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Transverse Linearization for Mechanical Systems with Several Passive Degrees of Freedom with Applications to Orbital Stabilization

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***Abstract*— A class of mechanical systems with many unactuated degrees of freedom is studied. An analytical method for computing coefficients of a linear controlled system, solutions of which approximate dynamics of transverse part of coordinates of an underactuated mechanical system along a feasible motion, is proposed. The procedure is constructive and is based on a particular choice of coordinates in a vicinity of the motion. It allows explicit introduction of the so-called moving Poincaré section associated with a finite-time or periodic motion. It is shown that the coordinates admit analytical linearization of transverse part of the system dynamics prior to any controller design. If the motion is periodic, then these coordinates are used for developing feedback controllers. Necessary and sufficient conditions for exponential orbital stabilization of a cycle for underactuated mechanical systems are derived. Two illustrative examples are elaborated in details.**

***Index Terms*— Moving Poincaré section; Periodic solutions; Orbital stability; Transverse linearization; Underactuated mechanical systems; Virtual holonomic constraints; Spherical pendulum; Synchronization of mechanical systems**

I. Introduction

Underactuated mechanical systems are common for robotics applications such as locomotion, grasping, juggling etc. Motion planning and stabilization tasks for these applications are often solved ad hoc. If a desired motion is found, it is the common case that nonlinearities in system dynamics cannot be removed by a feedback action, even in a vicinity of the target motion. To illustrate the challenges it is worth noting the series of investigations [15], [12], [4], devoted only to the particular case—controlled mechanical systems with *one passive link*, i.e. of underactuation degree one.

This work is focused on orbital stability and stabilization of motions for mechanical systems that have two (three, four, ...) less independent control input than the number of degrees of freedom. It turns out that the existing approaches and analytical arguments developed for system with one passive link are difficult to use when the degree of underactuation is two or higher. The main contribution here is a constructive solution. It is based on simple but important observation that for any motion of controlled mechanical system there is a generic choice of transverse coordinates. They are introduced through a constructive procedure and can be analytically linearized for the case when level of underactuation is arbitrary. This result is used further for verifying orbital exponential stability of a periodic motion of the closed-loop system and for synthesizing stabilizing controllers. Two illustrative examples are studied: stabilization of oscillations of a spherical pendulum on a puck around its upright (unstable) equilibrium and synchronization of oscillations of 3-cart-pendulum systems. The first example has 2 passive degrees of freedom and the second one has 3.

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A. Preliminaries: Transverse Coordinates

As known, the classical tool—the linearization of full-state dynamics of an autonomous system along its non-trivial solution—might not be appropriate for analyzing various local properties[1]. Instead, only the part of dynamics responsible for capturing local behavior in a vicinity of the solution should be considered. The geometrical interpretation of this observation is that one needs to find new variables such that the system's states are decomposed into:

1) A scalar variable representing position *along a trajectory of the target solution*. Local properties of system dynamics in a vicinity of the solution are independent on this variable; so that it can be safely disregarded.
2) The remaining coordinates representing the dynamics *transverse to a trajectory of the solution*. These coordinates define a moving Poincaré section [7].

A diffeomorphism to a new set of coordinates with this form always exists in a vicinity of any solution defined on a finite-time interval[2], but finding such coordinates in an explicit form is typically difficult. However, it is important to notice that for analysis and designing of a feedback controller modifying the system dynamics near the solution, linearizing the transverse dynamics (i.e. computing a *transverse linearization*) about the solution is often sufficient, and explicit formulae for the full change of variables are not required.

Consider the class of controlled Euler-Lagrange systems

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(q,\dot q)}{\partial \dot q}\right) - \frac{\partial \mathcal{L}(q,\dot q)}{\partial q} = B(q)\,u \qquad (1)$$

Here $q \in \mathbb{R}^n$, $\dot q \in \mathbb{R}^n$ are generalized coordinates and velocities, $u \in \mathbb{R}^m$ are control inputs, $\mathcal{L}(q,\dot q) = \frac{1}{2}\dot q^T M(q)\dot q - V(q)$ is the Lagrangian, $M(q)$ is a matrix of inertia, $V(q)$ is a potential energy, $B(q)$ is a matrix function of full-rank.

[1]For instance, a linearization around a periodic solution cannot be asymptotically stable [1], [16], [7].

[2]When the solution is a cycle, the appropriate statements can be found in [14], [5], [9], [6]. Similar arguments can be applied for any non-trivial and non-periodic solution defined on a finite-time interval.

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Definition 1: Let $q_\star(t)$ for $t \in [0,T]$ be a solution of the n-degrees-of-freedom Euler-Lagrange system (1) driven by a control signal $u_\star(t) \in C^1([0,T])$ with the initial conditions at $q_\star(0) = q_0$, $\dot{q}_\star(0) = \dot{q}_0$, such that $\left(|\dot{q}_\star(t)|^2 + |\ddot{q}_\star(t)|^2\right) > 0$ for all $t \in [0,T]$. The orbit of the trajectory is

$$\mathcal{O}_\star = \left\{[q;\dot{q}] : q = q_\star(\tau),\, \dot{q} = \dot{q}_\star(\tau),\, \tau \in [0,T]\right\} \qquad (2)$$

and its tabular neighborhood, the set of all points on a distance not bigger than some $\varepsilon > 0$, is

$$\mathcal{O}_\varepsilon(q_\star) = \{[q;\dot{q}] : \min_{\tau\in[0,T]} \|[q;\dot{q}] - [q_\star(\tau);\dot{q}_\star(\tau)]\| \le \varepsilon\} \quad (3)$$

1) A family of $(2n-1)$-dimensional C^1-smooth surfaces $\left\{S(t),\, t \in [0,T]\right\}$ is called a *moving Poincaré section* associated with the solution $q_\star(t)$, $t \in [0,T]$, if

(a) Surfaces $S(t)$ are locally disjoint, i.e. $\exists \varepsilon > 0$: $S(\tau_1) \cap S(\tau_2) \cap \mathcal{O}_\varepsilon(q_\star) = \emptyset$, $\forall \tau_1 \ne \tau_2 \in [0,T]$.

