2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009



# Performance Robustness of MRAC under Reduction in Actuator Effectiveness

*Qian Sang* and *Gang Tao*

Department of Electrical and Computer Engineering
University of Virginia
Charlottesville, VA 22904

*Abstract*— **The problem of performance robustness of model reference adaptive control (MARC) schemes under reduction in actuator effectiveness (loss-of-effectiveness) is studied in this paper. The reduction is modeled by an uncertain gain matrix representing the actuator effectiveness at the control inputs. MRAC systems are analyzed to determine the robustness of the desired closed-loop performance of stability and asymptotic tracking with respect to such reductions. Conclusions are drawn for direct multivariable MRAC as well as indirect MRAC for both continuous-time and discrete-time schemes. A simulation study is presented to illustrate some of the theoretical results.**

## I. INTRODUCTION

In aircraft flight control systems, actuators are driven by control signals to generate aerodynamic forces and moments on the aircraft to achieve the control objectives. Uncertain actuator failures and damages can occur, including reduction in effectiveness (loss-of-effectiveness, in some research papers), floating, lock-in-place, and hard-over. Lack of appropriate compensation can severely deteriorate system performance, or even lead to instability and cause catastrophic accidents.

Fault detection and diagnosis is one approach to actuator failure compensation [5]. However, false detection due to incomplete failure classification and discrepancy between the model and real plant restricts its application. Adaptive control, on the other hand, with the ability to cope with parametric, structural, and environmental uncertainties is promising and viable for successful compensation for the large uncertainties introduced by actuator failures and damages. In adaptive control, explicit fault detection and diagnosis is not needed for ensuring desired performance. Both direct and indirect adaptive control designs have been proposed in the literature. In [1], several algorithms were compared and simulation results demonstrated their successful compensation for reduction in actuator effectiveness. The direct adaptive control design in [7] achieved satisfactory performance under both reduction in actuator effectiveness and saturation. Indirect adaptive control schemes were used in [3] for compensation of reduction in actuator effectiveness based on parameter-adaptive and variable-structure approaches. Actuator failure modeling was performed in [11] for systems with floating and lock-in-place types of failure, and both state feedback and output feedback direct MRAC schemes were proposed. In [2], a multiple model based scheme was presented for fast and accurate flight control reconfiguration. One common feature of these designs is a proper parametrization of actuator failures and damages based on which direct or indirect adaptive control schemes are developed to achieve stability and tracking performance. Some other designs exist such as those using adaptive neural networks [4], robust control designs [12], and so on.

In this paper, we focus on reduction in actuator effectiveness, the type of actuator failure in which partial control surface is damaged and the effectiveness reduces to an uncertain fraction of the normal level (that without reduction in effectiveness). We conduct the performance robustness analysis of some MRAC schemes with respect to such reductions, which can be represented by an actuator effectiveness matrix at the control inputs. The objective is to determine whether or not an MRAC scheme designed for the normal case, i.e., that without reduction in actuator effectiveness, can still achieve the desired closed-loop stability and asymptotic tracking performance in the presence of reduction in effectiveness. Our recent results on the gain margins of MRAC systems [8], [9] are applied for solutions of this problem.

Next, we will formulate the performance robustness problem of MRAC under reduction in actuator effectiveness in Section II. We will show in Section III that the closed-loop performance of direct continuous-time multivariable MRAC designs is robust to arbitrary reduction in actuator effectiveness, while as proved in Section IV, that of their discrete-time counterparts is design-based. In particular, the performance of the multivariable MRAC design based on the LDS decomposition of the high frequency gain matrix may not be robust to some reduction in actuator effectiveness, but that of the LDU based design is. Indirect MRAC schemes are considered in Section V. The main features of the performance robustness properties of the LDS based MRAC design are illustrated by simulation results in Section VI.

## II. PROBLEM STATEMENT

The performance robustness problem of MRAC systems in the presence of reduction in actuator effectiveness is formulated in this section. The standard multivariable MRAC assumptions, including those based on the high frequency gain matrix decompositions, are also presented.

### A. *The Performance Robustness Issue for MRAC in the Presence of Reduction in Actuator Effectiveness*

We consider the $M$-input $M$-output linear time-invariant plant with transfer matrix representation:

$$y(t) = G(D)[u_p](t), \tag{1}$$

978-1-4244-4524-0/09/$25.00 ©2009 AACC

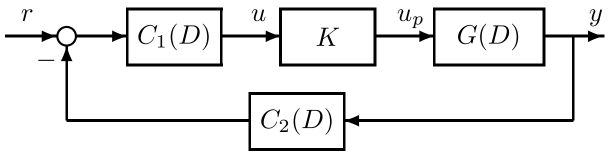


Fig. 1. Adaptive control system with reduction in actuator effectiveness.

where $u_p(t), y(t) \in \mathbb{R}^M$ are the plant input and output vectors, $G(D) = Z(D)P^{-1}(D)$ is strictly proper and full rank, and $Z(D)$, $P(D) \in \mathbb{R}^{M\times M}$ are right coprime polynomial matrices with $P(D)$ being column proper. [1]

Denoting $u(t) \in \mathbb{R}^M$ as the control signal generated from an adaptive controller, and $K \in \mathbb{R}^{M\times M}$ as the actuator effectiveness matrix that assumes the following form

$$K = \mathrm{diag}\{k_1, \dots, k_M\},\ 0 < k_i \le 1,\ i = 1, 2, \dots, M, \quad (2)$$

we have

$$u_p(t) = Ku(t). \quad (3)$$

In the absence of reduction in actuator effectiveness, $K$ is an identity matrix $I_M \in \mathbb{R}^{M\times M}$, and $u_p(t) = u(t)$.

