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Zero dynamics inverse design for asymptotic regulation of the heat equation: the non-colocated case

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***Abstract*— In this paper we present the results of our recent work on asymptotic regulation using zero dynamics inverse (ZDI) design for a one dimensional heat equation applied to the non-colocated case. In our previous works we have considered several different types of examples for the co-located case in which boundary control is used to track reference signals given at the same end of the rod or on the interior of the rod. In this work we show how this method can also be applied in the case of boundary control in order to achieve tracking a pair of signals prescribed at interior points of the rod.**

I. Introduction

Asymptotic regulation has attracted considerable attention in the literature on both lumped and distributed parameter systems using an impressive variety of technical approaches. Roughly speaking, the problem is to design a feedback controller, using only the error $e(t)$ and measured variables $y_m(t)$, to achieve $\lim_{t\to\infty} e(t) = 0$ together with appropriate boundedness conditions on the state of the system and the controller(s). In order to reflect various degrees of uncertainty about the signal $w(\cdot)$, we distinguish between *three* versions of the problem of asymptotic regulation, following tradition in robust control. In the first, and worst, case scenario, we refer to the situation where the exogenous signals $w(t)$ is unknown as *asymptotic regulation with unstructured uncertainty*. In fact, a more reasonable problem is to design a controller which attenuate the effect of w and the current approach of choice in this case is indisputably $H^\infty control$, which has received enormous attention in the literature as a method for the robust control of lumped linear and nonlinear cases ([7], [1], [8] and the references therein). For distributed parameter systems, see [2] and the extensive treatises [9] - [11]. The next most natural case is *asymptotic regulation with structured uncertainty* ; i.e., the problem of output regulation. In this approach one assumes that the signal w is the output of an exogenous, known autonomous system but that only w and perhaps some other functional of the state of the exosystem is known. Since unknown parameters θ can be incorporated into the states of the exosystem augmented by $\dot{\theta} = 0$, this formulation does indeed capture the essence of models with structured uncertainty. Our third delineation regarding uncertainty, asymptotic regulation with *measured signals* , lies between the cases of structured and unstructured uncertainty; viz., the case where the value $w(t)$ of the signal is available at any instant of time. In general, since w is available as an input, it can be filtered through a cascade controller which produces a feedforward contol, u_R. The ZDI design philosophy designs a cascade controller for a class of signals which are unknown, except at the instant when they are used as inputs to the cascade controller. In particular, this problem formulation is in sharp contrast to exact tracking, asymptotic tracking or dynamic inversion of a completely known trajectory.

In a series of earlier papers [3]–[6] we have solved the asymptotic regulation problem for some distributed parameter systems using the zero dynamics inverse (ZDI) design. In each of the earlier papers we consider the co-located case in which the actuator and sensor were placed at the same points in the spatial domain. The non-colocated case is somewhat more challenging for the ZDI method due to the fact that the spatial domain for the plant and the zero dynamics are different. Nevertheless we have been able to overcome this problem and, in this paper, we present some positive results in this direction for a MIMO example for the one dimensional heat equation with two inputs and two outputs.

In this work we consider a control system governed the one dimensional heat equation

$$z_t(x,t) = z_{xx}(x,t) \quad 0 \le x \le 1,\ t > 0, \tag{I.1}$$

$$z(x,0) = 0,$$

$$\mathcal{B}_0 z = z_x(0,t) - k_0 z(0,t) = u_0(t), \quad k_0 > 0, \tag{I.2}$$

$$\mathcal{B}_1 z = z_x(1,t) + k_1 z(1,t) = u_1(t), \quad k_1 > 0, \tag{I.3}$$

$$y_a(t) = (C_a z)(t) = z(a,t) \tag{I.4}$$

$$y_b(t) = (C_b z)(t) = z(b,t), \quad 0 < a < b < 1. \tag{I.5}$$

We then denote the input and output for this system by

$$u = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \quad y = \begin{bmatrix} y_a \\ y_b \end{bmatrix}.$$

We are also given a pair of reference signals $y_{r,a}(t)$ and $y_{r,b}(t)$. The control objective is to design a control u so that the error

$$e(t) = y(t) - y_r(t) \xrightarrow{t\to\infty} 0$$

where $y_r(t) = [y_{r,a}(t),\ y_{r,b}(t)]^T$.