(b) Each of $S(t)$ locally intersects the orbit only in one point, i.e. $\exists \varepsilon > 0$: $S(\tau) \cap \{[q_\star(t);\dot{q}_\star(t)],\ |t-\tau| < \varepsilon\} \cap \mathcal{O}_\varepsilon(q_\star) = \{[q_\star(\tau);\dot{q}_\star(\tau)]\}$ for each $\tau \in [0,T]$.

(c) The surfaces $S(t)$ are smoothly parametrized by time, i.e. $\exists f \in C^1(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R})$: $S(t) \cap \mathcal{O}_\varepsilon(q_\star) = \{[q;\dot{q}] \in \mathbb{R}^n \times \mathbb{R}^n : f_s(q,\dot{q},t) = 0\} \cap \mathcal{O}_\varepsilon(q_\star)$.

(d) The surfaces $S(t)$ are transversal to $q_\star(t)$.

2) Given a moving Poincaré section $\left\{S(t),\, t \in [0,T]\right\}$ associated with the motion $q_\star(t)$, the state coordinates $[q;\dot{q}]$ of (1) can be (locally) changed into $[\psi; x_\perp]$ where the scalar variable $\psi(t)$ parameterizes a position along the curve (trajectory) in $\mathbb{R}^n \times \mathbb{R}^n$ defined by $q_\star(t)$ and the $(2n-1)$-dimensional vector $x_\perp(t)$ defines location on the surface $S(t)$. $x_\perp$ is known as a vector of *transverse coordinates*.

3) The dynamics of (1) rewritten in $[\psi; x_\perp]$-coordinates and linearized along the solution $q_\star(t)$, $t \in [0,T]$ give rise to the linear time-varying control system of dimension $2n$ defined on $t \in [0,T]$. The $(2n-1)$-dimensional subsystem that corresponds to linearization of the dynamics of the transverse coordinates $x_\perp$ is called a *transverse linearization*[3]. ∎

B. Preliminaries: Re-parametrization of a Motion

If one wants to describe a solution of the system (1), there is a number of possible formats. The most immediate is a time-evolution of the generalized coordinates[4]:

$$q_1 = q_{1\star}(t), \quad \ldots, \quad q_n = q_{n\star}(t), \quad t \in [0,T] \qquad (4)$$

However, there are descriptions without an explicit reference to time, such as the orbit (2). One can introduce relations

$$q_1 = \phi_1(\theta_\star), \quad \ldots, \quad q_n = \phi_n(\theta_\star), \quad \theta_\star \in [\Theta_b, \Theta_e], \quad (5)$$

valid on the same orbit, where $\theta_\star$ could be some parameter such as the arc length[5] along the orbit or, in many cases, one of the coordinates, e.g. $\theta_\star(t) = q_n(t)$ so that $\phi_n(\theta_\star) = \theta_\star$. Identities, as in (5), are known as *virtual holonomic constraints* [15] since they express relations among the generalized coordinates $q_1, \ldots, q_n$. For a feasible motion the relations (5) can always be found. If the system is fully actuated, the dynamics along the orbit of the motion is controlled. However, for underactuated systems, it is not the case, the dynamics are fixed.

Lemma 1: *Consider the controlled mechanical system (1) of n-degrees of freedom with m independent control inputs, i.e. of underactuation $(n-m)$. Let $q = q_\star(t)$ be a motion (4) of (1) in response to a control signal $u = u_\star(t)$, both defined on the time interval $t \in [0,T]$. Let $\theta = \theta_\star(t)$ be a scalar parameter used in (5) to describe the motion $q_\star(t)$. Then $\theta_\star(t)$ is not any, but it is simultaneously a solution of $(n-m)$ second order differential equations*[6]

$$\alpha_i(\theta)\ddot{\theta} + \beta_i(\theta)\dot{\theta}^2 + \gamma_i(\theta) = 0, \quad i = 1, \ldots, n-m. \quad (6)$$

Lemma 2: *Let $\theta = \theta(t)$ be a C^2-smooth scalar function defined on the time interval $t \in [0,T]$ and be at the same time the solution of (6) with some i. The following statements are true for any nontrivial subinterval $[t_b, t_e] \subset [0,T]$:*

1) *If $\alpha_i(\theta(t)) \equiv 0$ and $\beta_i(\theta(t)) \equiv 0$, then $\theta(t)$ satisfies*

$$\alpha_i(\theta(t)) \equiv 0, \quad \beta_i(\theta(t)) \equiv 0, \quad \gamma_i(\theta(t)) \equiv 0 \quad (7)$$