Figure 1 depicts the scenario we consider in this paper. The adaptive controller, denoted by $C_1(D)$ and $C_2(D)$, has been designed for $K = I_M$, and ensures closed-loop stability (signal boundedness) and asymptotic output tracking performance, that is, $y(t)$ asymptotically tracks a desired trajectory $y_m(t)$ generated from a reference model system

$$y_m(t) = W_m(D)[r](t), \quad (4)$$

where $W_m(D) \in \mathbb{R}^{M\times M}$ is a rational transfer matrix and $r(t) \in \mathbb{R}^M$ is a bounded reference input signal. Our objective is to study whether or not the designed adaptive controller still ensures the desired closed-loop performance in the presence of reduction in actuator effectiveness, i.e., when $0 < K < I_M$.

The high frequency gain matrix is defined as $K_p = \lim_{D\to\infty} \xi_m(D)G(D)$ with $\xi_m(D)$ being the modified interactor matrix of $G(D)$. For MRAC designs based on decompositions of $K_p$, the reference model transfer matrix $W_m(D)$ in (4) is chosen to be $W_m(D) = \xi_m^{-1}(D)$.

**Assumptions**. The standard MRAC assumptions are: (A.1) All zeros of $G(D)$ are stable; (A.2) the observability index $\nu$ of $G(D)$ is known; (A.3) a modified interactor matrix $\xi_m(D)$, which has a stable inverse, of $G(D)$ is known; (A.4) all leading principal minors of $K_p$, $\Delta_i$, are nonzero with their signs known. Besides, for discrete-time designs, we assume: (A.4D) some upper bounds $d_i^0$ of $|d_i^*| = |\frac{\Delta_i}{\Delta_{i-1}}|$ with $\Delta_0 = 1$, such that $0 < |d_i^*| \le d_i^0, i = 1, 2, \dots, M$, are known.

For multivariable MRAC an important design condition is given for $K_p$ in Assumption (A.4). In the presence of reduction in actuator effectiveness, the high frequency gain matrix becomes $K_pK$ with $K$ in (2). Next we present some basic knowledge about decomposition of $K_p$ and the invariance under reduction in actuator effectiveness.

[1]The symbol $D$ is used, in the continuous-time case, as the time-differentiation operator: $D[x](t) = \dot{x}(t)$, $t \in [0, +\infty)$; or in the discrete-time case, as the time-advance operator: $D[x](t) = x(t+1)$, $t \in \{0, 1, 2, 3, \dots\}$.

### *B. High Frequency Gain Matrix Decomposition*

With Assumption (A.4), the LDU decomposition of $K_p$ is $K_p = LD^*U$ for some $M \times M$ unity lower triangular matrix $L$, unity upper triangular matrix $U$, and diagonal matrix

$$D^* = \mathrm{diag}\{d_1^*, \dots, d_M^*\} = \mathrm{diag}\left\{\Delta_1, \dots, \frac{\Delta_M}{\Delta_{M-1}}\right\}. \quad (5)$$

Its LDS decomposition, $K_p = L_sD_sS$, follows with $L_s = LD_s(U^T)^{-1}D_s^{-1}$, $S = U^TD_s^{-1}D^*U$, and

$$D_s = \mathrm{diag}\left\{\mathrm{sign}[\Delta_1]\gamma_1, \dots, \mathrm{sign}\left[\frac{\Delta_M}{\Delta_{M-1}}\right]\gamma_M\right\} \quad (6)$$

with $\gamma_i > 0$, $i = 1, 2, \dots, M$, which can be arbitrary.

**Decomposition of** $K_pK$. The following lemma relates the LDU decomposition of $K_p$ with that of $K_pK$, and is crucial for the performance robustness analysis of matrix decomposition based multivariable MRAC designs.

**Lemma 1** [9]. *The matrix $K_pK \in \mathbb{R}^{M\times M}$ with $K$ in (2) has a unique LDU decomposition*

$$K_pK = \bar{L}\bar{D}^*\bar{U},\ \bar{L} = L,\ \bar{D}^* = D^*K,\ \bar{U} = K^{-1}UK, \quad (7)$$

*where $K_p = LD^*U$ is the LDU decomposition of the non-singular matrix $K_p$ with nonzero leading principle minors.*

With Lemma 1, we can conclude that the leading principle minors of $K_pK$ are also nonzero, and their sign information is the same as that of $K_p$. In other words, the design conditions stated in Assumption (A.4) is invariant in the presence of reduction in actuator effectiveness. Furthermore, Lemma 1 is crucial for obtaining the performance robustness conditions for direct discrete-time MRAC designs.