It can be shown that the problem (I.1)-(I.5) is equivalent to the abstract problem in $\mathcal{Z} = L^2(0,1)$ given by

$$z_t = Az + Bu,\ t > 0, \tag{I.6}$$

$$z(0) = 0,$$

$$y(t) = Cz(t) \tag{I.7}$$

where $A = \partial^2/\partial x^2$ is the negative self-adjoint operator with domain

$$D(A) = \{\varphi \in H^2(0,1) : \mathcal{B}_0\varphi = 0, \mathcal{B}_1\varphi = 0\} \tag{I.8}$$

and where

$$Bu = B_0 u_0 + B_1 u_1, \quad B_0 = -\delta_0, \quad B_1 = \delta_1, \tag{I.9}$$

(δ_a is the Dirac delta function located at $x = a$)

$$Cz(t) = \begin{bmatrix} C_a z \\ C_b z \end{bmatrix} = \begin{bmatrix} z(a,t) \\ z(b,t) \end{bmatrix}. \tag{I.10}$$

Related to the plant given in (I.1)-(I.5) we also consider the so-called forced zero dynamics system

$$\xi_t(x,t) = \xi_{xx}(x,t) \quad a \le x \le b,\ t > 0, \tag{I.11}$$

$$\xi(x,0) = 0,$$

$$\xi(a,t) = C_a\xi = y_{r,a}(t), \tag{I.12}$$

$$\xi(1,t) = C_b\xi = y_{r,b}(t), \tag{I.13}$$

$$u_0(t) = \mathcal{B}_0\xi, \tag{I.14}$$

$$u_1(t) = \mathcal{B}_1\xi. \tag{I.15}$$

II. Transfer function for forced zero dynamics

In this section we compute the transfer function for the forced zero dynamics system given in (I.11)-(I.15). To this end we apply the Laplace transform to (I.11)-(I.15) to obtain

$$s\widehat{\xi}(x,s) = \widehat{\xi}_{xx}(x,s) \quad a \le x \le b,\ t > 0, \tag{II.1}$$

$$\widehat{\xi}(x,0) = 0,$$

$$\widehat{\xi}(a,s) = C_a\widehat{\xi} = \widehat{y}_{r,a}(s), \tag{II.2}$$

$$\widehat{\xi}(b,s) = C_b\widehat{\xi} = \widehat{y}_{r,b}(s), \tag{II.3}$$

$$\widehat{u}_0(s) = \mathcal{B}_0\widehat{\xi}, \tag{II.4}$$

$$\widehat{u}_1(s) = \mathcal{B}_1\widehat{\xi}. \tag{II.5}$$

The general solution to (II.1) is given by

$$\widehat{z}(x,s) = c_1\varphi_1(x) + c_2\varphi_2(x), \tag{II.6}$$

$$\varphi_1(x) = \frac{\sinh(\sqrt{s}x)}{\sqrt{s}}, \quad \varphi_2(x) = \cosh(\sqrt{s}x)$$

and also we have

$$\widehat{z}_x(x,s) = c_1\cosh(\sqrt{s}x) + c_2\sqrt{s}\sinh(\sqrt{s}x).$$

Applying the conditions in (II.2) and (II.3) we have

$$c_1(C_a\varphi_1) + c_2(C_a\varphi_2) = C_a\widehat{\xi} = \widehat{y}_{r,a}$$

$$c_1(C_b\varphi_1) + c_2(C_b\varphi_2) = C_b\widehat{\xi} = \widehat{y}_{r,b}$$

Let Δ denote the coefficient matrix:

$$\Delta(s) = \begin{vmatrix} (C_a\varphi_1) & (C_a\varphi_2) \\ (C_b\varphi_1) & (C_b\varphi_2) \end{vmatrix}$$

$$= \frac{1}{\sqrt{s}}\begin{vmatrix} \sinh(\sqrt{s}a) & \cosh(\sqrt{s}a) \\ \sinh(\sqrt{s}b) & \cosh(\sqrt{s}b) \end{vmatrix}$$

$$= \frac{[\sinh(\sqrt{s}a)\cosh(\sqrt{s}b) - \cosh(\sqrt{s}a)\sinh(\sqrt{s}b)]}{\sqrt{s}}$$