2) *If $\alpha_i(\theta(t)) \equiv 0$ but $\beta_i(\theta(t)) \ne 0$ for any $t \in [t_b, t_e]$, then $\theta(t)$ satisfies two identities:*

$$\alpha_i(\theta(t)) \equiv 0, \quad \dot{\theta}^2(t) + \gamma_i(\theta(t))/\beta_i(\theta(t)) \equiv 0 \quad (8)$$

3) *If $\alpha(\theta(t)) \ne 0$ for any $t \in [t_b, t_e]$, then $\theta(t)$ satisfies the identity $I_3^{(i)}(\theta(t), \dot{\theta}(t), \theta(t_b), \dot{\theta}(t_b)) \equiv 0$ with*

$$I_3^{(i)} = \dot{\theta}^2 - e^{\left\{-\int\limits_{\theta(t_b)}^{\theta} \frac{2\beta_i(\tau)}{\alpha_i(\tau)} d\tau\right\}} \dot{\theta}^2(t_b) + \int\limits_{\theta(t_b)}^{\theta} e^{\left\{\int\limits_{\theta}^{s} \frac{2\beta_i(\tau)}{\alpha_i(\tau)} d\tau\right\}} \frac{2\gamma_i(s)}{\alpha_i(s)} ds \quad (9)$$

The relations (5) together with (7)–(9) and/or their derivatives can be used for describing the orbit of the target motion $q = q_\star(t)$. Namely, introduce the quantities

$$y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_n = q_n - \phi_n(\theta) \qquad (10)$$

where $\phi_1(\cdot), \ldots, \phi_n(\cdot)$ are taken from (5) while θ is one of the new generalized coordinates for (1) to be found in a vicinity of the motion. By definition, the variables (10) and their time derivatives are zeros on (2). In the same way, for each of the $(n-m)$ equations (6) one can introduce at least one function of q and $\dot{q}$ that becomes zero on the orbit (2). Indeed, depending on conditions, one of 3 possible cases of Lemma 2 takes place. If for the i-th equation of (6), the case 1) hold, then there are three functions

$$I_{11}^{(i)}(\theta) = \alpha_i(\theta), \quad I_{12}^{(i)}(\theta) = \beta_i(\theta), \quad I_{13}^{(i)}(\theta) = \gamma_i(\theta) \quad (11)$$

that are zeros on the orbit (2). If the conditions of the case 2) are valid, then the functions

$$I_{21}^{(i)}(\theta) = \alpha_i(\theta), \quad I_{22}^{(i)}(\theta, \dot{\theta}) = \dot{\theta}^2 + \gamma(\theta)/\beta(\theta) \qquad (12)$$

[3]The concept of transverse linearization was used for feedback control of various systems, see e.g. [13], [2], [3], [8].

[4]Note that the other half of the states are derivatives of these functions.

[5]For this case, we must assume that there are no sub-intervals of time where $|\dot{q}_\star(t)|^2 + |\ddot{q}_\star(t)|^2 \equiv 0$.

[6]Some of these differential equations can be of lower order or even trivial.

are zeros on the orbit. For the case 3), there is at least one such function $I_3^{(i)}(\theta)$ defined in (9).

So, for any non-trivial motion $q = q_\star(t)$, $t \in [0, T]$ of the controlled mechanical system (1) with the degree of underactuation $(n-m)$, there is a large family of functions of the generalized coordinates and velocities including

$$y_1, \ldots, y_n, \quad \dot y_1, \ldots, \dot y_n, \quad I_{\chi_1}^{(1)}, \ldots, I_{\chi_{(n-m)}}^{(n-m)} \tag{13}$$

that are candidates for transverse coordinates in a vicinity of the part of the motion restricted to $t \in [t_b, t_e] \subset [0, T]$.

II. Main Results

Here we proceed with one of possible choices for $(2n-1)$-transverse coordinates from quantities (13) and show the steps for computing *transverse linearization* of the dynamics (1) based on these coordinates in a vicinity of the motion.

A. Changing Generalized Coordinates

Given a motion $q = q_\star(t)$ of (1) in the response to a C^1-smooth control input $u_\star(t)$ on $[0, T]$ and given the scalar functions $\phi_1(\cdot), \ldots, \phi_n(\cdot)$ defined by the alternative representation (5) of the motion, following [10], the quantities

$$\theta, \quad y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_n = q_n - \phi_n(\theta) \tag{14}$$

can be seen as excessive generalized coordinates for the n-DOF Euler-Lagrange system (1). Therefore, one of them can be expressed as a function of the others. Suppose that in (3)

$$y = (y_1, \ldots, y_{n-1})^T \quad \text{and} \quad \theta, \tag{15}$$

can be taken as new generalized coordinates for (1).

The dynamics (1) in the new generalized coordinates (15) can be partially written in the following form

$$\ddot y = R(\theta, \dot\theta, y, \dot y) + N(\theta, y)\, u, \tag{16}$$

where $N(\theta, y)$ and $R(\theta, \dot\theta, y, \dot y)$ are certain matrix functions.