The gain margin (GM) problem formulation in our recent work [8], [9] is similar to that of the performance robustness problem considered in this paper, with the difference in that the control gain variations are not bounded from top by 1, i.e., $k_i > 0, i = 1, 2, \dots, M$ in (2). Therefore, the GM results can be applied for solutions of the performance robustness problem under reduction in actuator effectiveness. As shown in the following sections, by comparing the GM results with the range of actuator effectiveness matrix defined in (2), we can determine whether or not the performance of an MRAC design is robust to and to what extent it can handle the reduction in actuator effectiveness.

## III. Continuous-Time MIMO Direct MRAC

There are some well-known direct multivariable MRAC designs based on decompositions of $K_p$ [6], [10]. To study their performance robustness under reduction in actuator effectiveness, the actuator effectiveness matrix $K$, together with the plant (1) is treated as the new controlled plant.

The gain margins of continuous-time MRAC designs are infinity [8], [9], that is, the closed-loop performance remains for any control gain at the control inputs, thus we have:

**Proposition 1**. *The performance of direct continuous-time multivariable MRAC designs based on high frequency gain matrix decompositions is robust with respect to reduction in actuator effectiveness.*

Proof: From (1) and (3), we have

$$y(t) = G(s)K[u](t)$$

as the new controlled plant, and its high frequency gain matrix is $K_pK$. From (5) and (7) in Lemma 1, we can see that the presence of reduction in actuator effectiveness, i.e., $0 < K < I_M$ does not violate the assumptions of nonzero leading principle minors of the high frequency gain matrix under (A.4), and their sign information is the same as that of $K_p$. Therefore, the adaptive law for $K = I_M$ can still be used for $0 < K < I_M$ to achieve closed-loop performance of signal boundedness and asymptotic tracking. This is also true for direct continuous-time SISO MRAC designs. $\nabla$

The gain matrix decomposition based continuous-time MRAC designs can adaptively compensate for arbitrary reduction in actuator effectiveness to ensure desired closed-loop performance. However, for discrete-time designs, although the performance of SISO MRAC systems is robust with respect to reduction in actuator effectiveness, the performance robustness of multivariable MRAC systems are design-based, which is shown in the next section.

## IV. Discrete-Time MIMO Direct MRAC

We will first analyze the performance robustness properties of the MRAC design based on the LDS decomposition of $K_p$ in this section. As a comparison, the LDU based design is analyzed to show that the performance robustness of the discrete-time MRAC systems under reduction in actuator effectiveness is design-based.

### A. *Performance Robustness of the LDS Based Design*

Among the gain matrix decomposition based MRAC designs, the LDS based design assumes a simple adaptive controller structure, and its error model parametrization and the auxiliary signal structures are simple for implementation.

#### *A.1. The LDS Based Design*

The controller for $K = I_M$ has the standard model reference control structure:

$$u(t) = \Theta_1^T \omega_1(t) + \Theta_2^T \omega_2(t) + \Theta_{20} y(t) + \Theta_3 r(t), \quad (8)$$

where $\omega_1(t) = F(z)[u](t)$, $\omega_2(t) = F(z)[y](t)$ with $F(z) = \frac{A(z)}{\Lambda(z)}$, $A(z) = [I_M, zI_M, \ldots, z^{\nu-2}I_M]^T$ for a stable monic polynomial $\Lambda(z)$ of degree $\nu - 1$. The controller parameter matrices $\Theta_i = \Theta_i(t)$, $i = 1, 2, 20, 3$, updated from an adaptive law, are the time-varying estimates of the nominal controller parameter matrices $\Theta_1^* = [\Theta_{11}^*, \ldots, \Theta_{1\nu-1}^*]^T$, $\Theta_2^* = [\Theta_{21}^*, \ldots, \Theta_{2\nu-1}^*]^T$, $\Theta_{ij}^*, \Theta_{20}^*, \Theta_3^* \in \mathbb{R}^{M \times M}$, $i = 1, 2, j = 1, \ldots, \nu - 1$, satisfying the plant-model transfer matrix matching equation

$$\begin{aligned} &\Theta_1^{*T} A(z)P(z) + \left(\Theta_2^{*T} A(z) + \Lambda(z)\Theta_{20}^*\right) Z(z) \\ &= \Lambda(z) \left(P(z) - \Theta_3^* \xi_m(z) Z(z)\right). \end{aligned} \quad (9)$$

**Error model**. With $\Theta_3^* = K_p^{-1}$ and $K_p = L_s D_s S$, the matching equation (9) leads to

$$\begin{aligned} &D_s S(u(t) - \Theta_1^{*T} \omega_1(t) - \Theta_2^{*T} \omega_2(t) - \Theta_{20}^* y(t) - \Theta_3^* r(t)) \\ &= L_s^{-1} \xi_m(z)[y - y_m](t). \end{aligned} \quad (10)$$

