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On the Controllability of Multiple Dynamic Agents with Fixed Topology

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Abstract— **This paper focuses on the controllability of multi-agent systems with fixed topology based on agreement protocols. We analyze three models of agents: single integrator, double integrator and high-order integrator. For a group of single-integrator agents, controllability is studied in a unified framework for both networks with leader-following structure and networks with undirected graph. Some new necessary/sufficient conditions for controllability of networks of single-integrator agents are established. For networks of double-integrator agents, we prove that controllability of the networks is equivalent to that of networks of single-integrator agents under the same topology and same prescribed leaders. This result is further extended to the case of networks of high-order-integrator agents. Moreover, two influencing factors of controllability of networks are investigated, that is, the selection of leaders and the link weights of graphs.**

I. INTRODUCTION

Distributed control and coordination of multi-agent systems have made a great progress in recent years due to the excellent developments in computing science and sensing & communication technologies ([3]-[18]). Research directions in distributed control and coordination of multi-agent systems include flocking motion of multiple autonomous agents ([1]-[4]), formation control of multiple mobile robots ([5]-[8]), rendezvous problems ([9]-[10]), agreement/consensus problems ([11]-[18]), and so on.

Recently, researchers in the control and dynamical system community have poured a huge amount of effort into studying the agreement of multi-agent systems, which is a type of coordination behavior and drives the states of all the agents to a common desired quantity by implementing appropriate agreement protocols. The mathematical models for agent dynamics include single-integrator model ([11]-[14]), double-integrator model ([15]-[16]), high-order-integrator model ([17]-[18]) and so on. Most agreement protocols are designed according to distributed control theory, where the control laws of each agent depend only on the local information available to it. Applications of this research pertain to cooperative control of unmanned air vehicles, autonomous formation flight, control of communication networks, distributed sensor fusion in sensor networks, swarm-based computing, to name a few. Moreover, controllability is a fundamental and important issue for controlled systems. It plays a basic and fundamental role in numerous research, such as pole-assignment, structure decomposition, optimal control and robust control. However, exploring controllability of multi-agent systems is a challenging task. This is because the behaviors of networks of dynamic agents are affected by a great number of factors, such as dynamics of agents, information flows among agents, and distributed control laws of networks (for example, agreement/consensus algorithms and flocking algorithms).

This paper studies the controllability for networks of dynamic agents with fixed topology based on agreement protocols. In the literature, the controllability of networks was mainly investigated for two kinds of topology structures, namely, leader-following structure ([20]-[23]) and undirected graph ([24]-[25]). We first recall the two kinds of topology structures and establish a unified framework for them via a new concept on graphs. Based on the framework, some new necessary/sufficient conditions for the controllability of networks of single-integrator agents are established. For networks of double-integrator agents, we consider the controllability under two kinds of agreement protocols. It is proved that network of double-integrator agents is completely controllable if and only if network of single-integrator agents is completely controllable under the same topology and same prescribed leaders. For networks of high-order-integrator agents, we mainly discuss why the dynamics of agents can be modeled as a high-order integrator when studying the controllability for multi-agent systems. There are four facts to support our idea. Moreover, it is shown that the controllability of networks of identical agents with dynamics $\dot{x} = Ax + Bu$, where $(A,\ B)$ is completely controllable, is equivalent to that of networks of single-integrator agents under the same topology and same prescribed leaders. At last, we show that the selection of leaders and the coupling weights of graphs have important influence on the controllability of networks.

The remainder of this paper is organized as follows. In the next section we recall two kinds of topology structures of networks. In Section III, IV and V, we study the controllability of networks of single-integrator agents, networks of double-integrator agents and networks of high-order-integrator agents, respectively. Section VI analyzes the influencing factors of controllability of networks, that is, the selection of leaders and the coupling weights of graphs. The last section states some conclusions. Note that all the proofs of theorems are omitted due to space limitations.

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II. TOPOLOGY STRUCTURES OF NETWORKS

In this section, we recall two typical topology structures in the study of controllability of networks in the literature ([20]-[25]), that is, leader-following structure and undirected graph. Through introducing a new concept on graphs, we establish a unified framework for both topology structures.