Given any smooth function $U(\theta, \dot\theta, y, \dot y)$ that coincides with the nominal input $u_\star(t)$ on the orbit (2)

$$u_\star(t) = U(\theta_\star(t), \dot\theta_\star(t), 0, 0), \qquad \forall\, t \in [0, T], \tag{17}$$

the feedback transform

$$u = v + U(\theta, \dot\theta, y, \dot y), \tag{18}$$

brings the y-dynamics (16) into the form

$$\ddot y = F(\theta, \dot\theta, y, \dot y) + N(\theta, y)\, v. \tag{19}$$

By construction the vector-function $F(\cdot)$ is zero on the orbit

$$F\left(\theta_\star(t), \dot\theta_\star(t), 0, 0\right) \equiv \mathbf{0}_{(n-1)\times 1}, \quad \forall\, t \in [0, T]. \tag{20}$$

Eqn. (19) is only a part of the dynamics of (1) in the new coordinates (15). It should be complemented by a 2^{nd}-order equation w.r.t. the θ-variable. One way to write it is to use one of the equations (6), for which the coefficient $\alpha_i(\theta_\star(t))$ is separated from zero on the orbit $\forall\, t$. Assuming this is the case for some i, the dynamics of θ can be rewritten as

$$\alpha_i(\theta)\ddot\theta + \beta_i \dot\theta^2 + \gamma_i(\theta) = g_i(\theta, \dot\theta, \ddot\theta, y, \dot y, \ddot y, v) \tag{21}$$

The right-hand side of this equation—the smooth function $g_i(\cdot)$—is not any, but equals zero on the orbit. Therefore, following the Hadamard's lemma, it can be represented as

$$g_i = g_y^{(i)}(\theta, \dot\theta, \ddot\theta, y, \dot y) y + g_{\dot y}^{(i)}(\theta, \dot\theta, \ddot\theta, y, \dot y)\dot y + g_v^{(i)}(\theta, \dot\theta, y, \dot y) v \tag{22}$$

where $g_y^{(i)}(\cdot)$, $g_{\dot y}^{(i)}(\cdot)$, $g_v^{(i)}(\cdot)$ are smooth vector functions.

Theorem 1: *Let $q_\star(t)$ be a solution of (1) with $u = u_\star(t)$ being C^1-smooth and suppose $\phi_1(\cdot), \ldots, \phi_n(\cdot)$ are C^2-smooth functions representing an alternative parametrization (5) of this motion. Then, under some mild technical assumptions, in some vicinity of the orbit (3) the dynamics (1) can be equivalently rewritten as (19) and (22).*

B. Transverse Coordinates and Transverse Linearization

It turns out that the dynamical system (19), (22) possesses a natural choice of $(2n-1)$-transverse coordinates

$$x_\perp^{(i)} = \left[I_3^{(i)}(\theta, \dot\theta, \theta_\star(0), \dot\theta_\star(0));\ y;\ \dot y\right]^T \tag{23}$$

for which computing a *transverse linearization* can be done analytically. Here the scalar function $I_3^{(i)}($ is defined by (9).

Theorem 2: *Consider the nonlinear dynamical system (19), (22) and its solution defined for $t \in [0, T]$*

$$y_{1\star} \equiv 0, \ldots, \quad y_{(n-1)\star} \equiv 0, \; \theta = \theta_\star(t), \; v_\star \equiv 0, \tag{24}$$

the linerization of dynamics of the transverse coordinates (23) along (24) is given by the following equations.

- *the linearized dynamics of the scalar quantity $I_3^{(i)}(\cdot)$ are*

$$\frac{dI_\bullet^{(i)}}{dt} = a_{11}^{(i)}(t) I_\bullet^{(i)} + a_{12}^{(i)}(t) Y_{1\bullet} + a_{13}^{(i)}(t) Y_{2\bullet} + b_1^{(i)}(t) V_\bullet \tag{25}$$

with $a_{11}^{(i)}(t) = \frac{2\dot\theta_\star(t)}{\alpha_i(\theta_\star(t))} \cdot \beta_i(\theta_\star(t))$ and

$$\begin{array}{rcl} a_{12}^{(i)}(t) & = & \frac{2\dot\theta_\star(t)}{\alpha_i(\theta_\star(t))} \cdot g_y^{(i)}(\theta_\star(t), \dot\theta_\star(t), \ddot\theta_\star(t), 0, 0) \\ a_{13}^{(i)}(t) & = & \frac{2\dot\theta_\star(t)}{\alpha_i(\theta_\star(t))} \cdot g_{\dot y}^{(i)}(\theta_\star(t), \dot\theta_\star(t), \ddot\theta_\star(t), 0, 0) \\ b_1^{(i)}(t) & = & \frac{2\dot\theta_\star(t)}{\alpha_i(\theta_\star(t))} \cdot g_v^{(i)}(\theta_\star(t), \dot\theta_\star(t), 0, 0) \end{array} \tag{26}$$

- *the linearized dynamics of $[y^T, \dot y^T]$-variables are*

$$\frac{d}{dt}\begin{bmatrix} Y_{1\bullet} \\ Y_{2\bullet} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n-1)\times 1} & \mathbf{0}_{n-1} & \mathbf{1}_{n-1} \\ A_{21}(t) & A_{22}(t) & A_{23}(t) \end{bmatrix} \begin{bmatrix} I_\bullet^{(i)} \\ Y_{1\bullet} \\ Y_{2\bullet} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{(n-1)\times 1} \\ B_2(t) \end{bmatrix} V_\bullet \tag{27}$$

where $\mathbf{0}$ and $\mathbf{1}$ are zero and identity matrices of appropriate dimensions; $A_{2j}(t)$, $B_2(t)$ are matrix functions defined as

$$A_{21} = \left.\frac{\left[\dot\theta \frac{\partial F}{\partial \dot\theta} - \ddot\theta \frac{\partial F}{\partial \theta}\right]}{2(\dot\theta^2 + \ddot\theta^2)}\right|_{\substack{\theta=\theta_\star(t)\\ \dot\theta=\dot\theta_\star(t)\\ \ddot\theta=\ddot\theta_\star(t)\\ y=\dot y=0}}, \quad A_{22} = \left.\frac{\partial F}{\partial y}\right|_{\substack{\theta=\theta_\star(t)\\ \dot\theta=\dot\theta_\star(t)\\ y=\dot y=0}} \tag{28}$$

$$A_{23} = \left.\frac{\partial F}{\partial \dot y}\right|_{\substack{\theta=\theta_\star(t),\, \dot\theta=\dot\theta_\star(t)\\ y=\dot y=0}}, \quad B_2 = N(\theta_\star(t), 0) \tag{29}$$