With $\Theta(t) = [\Theta_1^T(t), \Theta_2^T(t), \Theta_{20}(t), \Theta_3(t)]^T$, $\Theta^* = [\Theta_1^{*T}, \Theta_2^{*T}, \Theta_{20}^*, \Theta_3^*]^T$, (8) and (10) yield

$$\xi_m(z)[y - y_m](t) + \Theta_0^* \xi_m(z)[y - y_m](t) = D_s S \tilde{\Theta}^T(t) \omega(t),$$

where $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$, $\omega(t) = [\omega_1^T(t), \omega_2^T(t), y(t), r(t)]^T$, and $\Theta_0^* = L_s^{-1} - I_M$ has a special lower triangular form with zero diagonal elements. Denote the parameter vectors consisting of the nonzero parameters in each row of $\Theta_0^*$ to be $\theta_i^* = [\theta_{i1}^*, \ldots, \theta_{ii-1}^*]^T \in \mathbb{R}^{i-1}$, and let their estimates to be $\theta_i(t) = [\theta_{i1}(t), \ldots, \theta_{ii-1}(t)]^T \in \mathbb{R}^{i-1}$, $i = 2, 3, \ldots, M$. Introduce the estimation error as

$$\epsilon(t) = \bar{e}(t) + [0, \theta_2^T(t)\eta_2(t), \ldots, \theta_M^T(t)\eta_M(t)]^T + \Psi(t)\xi(t),$$

where $\bar{e}(t) = \xi_m(z)\frac{1}{f(z)}[y - y_m](t) = [\bar{e}_1(t), \ldots, \bar{e}_M(t)]^T$ with $f(z)$ being a chosen stable monic polynomial of the same degree as the maximum degree of $\xi_m(z)$, $\eta_i(t) = [\bar{e}_1(t), \ldots, \bar{e}_{i-1}(t)]^T$, $i = 2, 3, \ldots, M$, $\Psi(t)$ is the estimate of $\Psi^* = D_s S$, and $\xi(t) = \Theta^T(t)\zeta(t) - h(z)[\Theta^T \omega](t)$ with $\zeta(t) = h(z)[\omega](t)$. The following error equation is obtained

$$\begin{aligned} \epsilon(t) = &[0, \tilde{\theta}_2^T(t)\eta_2(t), \tilde{\theta}_3^T(t)\eta_3(t), \ldots, \tilde{\theta}_M^T(t)\eta_M(t)]^T \\ &+ D_s S \tilde{\Theta}^T(t)\zeta(t) + \tilde{\Psi}(t)\xi(t) \end{aligned} \quad (11)$$

with $\tilde{\theta}_i(t) = \theta_i(t) - \theta_i^*$ and $\tilde{\Psi}(t) = \Psi(t) - \Psi^*$.

**Adaptive law**. Based on the error model (11), we choose the following gradient adaptive laws:

$$\theta_i(t+1) - \theta_i(t) = -\frac{\Gamma_{\theta i}\epsilon_i(t)\eta_i(t)}{m^2(t)}, \quad (12)$$

$$\Theta^T(t+1) - \Theta^T(t) = -\frac{D_s \epsilon(t)\zeta^T(t)}{m^2(t)}, \quad (13)$$

$$\Psi(t+1) - \Psi(t) = -\frac{\Gamma \epsilon(t)\xi^T(t)}{m^2(t)}, \quad (14)$$

for $i = 2, 3, \ldots, M$, where $\epsilon(t) = [\epsilon_1(t), \epsilon_2(t), \ldots, \epsilon_M(t)]^T$, $0 < \Gamma_{\theta i} = \Gamma_{\theta i}^T < 2I_{i-1}$, $0 < \Gamma = \Gamma^T < 2I_M$, $m^2(t) = 1 + \zeta^T(t)\zeta(t) + \xi^T(t)\xi(t) + \sum_{i=2}^M \eta_i^T(t)\eta_i(t)$, and the adaptation gain matrix $D_s$ in (13) is chosen to satisfy

$$0 < D_s U^T D_s^{-1} D^* U D_s < 2I_M, \quad (15)$$

that is, $\gamma_i$, $i = 1, 2, \ldots, M$, in $D_s$ given by (6) are chosen such that $\gamma_i \in (0, \gamma_i^0)$ for some $\gamma_i^0 > 0$, which depend on the knowledge of the LDU decomposition of $K_p = LD^*U$.

The adaptive controller (8) with the adaptive laws (12)–(14) ensures closed-loop signal boundedness and asymptotic output tracking, i.e., $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$ [6], [10].

#### *A.2. Performance Robustness Analysis*

Similar to the analysis in Section III, the presence of reduction in actuator effectiveness does not violate the assumption of nonzero leading principle minors of the high frequency gain matrix, and their sign information is invariant with $0 < K < I_M$.

However, for the aforementioned LDS based design to ensure the closed-loop signal boundedness and asymptotic tracking performance, the new condition for $0 < K < I_M$, similar to (15) for $K = I_M$, is $0 < D_s \bar{U}^T D_s^{-1} \bar{D}^* \bar{U} D_s < 2I_M$, which needs to be satisfied for the chosen $D_s$ in (15).

From Lemma 1, this new condition is equivalent to

$$0 < D_s K U^T K^{-1} D_s^{-1} D^* U K D_s < 2 I_M, \qquad (16)$$

from which it can be seen that for $k_i$ of $K$, there is an upper bound which depends on $\gamma_j$, $k_j$, $j = 1, \ldots, i-1$, $d_j^*$, $d_j^0$, $j = 1, \ldots, i$ and the nonzero elements of $U$, such that the closed-loop performance is still achieved. Thus we have:

**Proposition 2**. *The stability and asymptotic tracking performance of direct discrete-time multivariable MRAC designs based on the LDS decomposition of $K_p$ may not be robust with respect to some reduction in actuator effectiveness.*

Proof: For clarity of presentation and without loss of generality, we consider the case for $M = 2$.