$$= -\frac{\sinh(\sqrt{s}(b-a))}{\sqrt{s}} \tag{II.7}$$

So we have

$$c_1 = \frac{1}{\Delta}\{(C_b\varphi_2)\widehat{y}_a - (C_a\varphi_2)\widehat{y}_b\}$$
$$= \frac{1}{\Delta}\left\{\left(\frac{\sinh(\sqrt{s}b)}{\sqrt{s}}\right)\widehat{y}_a - \left(\cosh(\sqrt{s}a)\right)\widehat{y}_b\right\}$$

and

$$c_1 = \{(C_a\varphi_1)\widehat{y}_b - (C_b\varphi_1)\widehat{y}_a\}$$
$$= \frac{1}{\Delta}\left\{\left(\frac{\sinh(\sqrt{s}a)}{\sqrt{s}}\right)\widehat{y}_b - \left(\cosh(\sqrt{s}b)\right)\widehat{y}_a\right\}$$

Therefore, we have

$$\widehat{\xi}(x,s) = \frac{1}{\Delta}\left[\{(C_b\varphi_2)\widehat{y}_a - (C_a\varphi_2)\widehat{y}_b\}\,\varphi_1(x) \right. \tag{II.8}$$
$$\left. + \{(C_a\varphi_1)\widehat{y}_b - (C_b\varphi_1)\widehat{y}_a\}\,\varphi_2(x)\right]$$
$$= \frac{1}{\Delta}\left[\left\{\left(\frac{\sinh(\sqrt{s}b)}{\sqrt{s}}\right)\widehat{y}_a - \left(\cosh(\sqrt{s}a)\right)\widehat{y}_b\right\}\varphi_1(x) \right.$$
$$\left. + \left\{\left(\frac{\sinh(\sqrt{s}a)}{\sqrt{s}}\right)\widehat{y}_b - \left(\cosh(\sqrt{s}b)\right)\widehat{y}_a\right\}\varphi_2(x)\right]$$

We note that

$$\mathcal{B}_0\varphi_1 = 1, \quad \mathcal{B}_0\varphi_2 = -k_0,$$

and

$$\mathcal{B}_1\varphi_1 = \left[\cosh(\sqrt{s}) + k_1\frac{\sinh(\sqrt{s})}{\sqrt{s}}\right],$$
$$\mathcal{B}_1\varphi_2 = \left[\sqrt{s}\sinh(\sqrt{s}) + k_1\cosh(\sqrt{s})\right].$$

From these we readily obtain

$$\widehat{u}_0 = \mathcal{B}_0\widehat{\xi} = \frac{1}{\Delta}\left[\{(C_b\varphi_2)\widehat{y}_a - (C_a\varphi_2)\widehat{y}_b\}\,(\mathcal{B}_0\varphi_1)\right.$$
$$\left. + \{(C_a\varphi_1)\widehat{y}_b - (C_b\varphi_1)\widehat{y}_a\}\,(\mathcal{B}_0\varphi_2)\right] \tag{II.9}$$
$$= -\left[\frac{\sqrt{s}\cosh(\sqrt{s}b) + k_0\sinh(\sqrt{s}b)}{\sinh(\sqrt{s}(b-a))}\right]\widehat{y}_a$$
$$+ \left[\frac{\sqrt{s}\cosh(\sqrt{s}a) + k_0\sinh(\sqrt{s}a)}{\sinh(\sqrt{s}(b-a))}\right]\widehat{y}_b.$$

and

$$\widehat{u}_1 = \mathcal{B}_1\widehat{\xi} = \frac{1}{\Delta}\left[\{(C_b\varphi_2)\widehat{y}_a - (C_a\varphi_2)\widehat{y}_b\}\,(\mathcal{B}_1\varphi_1)\right.$$
$$\left. + \{(C_a\varphi_1)\widehat{y}_b - (C_b\varphi_1)\widehat{y}_a\}\,(\mathcal{B}_1\varphi_2)\right] \tag{II.10}$$
$$= -\left[\frac{\sqrt{s}\cosh(\sqrt{s}(1-b)) + k_1\sinh(\sqrt{s}(1-b))}{\sinh(\sqrt{s}(b-a))}\right]\widehat{y}_a$$
$$+ \left[\frac{\sqrt{s}\cosh(\sqrt{s}(1-a)) + k_1\sinh(\sqrt{s}(1-a))}{\sinh(\sqrt{s}(b-a))}\right]\widehat{y}_b$$

III. The Control Problem – Harmonic Tracking

We now turn to using all these expressions to give simple formulas for the controls u_0, u_1 solving the tracking problem described in Section I.