The choice of transverse coordinates (23) generates a moving Poincaré section $\{S(t)\}_{t\in[0,T]}$ associated with the motion. Theoretically, it can be computed as follows.

1) Change the variables $[\theta, \dot{\theta}, y^T, \dot{y}^T]$ into $[\psi^{(i)}, I_3^{(i)}, y^T, \dot{y}^T]$, where $I_3^{(i)}(\cdot)$ is defined by (9). On this step the scalar variable $\psi^{(i)} = \psi^{(i)}(\theta, \dot{\theta})$ is introduced such that the target trajectory is $\left\{\psi^{(i)} = \psi_\star^{(i)}(t), I_3^{(i)} = 0, y = 0, \dot{y} = 0\right\}$, and $\psi_\star^{(i)}(t) := \psi^{(i)}(\theta_\star(t), \dot{\theta}_\star(t))$ monotonically[7] changes with time.

2) After that, the moving Poincaré section is defined by

$$S^{(i)} := \left\{[\theta; \dot{\theta}; y; \dot{y}] : \psi^{(i)}(\theta, \dot{\theta}) - \psi_\star^{(i)}(t) = 0\right\} \cap \mathcal{O}_\varepsilon(q_\star) \quad (30)$$

for $t \in [0, T]$ and can be expressed in $[q; \dot{q}]$-coordinates using the inverse transformation. Note that for computing such moving Poincaré section one requires the transformation $\psi^{(i)}(\theta, \dot{\theta})$, which is not computed above. However, it is easy to compute the tangential plane for each of $S^{(i)}(t)$

$$TS^{(i)} := \left\{[q; \dot{q}] : (q - q_\star)^T \dot{q}_\star + (\dot{q} - \dot{q}_\star)^T \ddot{q}_\star = 0\right\} \cap \mathcal{O}_\varepsilon(q_\star) \quad (31)$$

C. Orbital Stabilization of Cycles of Mechanical System

The stability concept requires the motion to be defined on infinite time interval. Meanwhile, if the motion is defined on an infinite time interval, then the important step of the smooth re-parametrization (5) of the motion via introducing a new degree of freedom might not be feasible. This step is, however, always possible if the motion is periodic

$$q_\star(t) = q_\star(t + T), \qquad \forall t \quad (32)$$

Moreover, a valid choice is taking θ to be one of the generalized coordinates. Below it is assumed that the target motion $q_\star(t)$ is T-periodic and is non-trivial, i.e. $T > 0$.

Theorem 3: *Given a T-periodic motion $q_\star(t)$ (32) of the controlled mechanical system (1) in response to a C^1-smooth control signal $u_\star(t)$ and given C^2-smooth functions $\phi_1(\cdot)$, ..., $\phi_n(\cdot)$ representing an alternative parametrization (5) of the motion, suppose the conditions of Theorem 1 hold. The following two statements are equivalent.*

1) There is a C^1-smooth T-periodic matrix gain $K(t)$ such that the feedback control law

$$V_\bullet = K(t)\,\left[I_\bullet^{(i)}, Y_{1\bullet}^T, Y_{2\bullet}^T\right]^T, \quad K(t) = K(t+T) \quad (33)$$

stabilizes the origin of the linear system (25)–(29).

2) There exists a C^1-smooth time-invariant feedback control law of the form (18) with

$$v = f(\theta, \dot{\theta}, y, \dot{y}) \quad (34)$$

making the motion (32) of (1) exponentially orbitally stable.

Furthermore, the matrix functions $K(\cdot)$ and $f(\cdot)$ can be constructed as follows:

a) Given (33), a possible choice for (34) is

$$v(t) = K(\tau)\, x_\perp^{(i)}(t), \ \tau = \{s : [q(t); \dot{q}(t)] \in S^{(i)}(s)\} \quad (35)$$

where $x_\perp^{(i)}(t)$ is the vector of transverse coordinates defined by (23), $\mathcal{O}_\varepsilon(q_\star)$ with a small $\varepsilon > 0$ is defined in (3), and τ is an index parameterizing the particular leaf of the moving Poincaré section $\{S(t)\}_{t\in[0,T]}$, see (30), to which the vector $x_\perp^{(i)}(\cdot)$ belongs at the time moments t.

b) Given (34), a possible choice for (33) is

$$V_\bullet = \left(\frac{\dot{\theta}\dfrac{\partial f}{\partial \dot{\theta}} - \ddot{\theta}\dfrac{\partial f}{\partial \theta}}{2(\dot{\theta}^2 + \ddot{\theta}^2)}\, I_\bullet^{(i)} + \frac{\partial f}{\partial y}\, Y_{1\bullet} + \frac{\partial f}{\partial \dot{y}}\, Y_{2\bullet} \right)_{\substack{y=\dot{y}=0 \\ \theta=\theta_\star(t) \\ \dot{\theta}=\dot{\theta}_\star(t) \\ \ddot{\theta}=\ddot{\theta}_\star(t)}} \quad (36)$$

