performance of the refined UTSFF presented in Section VI with the following RTSKF computing structure:

$$\Theta_k = \left(\tilde{L}_k, I, 0, \bar{u}_{k-1}, P^{\bar{u}}_{k-1}\right), \Xi_k = \left(\bar{D}_k, \tilde{F}_{k-1}, 0, 0\right). \quad (94)$$

We considered all the four simulation examples in [5], and list the root-mean-square-errors (rmse) in the state estimates of the OROF, the OUFF, and the TSFF[1] in [5], the DUFF in [19], and the UTSFF (RTSKF), i.e., (94), in Table I.

From Table I, we obtain the following results: 1) the RTSKF and the OUFF have similar filtering performances in the first three simulation cases, 2) the filtering performances of the RTSKF and the OUFF are much better than those of the TSFF[1] and the DUFF in the first two simulation cases, and 3) the OUFF is not available for the last case since (62) does not hold, while the RTSKF has the same filtering performance as those of the TSFF[1] and the DUFF. Based on the above results, we may conclude that the UTSFF with the RTSKF computing structure is the most preferable functional filter in the view point of filtering performance.

VIII. CONCLUSION

In this paper, the new RTSKF, which can be seen as a unified extension of the previously proposed one, is presented as a general filter structure to derive optimal unbiased minimum-variance filters for systems with unknown inputs. The equivalence of the RTSKF and the existing literature results is also addressed. The global optimality analysis of the unknown input and system state estimators of the RTSKF reveals that the global optimal LMVU joint input and state estimator can be obtained assuming that the direct feedthrough matrix G_k is of full column rank. Through the new proposed RTSKF, a refined version of the previously proposed UTSFF is also proposed. The filtering superiority of this refined UTSFF than the original one is further verified by a simulation example. This research shows that the RTSKF serves as a unified and refined filter structure for solving general unknown input filtering problems and unbiased minimum-variance functional filtering problems.

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REFERENCES

[1] B. Friedland, "Treatment of bias in recursive filtering", *IEEE Trans. Automat. Contr.,* vol. 14, pp. 359-367, 1969.

[2] C.-S. Hsieh and F.-C. Chen, "Optimal solution of the two-stage Kalman estimator," *IEEE Trans. Automat. Contr.,* vol. 44, pp. 194-199, 1999.

[3] P.K. Kitanidis, "Unbiased minimum-variance linear state estimation," *Automatica,* vol. 23, pp. 775-778, 1987.

[4] J. Chen and R.J. Patton, "Optimal filtering and robust fault diagnosis of stochastic systems with unknown disturbances," *IEE Proc.-Control Theory Appl.,* vol. 143, pp. 31-36, 1996.

[5] C.-S. Hsieh, "The unified structure of unbiased minimum-variance reduced-order filters," *IEEE Trans. Automat. Contr.*, vol. 52, no. 9, pp. 1716-1721, 2007.

[6] A.V. Borisov and A.R. Pankov, "Optimal filtering in stochastic discrete-time systems with unknown inputs," *IEEE Trans. Automat. Contr.,* vol. 39, pp. 2461-2464, 1994.

[7] M. Darouach and M. Zasadzinski, "Unbiased minimum variance estimation for systems with unknown exogenous inputs," *Automatica,* 33, pp. 717-719, 1997.

[8] M.E. Valcher, "State observers for discrete-time linear systems with unknown inputs," *IEEE Trans. Automat. Contr.,* vol. 44, no. 2, pp. 397-401, 1999.

[9] M. Darouach, M. Zasadzinski, and M. Boutayeb, "Extension of minimum variance estimation for systems with unknown inputs," *Automatica,* 39, pp. 867-876, 2003.

[10] M. Hou and P.C. Müller, "Disturbance decoupled observer design: a unified viewpoint", *IEEE Trans. Automat. Contr.,* vol. 39, pp. 1338-1341, 1994.

[11] M. Hou and R.J. Patton, "Optimal filtering for systems with unknown inputs", *IEEE Trans. Automat. Contr.,* vol. 43, pp. 445-449, 1998.

[12] R. Nikoukhah, S.L. Campbell, and F. Delebecque, "Kalman filtering for general discrete-time linear systems", *IEEE Trans. Automat. Contr.,* vol. 44, pp. 1829-1839, 1999.

[13] C.-S. Hsieh, "Robust two-stage Kalman filters for systems with unknown inputs," *IEEE Trans. Automat. Contr.,* vol. 45, pp. 2374-2378, 2000.

[14] S. Gillijns and B. De Moor, "Unbiased minimum-variance input and state estimation for linear discrete-time systems", *Automatica,* 43, pp. 111-116, 2007.

[15] S. Gillijns and B. De Moor, "Unbiased minimum-variance input and state estimation for linear discrete-time systems with direct feedthrough", *Automatica,* 43, pp. 934-937, 2007.

[16] C.-S. Hsieh, "Extension of unbiased minimum-variance input and state estimation for systems with unknown inputs", Submitted to *Automatica,* 2008.

[17] C.-S. Hsieh, "On the optimality of the two-stage Kalman filtering for systems with unknown inputs," Submitted to *Asia Journal of Control,* 2008.

[18] W.S. Kerwin and J.L. Prince, "On the optimality of recursive unbiased state estimation with unknown inputs," *Automatica,* 36, pp. 1381–1383, 2000.

[19] M. Darouach, "On the optimal unbiased functional filtering," *IEEE Trans. Automat. Contr.,* vol. 45, pp. 1374–1379, July 2000.

[20] C.-S. Hsieh and F.-C. Chen, "Optimal minimal-order least-squares estimators via the general two-stage Kalman filter," *IEEE Trans. Automat. Contr.,* vol. 46, pp. 1772–1776, Nov. 2001.