In this section we consider the problem of tracking a pair of sinusoids, i.e.,

$$y_{r,0}(t) = M_0\sin(\alpha_0 t), \quad y_{r,1}(t) = M_1\sin(\alpha_1 t).$$

In order to compute u_0 and u_1 we apply inverse Laplace transforms to (II.9) and (II.10). Namely we have

$$u_j(t) = \mathcal{L}^{-1}(\widehat{u}_j)(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,\widehat{u}_j(s)\,ds. \tag{III.1}$$

Here $\gamma > 0$ so that the integration is carried out in the complex plane and all the singularities of $M(s)$ are poles that lie to the left of the the vertical line Re $(s) = \gamma$ corresponding to our path of integration. Indeed, since the poles of $\widehat{w}_j$ lie (as conjugate pairs) on the imaginary axis and the remaining poles of $M(s)$ correspond to the nonzero zeros of $\sinh(\sqrt{s})$ which form infinite set of simple poles that lie on the negative real axis and tend to minus infinity. The calculation is easily carried out using the residue theorem from complex analysis.

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,\widehat{u}_j(s) = \sum \text{Res}\left(e^{st}\,\widehat{u}_j(s)\right)$$

where the summation is taken over all poles in the closed left half of the complex plane. In the case of tracking a sinusoid we have

$$\mathcal{L}(M\sin(\alpha t)) = \frac{M\alpha}{s^2+\alpha^2} = \frac{M\alpha}{(s-i\alpha)(s+i\alpha)}. \tag{III.2}$$

Then the poles correspond to $S = S_{\mathcal{M}} \cup S_a \cup S_b$

$$S_{\mathcal{M}} = \{s\,:\,\Delta(s) = 0\} =, \quad S_0 = \{\pm i\alpha_a\}, \quad S_1 = \{\pm i\alpha_b\}.$$

and we have

First we compute $u_0(t)$. To this end we set

$$u_0(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,\left(e^{st}\,\widehat{u}_0(s)\right)(s)\,ds$$
$$= \sum_{s=-j^2\pi^2} \text{Res}\left(e^{st}\left(e^{st}\,\widehat{u}_0(s)\right)(s)\right)$$
$$+ \sum_{s=\pm i\alpha_0} \text{Res}\left(e^{st}\left(e^{st}\,\widehat{u}_0(s)\right)(s)\right)$$
$$+ \sum_{s=\pm i\alpha_1} \text{Res}\left(e^{st}\left(e^{st}\,\widehat{u}_0(s)\right)(s)\right)$$
$$\equiv H_{\text{stable}}(t) + H_{00}(t) + H_{01}(t). \tag{III.3}$$

The terms contained in $H_{\text{stable}}(t)$ all contain expressions in the form

$$h_j(t) = f(\zeta_j)e^{-j^2\pi^2 t}.$$

The terms $h_j(t)$ tend to zero exponentially as t goes to infinity. Thus these terms do not contribute to the steady state response and therefore are ignored in the computation of $u_0(t)$. Certainly these terms do contribute to the transient response of the control but the point is that it does not hurt to ignore these terms in the sense that we still obtain a control law. What we claim here is that the control obtained in this way is exactly the same as that obtained by solving the regulator equations.

Let us now focus on the residues that provide $H_{00}(t)$ and $H_{01}(t)$. First we consider $H_{00}(t)$. We recall that for any $\alpha \in \mathbb{R}$ the Laplace transform of a sinusoid with frequency α and amplitude M is given (III.2). For $H_{00}(t)$ we will need to calculate the sum of the residues at $\pm i\alpha_a$ for the following term

$$T_{11}(s) = -\frac{\left[\dfrac{\sqrt{s}\cosh(\sqrt{s}b) + k_0 \sinh(\sqrt{s}b)}{\sinh(\sqrt{s}(b-a))}\right] M_0 \alpha_0 e^{st}}{(s - i\alpha_0)(s + i\alpha_0)}$$

and for $H_{01}(t)$ we need the sum of the residues at $\pm i\alpha_b$ for

$$T_{12}(s) = \frac{\left[\dfrac{\sqrt{s}\cosh(\sqrt{s}a) + k_0 \sinh(\sqrt{s}a)}{\sinh(\sqrt{s}(b-a))}\right] M_b \alpha_b e^{st}}{(s - i\alpha_b)(s + i\alpha_b)}.$$