Consider a multi-agent system composed of $N+n_l$ agents, which are labeled 1 through $N+n_l$. The dynamics of each agent is described by

$$\dot{x}_i = u_i, \quad i \in \underline{N+n_l}, \tag{1}$$

where $x_i \in \mathbb{R}^d$ is the state of agent i, $u_i \in \mathbb{R}^d$ is the control input, and $\underline{N+n_l} = \{1, \ldots, N+n_l\}$ is an index set. In the context of agreement for multi-agent systems, the control input is called an agreement protocol. The interactions or communication links among agents are realized in their control inputs. We employ a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to model the interactions among agents. (Note that all the terms and notations relating to graph theory used in this paper, are consistent with those in [29]. We do not review the details.) Each vertex v_i in the vertex set $\mathcal{V}$ represents an agent i of the multi-agent system, and each arc/edge e_{ij} in the arc/edge set $\mathcal{E}$ means there is an interaction or communication link from agent i to agent j. If for any $e_{ij} \in \mathcal{E}$, $e_{ji} \in \mathcal{E}$ as well, then the communication is said to be bidirectional, namely, when agent i can receive information from agent j, agent j can receive information from agent i as well; otherwise, the communication is said to be unidirectional. When there exist bidirectional communication between a pair of distinct agents, we use an edge (undirected) to depict the communication; while in the case of unidirectional communication, we use an arc (directed) to depict it.

The agreement protocol is taken in the typical form ([11], [20]):

$$u_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i), \quad i \in \underline{N+n_l}. \tag{2}$$

For a given multi-agent system, we refer to $\mathcal{G}_x = (\mathcal{G}, x)$ as a network with value $x \in \mathbb{R}^{d(N+n_l)}$ and graph/topology $\mathcal{G}$, where x is the state collection of all the agents and $\mathcal{G}$ captures the communication links among agents. The controllability problems for networks based on agreement protocols are called *the controlled agreement problems for networks*. For simplicity, we assume that the state dimension of agents $d = 1$. All the results in this paper are valid for any dimension d, just rewriting the expressions based on Kronecker products.

In the literature, the controlled agreement problems are mainly discussed for two kinds of networks: network with leader-following structure ([20]-[23]) and network with undirected graph ([24]-[25]). We start the recall of topology structures with the partition of agents into leaders and followers. For a given multi-agent system, an agent is called a leader if the agent is actuated by some exogenous control inputs besides the interactions coming from its neighboring agents; otherwise, the agent is called a follower.

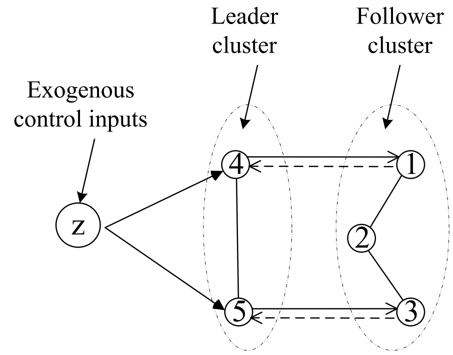

Fig. 1. Schematic diagram for topology structures of networks.

1) *Network with leader-following structure:* Communication links between leaders and followers are unidirectional, specifically, there only exist communication links from leaders to followers; while the communication links among followers are bidirectional. The dynamics of followers abides by the agreement protocol (2), while leaders' dynamics selects control inputs indifferently and freely. In Fig. 1, if we delete the links represented by dashed lines, the graph with vertices 1, 2, 3, 4 and 5 is an example of such a network.

Suppose there are N followers and n_l leaders over the network. Unless it is explicitly specified, we will assume that the followers have small labels and the leaders have large ones, that is, we will label the followers from 1 to N and the leaders from $N+1$ to $N+n_l$. For a network with leader-following structure, the associated Laplacian matrix of graph $\mathcal{G}$ can be written as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ \mathbf{0} & \mathcal{L}_l \end{bmatrix}, \tag{3}$$

where $\mathcal{L}_f$ corresponds to the indices of followers, and $\mathcal{L}_l$ corresponds to the indices of leaders.