For $K = I_2$, to ensure the desired closed-loop performance, $D_s = \mathrm{diag}\{\mathrm{sign}[d_1^*]\gamma_1, \mathrm{sign}[d_2^*]\gamma_2\}$ in (13) is chosen to satisfy (15), that is, $\gamma_1$ and $\gamma_2$ are chosen such that

$$0 < \gamma_1 < \frac{2}{d_1^0}, \qquad (17)$$

$$0 < \gamma_2 < \frac{4}{\sqrt{(d_2^0)^2 + \frac{16a^2|d_1^0|}{\alpha(\gamma_1)}} + |d_2^0|} \qquad (18)$$

with $\alpha(\gamma_1) = -\gamma_1(d_1^0\gamma_1 - 2)$ and the constant $a$ being the nonzero off-diagonal element of the $2 \times 2$ unity upper triangular matrix $U$. [2]

In the presence of reduction in actuator effectiveness, the performance robustness condition (16) yields

$$|d_1^*| k_1 \gamma_1 - 2 < 0,$$

$$(|d_1^*| k_1\gamma_1 - 2)(|d_2^*| k_2 \gamma_2 - 2) > 2a^2 |d_1^*| \frac{k_2^2 \gamma_2^2}{k_1 \gamma_1},$$

which, together with $0 < k_i \leq 1$, $i = 1, 2$, leads to

$$k_1 \in (0, 1], \qquad (19)$$

$$k_2 \in \begin{cases} (0, k_2^0) & \text{if } 0 < k_1 \leq k_{10}, \\ (0, 1] & \text{if } k_{10} < k_1 \leq 1, \end{cases} \qquad (20)$$

$$k_2^0 = \frac{4}{\left(\sqrt{(d_2^*)^2 + \frac{16a^2|d_1^*|}{\bar{\alpha}(k_1\gamma_1)}} + |d_2^*|\right)\gamma_2} \qquad (21)$$

with $\bar{\alpha}(k_1\gamma_1) = -k_1\gamma_1(|d_1^*| k_1 \gamma_1 - 2)$, and

$$k_{10} = \frac{1 - \sqrt{1 - \frac{2(|d_1^* | a \gamma_2)^2}{2 - |d_2^*|\gamma_2}}}{\gamma_1 |d_1^*|}. \qquad (22)$$

We can conclude from (19)–(22) that there exists an actuation level $k_{10}$ such that if $k_{10} < k_1 \leq 1$, the LDS based design can handle arbitrary actuator effectiveness reduction in $k_2$. However, for $0 < k_1 \leq k_{10}$, an upper bound $k_2^0 < 1$ depending on $k_1$, in addition to other design and plant parameters, is imposed on $k_2$ to ensure closed-loop performance. Moreover, both $k_{10}$ and $k_2^0$ depend on the unknown parameters $a$ in $U$, and $d_1^*$, $d_2^*$ in $D^*$ given in (5). In other words, the reduction in actuator effectiveness the LDS based design can handle, depends on the knowledge of the leading principle minors of $K_p$ and the nonzero elements of $U$. When the needed condition (16) is violated, the adaptive control system may become unstable. Therefore, the performance of the LDS based design may not be robust to some patterns of reduction in actuator effectiveness. $\nabla$

[2]The inequalities (17) and (18) can be obtained by solving (15) with the facts that $D^* = \mathrm{diag}\{d_1^*, d_2^*\}$, $\mathrm{sign}[d_i^*] d_i^* = |d_i^*|$, and the knowledge of $d_i^0$ such that $0 < |d_i^*| \leq d_i^0$, $i = 1, 2$, from Assumption (A.4D).

*Remark 1:* In general, explicit ranges of $k_i$ in which the LDS based design can still ensure the desired performance is difficult to derive from (16). Numerical methods can be used. By the Schur complement, (16) is equivalent to

$$\begin{bmatrix} 2I_M & L_d D^* K U^T \\ U K D^* L_d & K L_d \end{bmatrix} > 0 \qquad (23)$$

with $D_s = L_d D^*$ chosen to satisfy (15). A linear cost function can then be constructed, and the problem is converted to the optimization of the cost function subject to the LMIs in (23) and $0 < K < I_M$. $\square$

*Remark 2:* The performance robustness properties of the LDS based design becomes manifest when $U = I_M$, i.e., $K_p$ is lower triangular. The matrix inequalities in (15) and (16) can thus be decomposed into simple scalar inequalities decoupled in $\gamma_i$ and $k_i$, $i = 1, 2, \ldots, M$, respectively. To be specific, with (5) and (6), the condition in (15) leads to $0 < \gamma_i < \frac{2}{d_i^0}$. In the presence of reduction in actuator effectiveness, the condition in (16) yields $0 < k_i < \frac{2}{|d_i^*|\gamma_i}$, and we have

$$\frac{2}{|d_i^*|\gamma_i} > \frac{d_i^0}{|d_i^*|} \geq 1, \quad i = 1, 2, \ldots, M, \qquad (24)$$

from which we conclude that the performance of the LDS based design is robust with respect to arbitrary reduction in actuator effectiveness if $K_p$ is lower triangular. $\square$