The residues at $\pm i\alpha_a$ give

$$\begin{aligned} H_{00}(t) =& \operatorname{Res}_{s=i\alpha_a} T_{11}(s) + \operatorname{Res}_{s=-i\alpha_a} T_{11}(s) \\ =& 2\text{Re } (\operatorname{Res}_{s=i\alpha_a} T_{11}(s)) \\ =& M_a \left[\text{Re } (\mathcal{F}_a)\sin(\alpha_a t) + \text{Im } (\mathcal{F}_a)\cos(\alpha_a t)\right] \\ =& \text{Re } (\mathcal{F}_a) w_a^1(t) + \text{Im } (\mathcal{F}_a) w_a^2(t) \end{aligned} \tag{III.4}$$

where $\mathcal{F}_a$ is given by

$$-\left\{\frac{\sqrt{i\alpha_a}\,\cosh(\sqrt{i\alpha_a}\,b) + k_0 \sinh(\sqrt{i\alpha_a}\,b)}{\sinh(\sqrt{i\alpha_a}\,(b-a))}\right\}. \tag{III.5}$$

Similarly, the residues at $\pm i\alpha_b$ give

$$\begin{aligned} H_{01}(t) =& \operatorname{Res}_{s=i\alpha_b} T_{12}(s) + \operatorname{Res}_{s=-i\alpha_b} T_{12}(s) \\ =& 2\text{Re } (\operatorname{Res}_{s=i\alpha_b} T_{12}(s)) \\ =& M_b \left[\text{Re } (\mathcal{F}_b)\sin(\alpha_b t) + \text{Im } (\mathcal{F}_b)\cos(\alpha_b t)\right] \\ =& \text{Re } (\mathcal{F}_b) w_b^1(t) + \text{Im } (\mathcal{F}_b) w_b^2(t) \end{aligned} \tag{III.6}$$

where $\mathcal{F}_b$ is given by

$$\left\{\frac{\sqrt{i\alpha_b}\,\cosh(\sqrt{i\alpha_b}\,a) + k_0 \sinh(\sqrt{i\alpha_b}\,a)}{\sinh(\sqrt{i\alpha_b}\,(b-a))}\right\}. \tag{III.7}$$

Finally we have

$$\begin{aligned} u_0(t) &= H_{00}(t) + H_{01}(t) \\ &= \text{Re } (\mathcal{F}_a) w_a^1(t) + \text{Im } (\mathcal{F}_a) w_a^2(t) \\ &\quad + \text{Re } (\mathcal{F}_b) w_b^1(t) + \text{Im } (\mathcal{F}_b) w_b^2(t) \\ &= \Gamma_{11} w_a(t) + \Gamma_{12} w_b(t), \end{aligned} \tag{III.8}$$

where $\mathcal{F}_a$ and $\mathcal{F}_b$ are given, respectively, by

$$-\left\{\frac{\sqrt{i\alpha_a}\,\cosh(\sqrt{i\alpha_a}\,b) + k_0 \sinh(\sqrt{i\alpha_a}\,b)}{\sinh(\sqrt{i\alpha_a}\,(b-a))}\right\},$$

and

$$\left\{\frac{\sqrt{i\alpha_b}\,\cosh(\sqrt{i\alpha_b}\,a) + k_0 \sinh(\sqrt{i\alpha_b}\,a)}{\sinh(\sqrt{i\alpha_b}\,(b-a))}\right\}.$$

Repeating all the above we can compute $u_1(t)$.

In particular from Section II equation (II.10) we have

$$\begin{aligned} \widehat{u}_1 = &-\left[\frac{\sqrt{s}\cosh(\sqrt{s}(1-b)) + k_1 \sinh(\sqrt{s}(1-b))}{\sinh(\sqrt{s}(b-a))}\right] \widehat{y}_a \\ &+ \left[\frac{\sqrt{s}\cosh(\sqrt{s}(1-a)) + k_1 \sinh(\sqrt{s}(1-a))}{\sinh(\sqrt{s}(b-a))}\right] \widehat{y}_b \end{aligned}$$

We obtain $u_1(t)$ by applying the inverse Laplace transform to (II.10) which has simple poles at $s = -j^2\pi^2$, $s = \pm i\alpha_a$ and $s = \pm i\alpha_b$. So it is clear that we can simply repeat what we did to find $u_0(t)$ to obtain the desired result. Rather than repeat the details we just give the final results.