Assume the n_l leaders are governed by some exogenous control input $z \in \mathbb{R}^{n_l}$ which can steer the states of the leaders to be arbitrary values. Based on the partition of agents, we can write the agreement dynamics (1)-(2) as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -\begin{bmatrix} \mathcal{L}_f & l_{fl} \\ \mathbf{0} & \mathcal{L}_l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ z \end{bmatrix},$$

where x is the stacked vector of followers' states and y is the stacked vector of leaders' states. Then the dynamics of the followers can be viewed as a controlled linear time-invariant system

$$\dot{x} = -\mathcal{L}_f x - l_{fl} y \tag{4}$$

with the control input being the leaders' states y. We call the above controlled linear time-invariant system to be *a controlled agreement system of the network*. The following definition presents the concept of a network being completely controllable.

Definition 1: For a given network $\mathcal{G}_x$, we say the network is completely controllable under some prescribed leaders, if its associated controlled agreement system is completely controllable.

2) *Network with undirected graph:* Communication links of the whole network are bidirectional (Actually, in the agreement context there is no partition of leader and follower over this kind of networks). In order to transform the agreement dynamics (1)-(2) into a controlled agreement system, some agents are appointed leaders, the movements of which are dominated by some exogenous control inputs besides the states of their neighbors. In Fig. 1, if we view the two pairs of arcs in opposite directions, connecting 4, 1 and 5, 3, as two edges, then the corresponding graph is an example of such a network.

Suppose there are N followers and n_l leaders over the network, then the Laplacian matrix of its graph $\mathcal{G}$ has the form

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{bmatrix}, \tag{5}$$

where $\mathcal{L}_f$ and $\mathcal{L}_l$ have the same meanings as in (3). Then from the dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = - \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ z \end{bmatrix}$$

we can obtain that the controlled agreement system of the network is taken in the form (4) as well.

Remark 1: Observing the properties of the two kinds of networks, we find that a network with leader-following structure has the same controlled agreement system as a network with undirected graph, under the same partition of leaders and followers and same block matrices $\mathcal{L}_f$ and l_{fl} in their respective Laplacian matrices. Thus it is natural to think that there may be some connections between the controllability of networks with leader-following structure and that of networks with undirected graph.

Based on the observation, we next establish a unified framework for the controllability of the two kinds of networks. To this end, we provide the following definition on the underlying undirected graph of a graph.

Definition 2: For a given graph $\mathcal{G} = (\mathcal{V},\ \mathcal{E})$, we say an undirected graph to be *the underlying undirected graph of* $\mathcal{G}$, denoted by $\mathcal{G}^u = (\mathcal{V}^u,\ \mathcal{E}^u)$, if $\mathcal{V}^u = \mathcal{V}$, and $\mathcal{E}^u$ is an edge set of unordered pairs of distinct vertices of $\mathcal{V}^u$, where an edge $e_{ij}^u \in \mathcal{E}^u$ if $e_{ij} \in \mathcal{E}$ or $e_{ji} \in \mathcal{E}$.

Given a network $\mathcal{G}_x$ with some prescribed leaders, denote the corresponding Laplacian matrix of $\mathcal{G}$ as $\mathcal{L} = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{lf} & \mathcal{L}_l \end{bmatrix}$. Let $\mathfrak{S}$ be a collection of graphs, where $\mathcal{G} \in \mathfrak{S}$ means that the Laplacian matrix associated with its underlying undirected graph $\mathcal{G}^u$ has the form

$$\mathcal{L}^u = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \widetilde{\mathcal{L}}_l \end{bmatrix}, \tag{6}$$

where $\mathcal{L}_f$ and $\widetilde{\mathcal{L}}_l$ are symmetric.

Remark 2: It is evident that networks with leader-following structure studied in [20]-[23] and networks with undirected graph studied in [24]-[25] belong to the collection $\mathfrak{S}$. This brings the two kinds of networks into a unified framework. In addition, for a given network $\mathcal{G}_x$ with $\mathcal{G} \in \mathfrak{S}$, the controlled agreement system of the network $\mathcal{G}_x^u$ is the same as that of the network $\mathcal{G}_x$. Consequently, the controllability of $\mathcal{G}_x$ is equivalent to that of $\mathcal{G}_x^u$.

Under the unified framework, we derive some new necessary/sufficient conditions for the controllability of networks with $\mathcal{G} \in \mathfrak{S}$ in the next section, which extend and improve some results in the literature to a certain extent.