Theorem 3 implies that all possible linear control systems with periodic coefficients (25)–(29) generated by linearizing dynamics of different transverse coordinates (23) are equivalent: stabilization of one implies stabilization of any other. However, it might be difficult to implement the control law (36) due to necessity to solve a nonlinear optimization problem $\left\{s : [q(t); \dot{q}(t)] \in S^{(i)}(s)\right\}$. It can be shown that exponential orbital stabilization is also achieved with

$$v(t) = K(\tau)\, x_\perp^{(i)}(t), \ \tau = \{s : [q(t); \dot{q}(t)] \in TS^{(i)}(s)\} \quad (37)$$

using the easier to compute bundle of hyperplanes (31).

III. Examples

A. Oscillations of an Inverted Spherical Pendulum on a Puck

Consider a point mass spherical pendulum, whose suspension point is moving in the horizontal plane. This mechanical system has 4-dof: x_1, x_2 define a position of the suspension point (a puck) in the horizontal plane; two angles ε_1 and ε_2 define orientation of the spherical pendulum with respect to the vertical, see Fig. 1. The

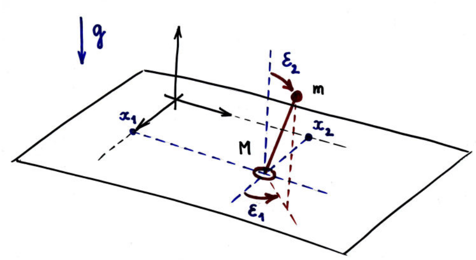

Fig. 1. A spherical pendulum on a puck. The coordinates x_1 and x_2 represent the position of the puck in the horizontal plane; the angles ε_1 and ε_2 give the orientation with respect to the inertia frame.

Lagrangian is $\mathcal{L}(\cdot) = K_{puck}(\cdot) + K_{pend}(\cdot) - \Pi_{pend}(\cdot)$, where $\Pi_{pend} = mgL\cos(\varepsilon_2)$ and the kinetic energies of the puck and the pendulum are $K_{puck} = \frac{M}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right)$ and $K_{pend} = \frac{1}{2}m\Big(\left[\frac{d}{dt}\left\{x_1 + L\cos(\varepsilon_1)\sin(\varepsilon_2)\right\}\right]^2 + \left[\frac{d}{dt}\left\{x_2 + L\sin(\varepsilon_1)\sin(\varepsilon_2)\right\}\right]^2\Big)$. Here M is the mass of the puck; m is the mass of the pendulum; L is the distance to the center of mass of the pendulum from the suspension point; g is the acceleration due to gravity. The dynamics of the spherical pendulum are given by

$$\begin{aligned} &\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varepsilon}_1}\right] - \frac{\partial \mathcal{L}}{\partial \varepsilon_1} = 0, \quad \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varepsilon}_2}\right] - \frac{\partial \mathcal{L}}{\partial \varepsilon_2} = 0, \\ &\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right] = \tau_1, \qquad \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right] = \tau_2 \end{aligned} \quad (38)$$

[7]This can be done under the assumption: $\dot{q}_\star^2(t) + \ddot{q}_\star^2(t) > 0$ for $\forall t$.

Here τ_1 and τ_2 are the controlled forces that can be applied to the puck along x_1 and x_2 axises on horizontal plane.

Planning a periodic motion of the spherical pendulum around its unstable equilibrium is as follows: First, we fix the shape of a curve on horizontal plane (x_1, x_2), and then analyze motions of the pendulum compliant with it. To this end, suppose that variables x_1 and x_2 satisfy the relations

$$x_1(t) = R \cdot \cos(\varepsilon_1(t)), \qquad x_2(t) = R \cdot \sin(\varepsilon_1(t)) \quad (39)$$

where R is a positive constant. With these identities the first two equation of (38) are then

$$\begin{aligned} &\big(M \cdot R^2 + m \cdot (R + L \cdot \sin\varepsilon_2)\big)\, \ddot{\varepsilon}_1 + \\ &\qquad 2m \cdot L \cdot \cos(\varepsilon_2) \cdot (R + L \cdot \sin\varepsilon_2) \cdot \dot{\varepsilon}_1 \dot{\varepsilon}_2 = 0 \\ &L\ddot{\varepsilon}_2 - \cos(\varepsilon_2)\left\{R + L \cdot \cos(\varepsilon_2)\right\} \dot{\varepsilon}_1^2 - g \cdot \sin(\varepsilon_2) = 0 \end{aligned} \quad (40)$$

admitting the family of solutions $\varepsilon_2(t) \equiv \varepsilon_{2\star} \in (-\frac{\pi}{2}, 0)$,

$$\varepsilon_1(t) = \omega_{1\star} \cdot t = \sqrt{\frac{g \cdot \sin(-\varepsilon_2(t))}{\{R + L\cos\varepsilon_2(t)\}\cos\varepsilon_2(t)}} \cdot t \quad (41)$$

Each of them together with relations (39) represents the perpetual rotation (relative equilibrium) of the system above the horizontal when the puck travels with constant angular velocity along the circle of radius R and the pendulum returns to its original position over the period keeping the constant angle $\varepsilon_{2\star}$ with the vertical.