*Remark 3:* The advantage of online computational efficiency of the LDS based design is at the expenses of much design effort, as the adaptation gain matrix $D_s$ for (13) should be carefully chosen such that (15), which is not in explicit form of $\gamma_i$ and requires knowledge of $U$, is satisfied. The choices of the adaptation gains, $\gamma_i$, are not straightforward even for $M = 2$, as shown in (17) and (18), and require the knowledge of the off-diagonal element $a$ of $U$. Furthermore, the choices affect the performance robustness properties of the design. $\square$

### *B. Performance Robustness of the LDU Based Design*

For the LDU based design [6], [10], the sign and upper bound information stated in Assumptions (4.1) and (4.1D) are used explicitly for the choice of adaptation gains, in contrast to the LDS based design. The parametrization of the error dynamics and the adaptation mechanism of the LDU based design are an expansion in dimension from the well-known SISO MRAC design, and it shares the same performance robustness property with its SISO counterpart.

**Proposition 3**. *The stability and tracking performance of direct discrete-time multivariable MRAC designs based on the LDU decomposition of high frequency gain matrix is robust with respect to reduction in actuator effectiveness.*

Proof: According to (7) and $\bar{D}^* = \operatorname{diag}\{\bar{d}_1^*, \ldots, \bar{d}_M^*\}$, we have $\bar{d}_i^* = d_i^* k_i$. It can be seen that $\bar{d}_i^* \neq 0$ and $\operatorname{sign}[\bar{d}_i^*] = \operatorname{sign}[d_i^*]$. For the LDU based design for $K = I_M$ to ensure stability and asymptotic tracking performance under reduction in actuator effectiveness, $|\bar{d}_i^*|$ needs to satisfy $|\bar{d}_i^*| \leq d_i^0$ (to ensure the stability condition on the adaptation gain matrix), so we have $k_i \in \left(0, \frac{d_i^0}{|d_i^*|}\right]$, $i = 1, 2, \ldots, M$. Noting that $d_i^0 \geq |d_i^*|$ from Assumption (A.4D), the above range for $k_i$ contains $(0, 1]$, which implies that the closed-loop performance of the LDU based design is robust with respect to reduction in actuator effectiveness. $\nabla$

Compared with the LDS based design, the auxiliary signal structures for the LDU based design are more complex, which need extra signal manipulations and regrouping. This effort, however, eases the design procedure, especially for the discrete-time cases, in which the choices of adaptation gains are straightforward. Besides, as an extension from the SISO discrete-time MRAC design, the LDU based design can accommodate for reduction in actuator effectiveness while the LDS based design cannot, in general.

## V. Indirect MRAC Designs

For indirect multivariable MRAC, the problem of non-singular estimation of a general plant high frequency gain matrix using system input and output measurements is still a problem to be solved. Thus, we consider the indirect MRAC design for SISO systems [10].

One key assumption in indirect SISO MRAC design is the knowledge of the sign information of the plant high frequency gain $k_p$, $\operatorname{sign}[k_p]$, and a lower bound $k_{p0}$ of $|k_p|$ such that $0 < k_{p0} \leq |k_p|$, for avoiding singularity estimation of the unknown plant parameters. Based on this assumption, we obtain the following performance robustness properties.

**Proposition 4**. *The performance of indirect MRAC designs for SISO systems may not be robust with respect to some reduction in actuator effectiveness in the sense that for the reduction greater than some level, stability and asymptotic tracking performance may no longer hold.*

Proof: In the presence of reduction in actuator effectiveness, i.e., with an actuator effectiveness gain $k \in (0, 1]$ at the control input, the plant high frequency gain becomes $k_p k$. The indirect MRAC design for $k = 1$ still ensures desired closed-loop performance if the lower bound of $|k_p k|$ is no less than $k_{p0}$, i.e., $k \geq \frac{k_{p0}}{|k_p|}$, which is derived from the assumption based on which a parameter projection law is designed for $k = 1$. Therefore, the indirect SISO MRAC design may not be able to compensate for the reduction in actuator effectiveness with $k \in \left(0, \frac{k_{p0}}{|k_p|}\right)$. $\nabla$

*Remark 4:* The performance robustness result in Proposition 4 applies directly to the class of indirect multivariable MRAC systems in [9], that is, lower bounds exist for the effectiveness gains of each control channel such that the indirect MRAC design can adaptively compensate for the reductions not below these bounds. $\square$

## VI. An Illustrative Example

In this section we present an $M = 2$ example to illustrate the performance robustness properties of the discrete-time LDS based direct MRAC design.

We consider a controlled plant (1) with $u_p(t), y(t) \in \mathbb{R}^2$, and the transfer matrix

$$G(z) = \begin{bmatrix} \frac{1}{z-0.5} & \frac{0.6}{z+0.3} \\ \frac{1}{z+0.5} & \frac{1}{z+0.7} \end{bmatrix},$$

which is expressed as $G(z) = Z(z)P^{-1}(z)$, where

$$Z(z) = \begin{bmatrix} z+0.5 & 0.6(z+0.7) \\ z-0.5 & z+0.3 \end{bmatrix},$$

$$P(z) = \operatorname{diag}\{(z-0.5)(z+0.5), (z+0.3)(z+0.7)\}$$

with observability index $\nu = 2$. It can be verified that $G(z)$ is minimum-phase, and $Z(z)$ and $P(z)$ are right coprime with $P(z)$ column proper.