III. NETWORKS OF SINGLE-INTEGRATOR AGENTS

Assumption 1: ([22]) For a given graph $\mathcal{G} \in \mathfrak{S}$, let $\mathcal{G}_f$ and $\mathcal{G}_l$ be the respective induced subgraphs by the follower vertices and the leader vertices; $\mathcal{G}_f$ and $\mathcal{G}_l$ are called follower subgraph and leader subgraph, respectively. We assume that the leader subgraph $\mathcal{G}_l$ is linked to all the connected components of the follower subgraph $\mathcal{G}_f$. That is to say, for each of the connected components of $\mathcal{G}_f$, there exists at least one leader in $\mathcal{G}_l$ and one follower in the connected component such that there is a path from the leader to the follower.

Assumption 1 indicates that the state of each follower has direct or indirect connection with the control inputs, i.e., the states of leaders. In the context of control theory of linear systems, this assumption is necessary for the controllability of controlled linear system. We will extend the necessary condition to the case of networks with general digraph in Section VI.

Proposition 1: For a given network $\mathcal{G}_x$ with dynamics (1)-(2) and $\mathcal{G} \in \mathfrak{S}$, suppose there are $n_l \geq 1$ leaders and N followers over the network, and the underlying undirected graph is $\mathcal{G}^u$ with Laplacian matrix $\mathcal{L}^u$. If Assumption 1 is satisfied, then the corresponding controlled agreement system (4) is completely controllable if and only if there are no common eigenvalues of $\mathcal{L}^u$ and $\mathcal{L}_f$.

Proposition 1 presents a necessary and sufficient condition for the controllability of the network $\mathcal{G}_x$ with $n_l \geq 1$ leaders based on the eigenvalues of $\mathcal{L}_f$. We next establish a necessary condition for the controllability of the network characterized by the eigenvalues and the eigenvectors of $\mathcal{L}_f$. Note that there is no any theoretical result on the controllability of networks with leader-following structure and multiple leaders in the literature, although the problem was first proposed in [20].

Theorem 1: For a given network $\mathcal{G}_x$ with dynamics (1)-(2) and $\mathcal{G} \in \mathfrak{S}$, suppose there are $n_l \geq 1$ leaders and N followers over the network, and Assumption 1 is satisfied. If the associated controlled agreement system (4) is completely controllable, then ① there exists no eigenvalue of $\mathcal{L}_f$ with multiplicity more than n_l; ② if there exists an eigenvalue of $\mathcal{L}_f$ with multiplicity $k \leq n_l$, then the product matrix Ml_{fl} has full row rank, where $M \in \mathbb{R}^{k \times N}$ is composed of the k linearly independent left eigenvectors corresponding to the eigenvalue.

Remark 3: In the case of networks with one leader, the result of Theorem 1 is consistent with that of Theorem IV.1 of [20].

Remark 4: Proposition 1 and the proof of Theorem 1 are derived based on Remark 2, i.e., the equivalence in the context of modeling the controlled agreement systems for the network $\mathcal{G}_x \in \mathfrak{S}$ and the network $\mathcal{G}_x^u$. In the viewpoint of networks with leader-following structure, Proposition 1 provides a simpler criterion for the controllability of (4) than the conditions of Theorem 1. This improves the results of [20]-[22]. In the viewpoint of networks with undirected graph, Proposition 1 expands the applicable ranges for the results of [25].

IV. NETWORKS OF DOUBLE-INTEGRATOR AGENTS

In this section, we consider the controlled agreement problems for a network of $N + n_l$ double-integrator agents. The dynamics of each agent is given by

$$\dot{x}_i = v_i, \quad \dot{v}_i = u_i, \quad i \in \underline{N + n_l}, \tag{7}$$

where $x_i \in \mathbb{R}$ and $v_i \in \mathbb{R}$ are the position information and the velocity information of agent i respectively, and $u_i \in \mathbb{R}$ is the control input. We will study the controlled agreement problems for the network under two agreement protocols: one is with the feedbacks of relative velocities ([15])

$$u_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i) + k \sum_{j \in \mathcal{N}_i} (v_j - v_i), \quad i \in \underline{N + n_l}, \tag{8}$$

and the other is with the feedback of absolute velocity ([16])

$$u_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i) + k v_i, \quad i \in \underline{N + n_l}, \tag{9}$$

where $k \neq 0$ is a feedback gain.