$$u_1(t) = H_{10}(t) + H_{11}(t) \tag{III.9}$$

$$\begin{aligned} &= \text{Re } (\mathcal{G}_a) w_a^1(t) + \text{Im } (\mathcal{G}_a) w_a^2(t) \\ &\quad + \text{Re } (\mathcal{G}_b) w_b^1(t) + \text{Im } (\mathcal{G}_b) w_b^2(t) \\ &= \Gamma_{21} w_a(t) + \Gamma_{22} w_b(t), \end{aligned} \tag{III.10}$$

where $\mathcal{G}_a$ and $\mathcal{G}_b$ are given, respectively, by

$$-\left\{\frac{\sqrt{i\alpha_a}\,\cosh(\sqrt{i\alpha_a}\,(1-b)) + k_1 \sinh(\sqrt{i\alpha_a}\,(1-b))}{\sinh(\sqrt{i\alpha_a}\,(b-a))}\right\},$$

and

$$\left\{\frac{\sqrt{i\alpha_b}\,\cosh(\sqrt{i\alpha_b}\,(1-a)) + k_1 \sinh(\sqrt{i\alpha_b}\,(1-a))}{\sinh(\sqrt{i\alpha_b}\,(b-a))}\right\}.$$

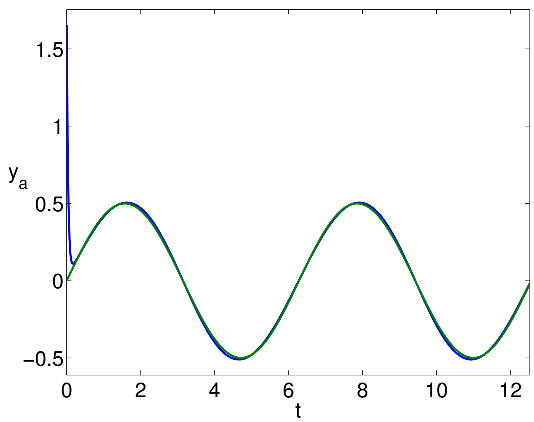
IV. Example

In this section we present a numerical example in which we have set $k_0 = k_1 = 10$, $\alpha_a = 1$, $M_a = .5$, $\alpha_b = 2$, $M_b = 1$, $a = .2$ and $b = .9$. We have also taken initial condition for the plant as $\varphi(x) = 2\cos(\pi x)$. The controls

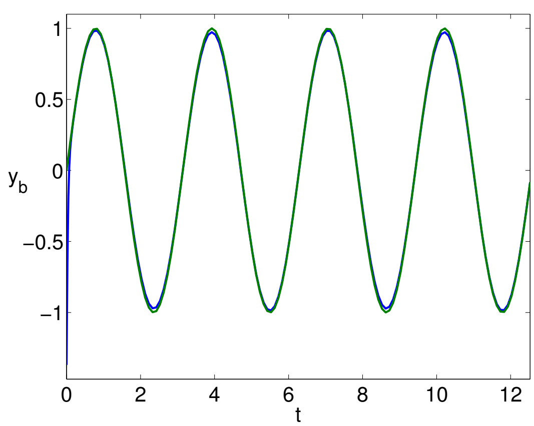
u_0 and u_1 are obtained from (III.8)

rre3.19 with

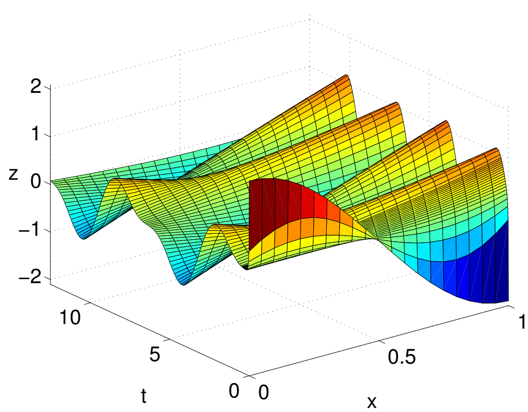
$$\Gamma = \begin{bmatrix} -14.2986 & -1.1468 & 4.2214 & -0.5982 \\ -2.8446 & 0.2232 & 12.9105 & 1.2471 \end{bmatrix}.$$



Plot of $y_a(t)$ with $a = .2$, $\alpha = 1$ and $M_a = .5$



Plot of $y_b(t)$ with $a = .9$, $\alpha = 2$ and $M_a = 1$



Plot of Solution Surface

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