Orbital stabilization of one of the periodic motions (39), (41) can be achieved based on stabilization of transverse linearizion of dynamics as discussed in Theorems 1–3. To start with the procedure, rewrite the motion (39), (41) in the form of virtual holonomic constraints (5)

$$x_1 = R \cdot \cos(\theta),\ x_2 = R \cdot \sin(\theta),\ \varepsilon_1 = \theta,\ \varepsilon_2 = \varepsilon_{2\star}. \quad (42)$$

Invariance of these relations along solutions of the dynamics of the spherical pendulum (38) under appropriate control inputs results into two "$\alpha(\cdot) - \beta(\cdot) - \gamma(\cdot)$" equations with respect to θ, see Lemma 1 and the system (6),

$$\ddot{\theta} = 0,\ \cos(\varepsilon_{2\star})\big(R + L\sin(\varepsilon_{2\star})\big)\,\dot{\theta}^2 + g\sin\varepsilon_{2\star} = 0 \quad (43)$$

A possible choice of the new generalized coordinates (10) instead of $[x_1, x_2, \varepsilon_1, \varepsilon_2]$ is θ and

$$y_1 = x_1 - R\cos\theta,\ y_2 = x_2 - R\sin\theta,\ y_3 = \varepsilon_2 - \varepsilon_{2\star}. \quad (44)$$

Let us define the feedback transformation (18) as

$$[\tau_1, \tau_2]^T = U(\theta, \dot{\theta}, y, \dot{y}) + [v_1, v_2]^T \quad (45)$$

meeting the condition of trivial dynamics for y_1 and y_2

$$\ddot{y}_1 = 0, \quad \ddot{y}_2 = 0 \quad \text{when} \quad v_1 \equiv 0, \quad v_2 \equiv 0,$$

so that $U(\cdot)$ satisfies the interpolation relation (17). Then, the equations of motion (38) in coordinates $[\theta, y]$ and with control inputs $[v_1, v_2]$ take the form of (19)–(22). **Transverse coordinates** $x_\perp$ along the solution

$$\theta_\star(t) = \omega_{1\star} \cdot t, \quad y_{1\star}(t) = y_{2\star}(t) = y_{3\star}(t) = 0 \quad (46)$$

with $v_1(t) = v_2(t) = 0$, are uniquely defined by (23) with

$$I(\theta(t), \dot{\theta}(t), \theta_\star(0), \dot{\theta}_\star(0)) = \dot{\theta}^2(t) - \omega_{1\star}^2 \quad (47)$$

where the constant $\omega_{1\star}$ is from (41). It is straightforward to compute the **transverse linearization** (25)–(29).

It follows from Theorem 3 that the motion (46) of the system (38) can be exponentially orbitally stabilized if and only if this linear system with periodic coefficients is stabilizable over the period. If the parameters of the system and the target motion are $M = 10\ [kg]$, $m = 5\ [kg]$, $L = 2\ [m]$, $R = 5\ [m]$, $\varepsilon_{2\star} = -0.1\ [rad]$ then the period of the target motion (46) is $T = 2\pi/\omega_{1\star} \approx 13.88\ [sec]$ and the linear system is controllable over the period. To stabilize the linear system we found numerically a stabilizing solution of the periodic Riccati differential equation (PRDE). The linear controller was modified into the nonlinear one (35) according to Theorem 3 using the fact that the coordinate needed for computation of the moving Poincaré section in the form of (30) can be taken as $\psi(\theta, \dot{\theta}) = \varepsilon_1 = \theta$ with $\psi_\star(t) = \omega_{1\star} \cdot t$ defined by (41). Fig. 2 illustrates the behavior of the solution of the dynamics of the spherical pendulum with the nonlinear feedback controller, where the evolution of the transverse coordinates (44), (47), see also (23), is depicted versus time.

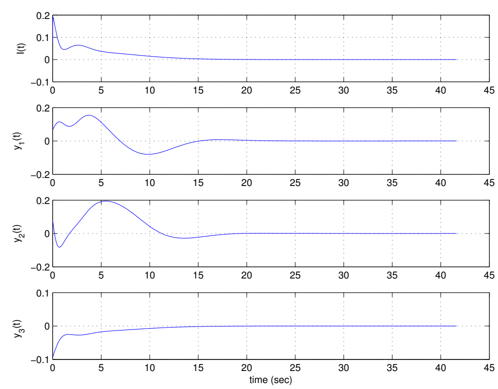

Fig. 2. An evolution of the transverse coordinates—the I-and y_1, y_2, y_3-variables defined by (44), (47)—of the spherical pendulum along the solution of the closed-loop system with a choice of randomly generated displacements of the initial conditions from the ones for the target motion. The components of the transverse coordinates are reduced to about 10^{-5} after 42 seconds, which is about the time of three periods of the cycle.

B. Synchronization of Oscillations of Pendulums on Carts

Consider the problem of synchronization of oscillations of 3-identical pendulum-cart systems around their unstable equilibriums[8], see Fig. 3. Assuming that for each system masses of the cart and the pendulum are 1 [kg], and the distance from the suspension to the center of mass of the pendulum is 1 [m], the dynamics of the system are

$$2\ddot{x}_i + \cos(\theta_i)\ddot{\theta}_i - \sin(\theta_i)\dot{\theta}_i^2 = u_i,\ \cos(\theta_i)\ddot{x}_i + \ddot{\theta}_i - g\sin(\theta_i) = 0, \quad (48)$$

[8]The way to plan a cycle for one cart-pendulum system and to make it then orbitally stable is described in [11]. Note that if such a strategy is applied to each system, there will be no synchronization and the differences in the phases of oscillations would depend on initial conditions.