Next, we investigate the controlled agreement problems for the network in (7) with topology modeled by a digraph $\mathcal{G}$. Assume there are $n_l \geq 1$ leaders and N followers. The movement of these leaders are dominated by some exogenous control inputs $z \in \mathbb{R}^{n_l}$, which can drive the states of the leaders to arbitrary values. Label the followers 1 through N, and the leaders $N+1$ through $N + n_l$. Then the associated Laplacian matrix of $\mathcal{G}$ has the form

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{lf} & \mathcal{L}_l \end{bmatrix},$$

where $\mathcal{L}_f \in \mathbb{R}^{N \times N}$ and $\mathcal{L}_l \in \mathbb{R}^{n_l \times n_l}$ have the same meanings as those in (3), and l_{lf} indicates the communication links from the follower group to the leader group.

According to the partition of leaders and followers, the multi-agent system (7) under protocol (8) can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & I_N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_m \\ -\mathcal{L}_f & -l_{fl} & -k\mathcal{L}_f & -kl_{fl} \\ -l_{lf} & -\mathcal{L}_l & -l_{lf} & -k\mathcal{L}_l \end{bmatrix} \begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ z \end{bmatrix},$$

where $x = [x_1 \ \cdots \ x_N]^T$ is the stacked vector of the followers' positions, $y = [x_{N+1} \ \cdots \ x_{N+n_l}]^T$ is the stacked vector of the leaders' positions, $v_x = [v_1 \ \cdots \ v_N]^T$ and $v_y = [v_{N+1} \ \cdots \ v_{N+n_l}]^T$ are the velocity vector of the followers and the velocity vector of the leaders respectively, and $z \in \mathbb{R}^{n_l}$ is the collection of exogenous control inputs of the leaders. Consequently, we can describe the dynamics of the followers to be the following controlled linear time-invariant system

$$\begin{bmatrix} \dot{x} \\ \dot{v}_x \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_N \\ -\mathcal{L}_f & -k\mathcal{L}_f \end{bmatrix} \begin{bmatrix} x \\ v_x \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -l_{fl} & -kl_{fl} \end{bmatrix} \begin{bmatrix} y \\ v_y \end{bmatrix}, \tag{10}$$

where the control inputs are the positions and velocities of the leaders.

For the multi-agent system (7) under protocol (9), the dynamics of the whole closed-loop system is represented by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & I_N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_m \\ -\mathcal{L}_f & -l_{fl} & kI_N & \mathbf{0} \\ -l_{lf} & -\mathcal{L}_l & \mathbf{0} & kI_m \end{bmatrix} \begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ z \end{bmatrix},$$

where x, y, v_x, v_y and z are defined as above. Then the dynamics of the followers can be written as the following controlled linear time-invariant system

$$\begin{bmatrix} \dot{x} \\ \dot{v}_x \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_N \\ -\mathcal{L}_f & kI_N \end{bmatrix} \begin{bmatrix} x \\ v_x \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -l_{fl} \end{bmatrix} y, \tag{11}$$

where the control inputs are the positions of the leaders.

The terminologies and notations which appear in this section have the same meanings as those in Section II.

Theorem 2: For a given network $\mathcal{G}_x$ with dynamics (7)-(8) and a digraph $\mathcal{G}$, suppose there are n_l leaders and N followers. Then the associated controlled agreement system (10) is completely controllable if and only if the controlled agreement system (4), i.e., $(-\mathcal{L}_f, \ -l_{fl})$, is completely controllable.

Analogously, we derive the following necessary and sufficient condition for the controllability of the system (7) under protocol (9).

Theorem 3: For a given network $\mathcal{G}_x$ with dynamics (7)(9) and a digraph $\mathcal{G}$, suppose there are n_l leaders and N followers. Then the associated controlled agreement system (11) is completely controllable if and only if the controlled agreement system (4) is completely controllable.

The above results are very interesting as the controllability of networks of double-integrator agents is equivalent to that of networks of single-integrator agents under the same topology and same prescribed leaders. In other words, the controllability of networks (7)-(8) (or (7)(9)) is independent of the dynamics of agents, and only determined by their topologies. In this sense, for network (7)-(8) (or (7)(9)) with graph $\mathcal{G} \in \mathfrak{S}$, the criteria for the controllability of networks of single-integrator agents established in Proposition 1 and Theorem 1 are suitable for determining the controllability of the network.