The modified interactor matrix and the associated high frequency gain matrix $K_p$ are

$$\xi_m(z) = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}, \; K_p = \begin{bmatrix} 1 & 0.6 \\ 1 & 1 \end{bmatrix},$$

from which we have $d_1^* = 1$, $d_2^* = 0.4$, $\operatorname{sign}[d_1^*] > 0$ and $\operatorname{sign}[d_2^*] > 0$. With $d_1^0 = 1.2$ and $d_2^0 = 1.0$, the standard MRAC assumptions (A.1)–(A.4D) are all satisfied. Moreover, the unique LDU decomposition of $K_p$ is

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; D^* = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}, \; U = \begin{bmatrix} 1 & 0.6 \\ 0 & 1 \end{bmatrix}. \tag{25}$$

The reference model is chosen to be (4) with the transfer matrix $W_m(z) = \xi_m^{-1}(z)$ and the reference input $r(t) = [10\cos(0.15t), 10\sin(0.1t) + 20\sin(0.2t)]^T$.

The controller structure is in (8). With the specifications of $\Lambda(z) = z$, $\xi_m(z)$, $P(z)$, and $Z(z)$ as above, its nominal parameter matrices $\Theta_1^*$, $\Theta_2^*$, $\Theta_{20}^*$, $\Theta_3^* = K_p^{-1}$ can be computed from (9). With (25), $D_s = I_2$, $L_s = LD_s(U^T)^{-1}D_s^{-1}$, and $S = U^T D_s^{-1} D^* U$, we have $\Psi^* = D_s S$, and $\theta_2^*$ can obtained from $\Theta_0^* = L_s^{-1} - I_2$.

The adaptive laws are (12)–(14) with $D_s$ chosen to be $D_s = \operatorname{diag}[\gamma_1, \gamma_2] = I_2$ satisfying the design condition (15), and in the estimation error model, $f(z)$ is chosen as $f(z) = z$. The initial estimates of $\theta_2^*$, $\Theta^*$, $\Psi^*$ are taken to be $\theta_2(0) = 0.95\theta_2^*$, $\Theta(0) = 0.95\Theta^*$, and $\Psi(0) = 0.8\Psi^*$.

From (22) we obtain $k_{10} = 0.2584$, and by (20), we know that for a reduction in actuator effectiveness of the first control channel such that $k_1 > k_{10}$, the performance is robust with respect to arbitrary $k_2 \in (0, 1]$. This is shown in Fig. 2 where the tracking error approaches zero asymptotically for $K = \operatorname{diag}\{0.7, 0.9\}$. Furthermore, for a $k_1 \leq k_{10}$, the stability and asymptotic tracking performance may not hold for some $k_2$. As is shown in Fig. 3, with $k_1 = 0.15 < k_{10}$, for $k_2 = 1.0$, the asymptotic tracking performance is lost (simulation was performed for a much longer time, and only the first 1000 steps are shown here for clarity). However, as shown in Fig. 4, asymptotic tracking is achieved for $K = \operatorname{diag}\{0.15, 0.65\}$, as $0.65 < k_2^0 = 0.8043$ from (21).

The simulation results illustrate the main features of the performance robustness of the LDS based design.

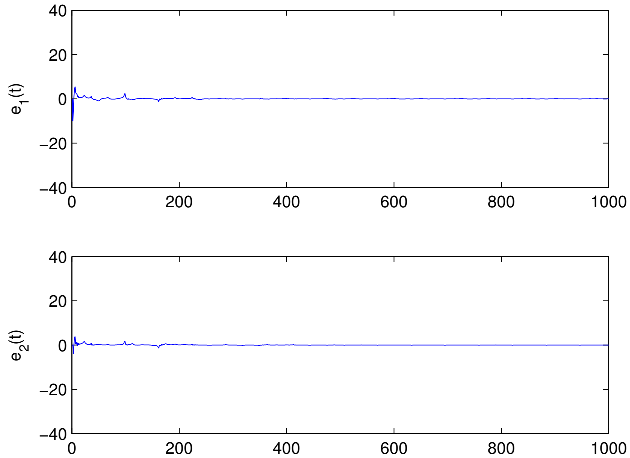


Fig. 2. Tracking error $e(t)$ for $K = \mathrm{diag}\{0.7, 0.9\}$.

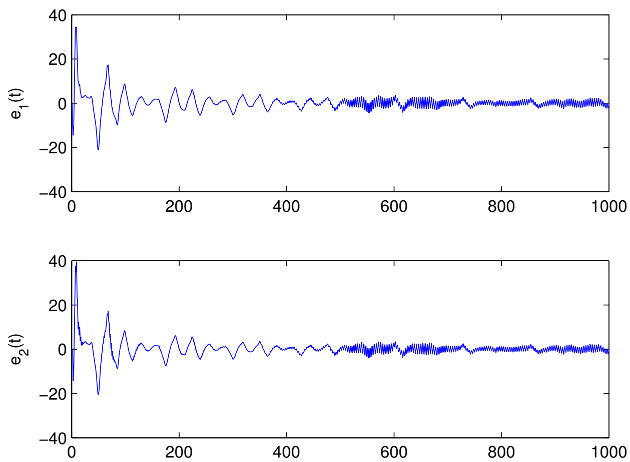


Fig. 3. Tracking error $e(t)$ for $K = \mathrm{diag}\{0.15, 1.0\}$.