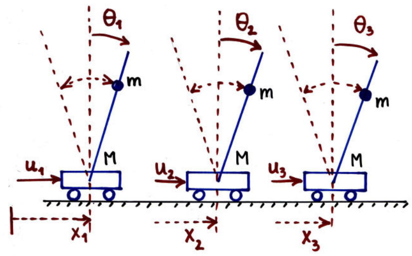

Fig. 3. Three identical cart-pendulum systems. The coordinates x_1, x_2 and x_3 represent positions of the carts along the horizontal, and θ_1, θ_2 and θ_3 give the angles of the pendulums with respect to the vertical.

with $i = 1, 2, 3$. So, the underactuation degree is 3.

Motion planning: Suppose the C^2-smooth function $\phi(\cdot)$ is chosen such that the invariance of the relations

$$x_1 = \phi(\theta_1), \quad x_2 = \phi(\theta_2), \quad x_3 = \phi(\theta_n) \tag{49}$$

results in 3 identical equations with $\theta = \theta_i$, $i = 1, \ldots, 3$,

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0$$

having a T-periodic solution $\theta_\star(t) = \theta_\star(t+T)$. Here

$$\alpha = \phi'(\theta)\cos\theta - 1, \ \beta = \phi''(\theta)\cos\theta, \ \gamma = -g\sin\theta \tag{50}$$

The solution written as 3 pairs

$$\{\theta_i = \theta_\star(t), \quad x_i = \phi\left(\theta_\star(t)\right)\}, \qquad i = 1, 2, 3 \tag{51}$$

is the target synchronous oscillations of all 3 systems.

Orbital stabilization of (51) can be achieved based on stabilization of transverse linearizion of dynamics using Theorems 1–3. Introducing the new coordinates $[\theta, y]$ by

$$\begin{array}{lll} y_1 = x_1 - \phi(\theta_1), & y_2 = x_2 - \phi(\theta_1), & y_3 = x_3 - \phi(\theta_1) \\ y_4 = \theta_1 - \theta_2, & y_5 = \theta_1 - \theta_3, & \theta = \theta_1 \end{array}$$

one can readily check that the dynamics of (48) can be rewritten in the form (19), (22). The feedback transform (18) from $[u_1, u_2, u_3]$ to $[v_1, v_2, v_3]$ can be defined by the following 3 targeted equations

$$\ddot{x}_i - \phi''(\theta_i)\dot{\theta}_i^2 - \phi'(\theta_i)\ddot{\theta}_i = v_i.$$

Transverse coordinates $x_\perp$ along the solution, see (51),

$$\theta = \theta_\star(t), \quad y_{1\star}(t) = y_{2\star}(t) = y_{3\star}(t) = y_{4\star}(t) = y_{5\star}(t) = 0$$

are defined by (23) and (9) with (50). The coefficients of the **transverse linearization** (25)–(29) are straightforward to compute. As argued in [11], the function $\phi(\cdot)$ in (49) can be chosen to meet various specifications on a periodic motion. For instance, with the choice

$$\phi(\theta) = -\left[1 + g/\omega^2\right] \cdot \log[(1+\sin\theta)/\cos\theta] \tag{52}$$

there are oscillations of each of the cart-pendulum systems around their unstable equilibria of period $T = 2\pi/\omega$. In Figs. 4 the motion of three cart-pendulum system (48) with the transverse linearization based nonlinear controller (37) based on Theorem 3 are shown for the case of the constraint function (52). Here the target trajectory is of the period $T \approx 5$ [sec] and with the amplitude 0.2 [rad].

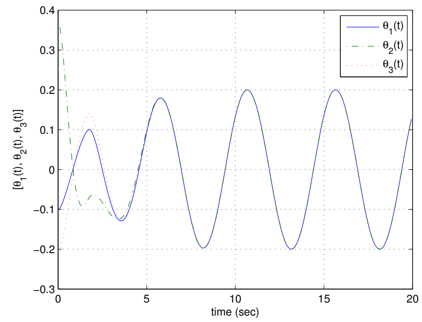

Fig. 4. Synchronization of oscillations for 3 cart-pendulum systems: An evolution of the angles—the θ_1, θ_2, θ_3-variables—along the solution of the closed-loop system. They are synchronized in about one period and have reached after transition the target amplitude of oscillations of 0.2 [rad].

IV. Conclusions

We have described a constructive procedure for computing transverse coordinates for a motion of controlled Euler-Lagrange system of n-degrees of freedom and with m external control variables, $m \le n$. As shown, the case when $m \le n-2$ can be treated extending the arguments from [12] recently elaborated for the case of underactuation one, i.e. when $m = n-1$. The presented technique allows in particular to synthesize orbitally exponentially stabilizing controllers for periodic motions of mechanical system based on static feedback control designs for linear periodic systems.

Conceptually, computing transverse linearization is similar to computing a linearization of a nonlinear controlled system around an equilibrium. However, the result is a linear time-varying control system of reduced order, whose stability/instability/stabilization is an decisive indicator for exponential orbital stability/instability/stabilization of the motion of the nonlinear mechanical system.

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