V. NETWORKS OF HIGH-ORDER-INTEGRATOR AGENTS

In this section, we consider the controlled agreement problems for a network of $N + n_l$ high-order-integrator agents. The dynamics of each agent is given by the following mth-order integrator

$$\begin{aligned} &\dot{x}_i^{(1)} = x_i^{(2)}, \ \cdots, \ \dot{x}_i^{(m-1)} = x_i^{(m)}, \\ &\dot{x}_i^{(m)} = u_i, \quad i \in \underline{N + n_l}, \end{aligned} \tag{12}$$

where m is a positive integer which denotes the order of the differential equations (12); $x_i^{(k+1)},\ k \in \underline{m-1}$ is the kth-order derivative of $x_i^{(1)}$; $u_i \in \mathbb{R}$ is the control input. We will study the controlled agreement problems for such a network under the following two agreement protocols: one is with the feedback of all relative information among agents (see [17])

$$u_i = \sum_{k=0}^{m-1} \sum_{j \in \mathcal{N}_i} c_k (x_j^{(k+1)} - x_i^{(k+1)}),\ i \in \underline{N + n_l}; \quad (13)$$

the other is with the feedback of partial relative information among agents (see [18])

$$u_i = \sum_{k=1}^{m-1} c_k x_i^{(k+1)} + \sum_{j \in \mathcal{N}_i} (x_j^{(1)} - x_i^{(1)}),\ i \in \underline{N + n_l}, \quad (14)$$

where $c_0,\ c_1,\ \cdots,\ c_{m-1}$ are nonzero feedback gains.

We first explain why the high-order integrator is employed to describe the dynamics of agents. The idea is inspired by the following four facts. First, any completely controllable continuous-time linear time-invariant (LTI) system, having the state-space equation $\dot{x} = Ax + Bu$, can be equivalently brought into a collection of decoupled and independently controlled chains of integrators, under an appropriate non-singular linear transformation and a suitable state feedback (see [28]). Second, denoting the controlled system $\dot{x} = Ax + Bu$ as the matrix pair (A, B), the set of all completely controllable pairs (A, B) is open and dense in the space composed of all matrix pairs (A, B) (see [26] and the references therein). Third, for a group of autonomous agents with dynamics $\dot{x}_i = Ax_i + bv_i,\ i \in \underline{N + n_l}$, which is completely controllable and can be transformed into the high-order integrator (12), if the multi-agent system (12) is completely controllable, then the group of agents with dynamics $\dot{x}_i = Ax_i + bv_i$ is completely controllable. Note that we take a single-input LTI system for example. This is because any completely controllable multi-input LTI system can be transformed into a completely controllable single-input LTI system (see [19]). Finally, the high-order-integrator model of agents is a generalization of the single-integrator and the double-integrator model, which are widely studied in the literature such as [11], [12], [15] and [16]. Hence it is of physical interest and of theoretical interest to investigate the controllability for networks of high-order-integrator agents. Due to space limitations, we will not review the first and the third facts.

Following the same manner as in Section IV, we can transform system (12)-(13) and system (12)(14) into their respective controlled linear time-invariant systems (We leave out the details due to space limitations). Moreover, given a network $\mathcal{G}_x$ with dynamics (12)-(13) (or (12)(14)) and digraph $\mathcal{G}$, suppose there are n_l leaders and N followers, then all the results established in Section IV are suitable for the network. This further implies that the controllability of networks of identical agents with dynamics $\dot{x}_i = Ax_i + bv_i$, where $(A,\ b)$ is completely controllable, is equivalent to that of networks of single-integrator agents under the same topology and same prescribed leaders.

VI. Some Discussions

In this section, we investigate some influencing factors of the controllability. Unless otherwise stated, a network means network of single-integrator agents throughout this section.

One influencing factor of the controllability is the selection of leaders, namely, the positions and the number. In the literature [22], a necessary condition for the controllability of networks with leader-following structure has been established. We next extend the result to the case of networks with digraph.