## VII. Conclusions

In this paper, the performance robustness property of some standard model reference adaptive control (MRAC) designs with respect to reduction in actuator effectiveness was studied. The reduction in effectiveness is modeled as an actuator effectiveness gain matrix $K$ at the control inputs which is diagonal with the elements inside the interval $(0, 1]$. In the presence of reduction in actuator effectiveness, the effectiveness gain reduces to a fraction of the normal level (that without reduction in actuator effectiveness). Analysis was performed for several direct and indirect MRAC schemes. It was shown that the performance of continuous-time multivariable direct MRAC designs is robust under arbitrary reduction in actuator effectiveness, while that of their discrete-time counterparts is design-based. In particular, the performance of the multivariable MRAC design based on the LDS decomposition of the high frequency gain matrix may not be robust with respect to some reduction in actuator effectiveness, but that of the LDU based design is. The

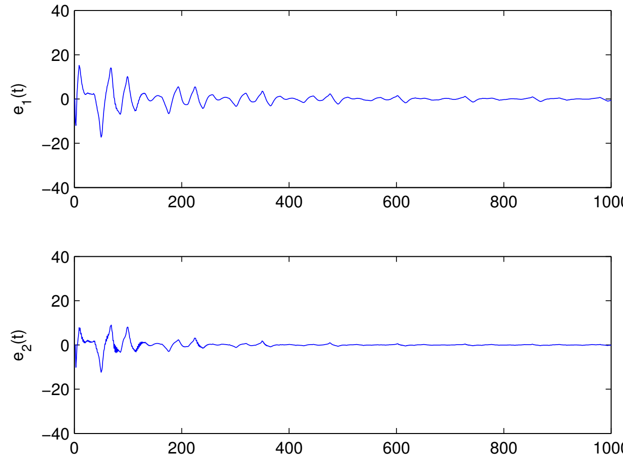


Fig. 4. Tracking error $e(t)$ for $K = \mathrm{diag}\{0.15, 0.65\}$.

performance robustness property of the SISO indirect MRAC design was also studied, and it was shown that the desired performance may no longer be achieved for large reductions in actuator effectiveness. A simulation study was presented to verify some of the theoretical results.


## Acknowledgments

This research was supported in part by NASA Langley Research Center under grant NNX08AB99A and by NSF under grant ECS0601475.


## References


[1] M. Bodson and J. E. Groszkiewicz, "Multivariable adaptive algorithms for reconfigurable flight control," *IEEE Trans. Contr. Syst. Technol.*, vol. 5, no. 2, pp. 217–229, March 1997.

[2] J. D. Bošković and R. K. Mehra, "Stable multiple model adaptive flight control for accommodation of a large class of control effector failures," in *Proc. American Contr. Conf.*, San Diego, California, 1999.

[3] J. D. Bošković and R. K. Mehra, "An adaptive scheme for compensation of loss of effectiveness of flight control effectors," in *Proc. 40th IEEE Conf. Decision Control*, Orlando, Florida, December 2001.

[4] A. J. Calise, S. Lee, and M. Sharma, "Development of a reconfigurable flight control law for tailless aircraft," *AIAA J. Guid., Control, Dynamics*, vol. 25, no. 5, pp. 896–902, 2001.

[5] J. Chen and R. Patton, *Robust Model-Based Fault Diagnosis for Dynamic Systems*, Norwell, MA: Kluwer, 1998.

[6] A. K. Imai, R. R. Costa, L. Hsu, G. Tao and P. V. Kokotović, "Multivariable MRAC using high frequency gain matrix factorization," *IEEE Trans. Automat. Contr.*, vol. 49, no. 7, pp. 1152–1157, 2004.

[7] M. Schwager, A. M. Annaswamy, and E. Lavretsky, "Adaptation-based reconfiguration in the presence of actuator failures and saturation," in *Proc. American Contr. Conf.*, Portland, OR, June, 2005.

[8] Q. Sang and G. Tao, "Gain margins of model reference adaptive control systems," in *Proc. 7th World Congress on Intelligent Control and Automation*, Chongqing, China, 2008.

[9] Q. Sang and G. Tao, "Gain margins of multivariable MRAC systems," in *Proc. 47th IEEE Conf. Decision Control*, Cancun, Mexico, 2008.

[10] G. Tao, *Adaptive Control Design and Analysis*, John Wiley and Sons, New York, 2003.

[11] G. Tao and S. M. Joshi, "Direct adaptive control of systems with actuator failures: state of the art and continuing challenges," in *Proc. AIAA Guid. Navi. Contr. Conf.*, Honolulu, Hawaii, 2008.

[12] Q. Zhao and J. Jiang, "Reliable state feedback control system design against actuator failures," *Automatica*, vol. 34, no. 10, 1998.