Theorem 4: For a given network $\mathcal{G}_x$ with dynamics (1)-(2) and digraph $\mathcal{G}$, suppose there are n_l leaders and N followers. Denote the induced subgraph of the followers as $\mathcal{G}_f$ and the induced subgraph of the leaders as $\mathcal{G}_l$. If the network is completely controllable, then for every strongly connected component of $\mathcal{G}_f$, there exists at least one leader such that there is a path from the leader to the strongly connected component.

Theorem 4 indicates that the positions of leaders have important effect on the controllability.

In addition, Proposition 1 implies that the number of leaders has great influence on the controllability as well. Specifically, for a given network of N agents with topology $\mathcal{G} \in \mathfrak{G}$, let $\mathcal{G}^u$ be the underlying undirected graph of $\mathcal{G}$ and $\mathcal{L}^u$ be the associated Laplacian to $\mathcal{G}^u$. Take $k\ (< N)$ agents arbitrarily as the leaders of the network. Denote the principle sub-matrix of $\mathcal{L}^u$ as $\mathcal{L}_f \in \mathbb{R}^{(n-k)\times(N-k)}$, which is obtained by deleting the rows and the columns indicated by the indices of the leaders. If there is a principle sub-matrix $\mathcal{L}_f$ has no common eigenvalue with $\mathcal{L}^u$, then the network with k leaders is completely controllable, according to Proposition 1. If any principle sub-matrix with order $N-k$ of $\mathcal{L}^u$ shares common eigenvalue with it, then the network is not controlled by any k leaders, in other words, any k agents of the network are not able to control the remainder completely.

It is known that the controllability of a LTI system depends upon the structures and the parameters of its coefficient matrices. In the proceeding sections, we have discussed the controlled agreement problems for networks with unweighted graph. However, agreement protocols are given in the forms of weighted ones in most literature, i.e., there exist link weights distinct to 1 (see [11]-[12], [15]-[18]). Ignoring link weights of the network means ignoring the parameters of the system.

As a matter of fact, for a given unweighted graph and some prescribed leaders, we can turn an uncontrollable network with the unweighted graph into a controllable network by selecting appropriate weights for the communication links. As a trivial example, consider the unweighted complete graph with three vertices. (Note that in [20], it is proved that a network with unweighted complete graph is uncontrollable.) If we put the following weighted adjacency matrix on the graph $\mathcal{A} = \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix}$, then by appointing any one agent as leader, the resulting controlled agreement system $(-\mathcal{L}_f,\ -l_{fl})$ is completely controllable. This example shows that the link weights have important effect on the controllability of networks. Notice that the results established in the previous sections are valid for networks with weighted graph as well. Therefore, for a given unweighted graph and some prescribed leaders, we wonder how to select appropriate link weights such that the resulting weighted network is completely controllable, and whether or not there are some connections between the controllability of multi-agent systems and the structural controllability of linear systems (see [26], [27]). In the light of this investigation, we derive the following result on the structural controllability of networks.

Theorem 5: For a given network $\mathcal{G}_x$ with dynamics (1)-(2) and digraph $\mathcal{G}$, suppose there are n_l leaders and N followers. Denote the induced subgraph of the followers as $\mathcal{G}_f$ and the induced subgraph of the leaders as $\mathcal{G}_l$. Then the network is structurally controllable if and only if for every strongly connected component of $\mathcal{G}_f$, there exists at least one leader such that there is a path from the leader to the strongly connected component.

Note that the network is called to be structural controllable, if the system $(-\mathcal{L}_f, \ -l_{fl})$ is structural controllable.

In summary, for a network with unweighted graph, the link weights, the positions and the number of leaders have important influence on the controllability. The research on controllability of multi-agent systems needs more exploration of the properties of graphs. Moreover, there are some prospectives for further research on this subject, for example, selecting appropriate link weights such that the network is completely controllable; choosing proper positions and suitable number of leaders to make the network completely controllable.

VII. CONCLUSIONS

This paper has investigated the controllability of multi-agent systems. Some new necessary/sufficient conditions for controllability of networks of single-integrator agents have been established. For a group of double-integrator agents, it has been proved that the controllability of the entire group is equivalent to that of networks of single-integrator agents under the same topology and same prescribed leaders. This result has been extended to networks of high-order-integrator agents. At last, it has been shown that the selection of leaders and the link weights have great effect on the controllability.

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