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Finite-Time Control for Linear Systems with Input Constraints

Hiroyuki Ichihara† and Hitoshi Katayama‡

Abstract— **This paper deals with finite-time boundedness (FTB) control with input constraints for linear time-invariant systems. A design method of state feedback FTB controllers with input constraints is proposed based on reachable sets of the state in finite-time period. A design method of observer-based output feedback FTB controllers is also proposed based on reachable sets of the estimated state. Both design methods include controller designs for finite-time stabilization. All design conditions are reduced to linear matrix inequalities. Numerical examples are shown to illustrate the proposed design methods.**

I. Introduction

Finite-time stability (FTS) requires that the state of a system does not exceed a certain bound during a specified time interval for given bound on the initial state. While Lyapunov stability is used to deal with the behavior of a system within a sufficiently long (or infinite) time interval, FTS is used to deal with the behavior of a system within a finite (or very short) time interval. Therefore there are real applications such as operations of missiles and space vehicles from an initial point to a final point in a specified time interval. Computationally tractable check conditions that guarantee FTS have been obtained and state feedback finite-time stabilization are considered for linear time-invariant systems using linear matrix inequalities (LMIs) [1]. The concept of FTS is also extended to that of FTB by introducing an exogenous input (2) and sufficient conditions for FTB are also given [2], [3]. Sufficient conditions for the existence of state feedback laws that guarantee FTB of a closed-loop system are given for linear continuous-time systems [2], [4] and for linear discrete-time systems [3]. Moreover sufficient conditions for the existence of output feedback controllers that guarantee FTS and FTB of a closed-loop system are given both for linear continuous-time and discrete-time systems [2]. In finite-time control problems, boundedness of the physical state of a system is of interest from the practical point of view and finite-time stabilization with observer-based output feedback controllers is considered for both linear continuous-time systems [3] and discrete-time systems [5].

In the above literatures on finite-time control, input signals could be larger as time has passed. Since trajectories do not always converge to the origin, input signals by state feedback laws could be larger and exceed a physical limitation on control. Similar situations may arise in the case of output feedback control. Input constraints in finite-time period are required to finite-time control from practical viewpoint. As far as Lyapunov stability, there exists literature on input constraint conditions using LMI [1], [6], [7]. However, they have not discussed any constraint conditions of finite-time period. In this paper, we give sufficient conditions for the existence of FTB (or FTS) controllers that satisfy a $\mathcal{T}$-period input constraint using reachable sets in finite-time periods. The obtained sufficient conditions are reduced to LMI conditions.

This paper is organized as follows. Section II gives preliminary results on a design method of state feedback FTB controller without input constraints. An extension to observer-based output feedback controller design is also given. In section III, $\mathcal{T}$-period input constraint is defined and state feedback controller design with input constraint is discussed. Section IV discusses output feedback controller design with input constraint. Section V gives numerical examples. Finally, section VI concludes with remarks.

Notations: Throughout this paper, let M_j be j-th row of a matrix M. $\mathsf{He}\{A\} := A + A^T$.

II. Preliminary Results

Consider

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t),\ x(0) = x_0 \quad (1)$$

$$\dot{w}(t) = Sw(t),\ w(0) = w_0 \quad (2)$$

$$y(t) = Cx(t) \quad (3)$$

where $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^{m_1}$ is the disturbance generated by the exosystem (2), $u \in \mathbf{R}^{m_2}$ is the control input, $y \in \mathbf{R}^p$ is the measurement and all matrices are of compatible dimensions. Then the following concepts are known.

Definition 2.1 ([4]): For given positive definite matrix Γ, $0 \le \delta_x < \varepsilon$ and $T > 0$, if $x^T(t)\Gamma x(t) < \varepsilon$, $t \in [0, T]$ whenever $x_0^T \Gamma x_0 \le \delta_x$, then the system $\dot{x}(t) = Ax(t)$ is said to be finite-time stable (FTS) with respect to $(\delta_x, \varepsilon, \Gamma, T)$.

Definition 2.2 ([2]): For given positive definite matrices Γ, Π, $0 \le \delta_x < \varepsilon$, $0 \le \delta_w$ and $T > 0$, if $x^T(t)\Gamma x(t) < \varepsilon$, $t \in [0, T]$ whenever $x_0^T \Gamma x_0 \le \delta_x$ and $w_0^T \Pi w_0 \le \delta_w$, then the system $\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t)$ and $\dot{w}(t) = Sw(t)$ is said to be finite-time bounded (FTB) with respect to $(\delta_x, \delta_w, \varepsilon, \Gamma, \Pi, T)$.

For the system (1) and (2), we also consider state feedback controllers

$$u(t) = Fx(t). \quad (4)$$

Then the closed-loop system (1), (2) and (4) is given by

$$\dot{x}(t) = A_F x(t) + B_1 w(t),\ x(0) = x_0, \quad (5)$$

†Department of Systems Design and Informatics, Kyushu Institute of Technology, Iizuka 820 8502, Japan.

‡Department of Electrical and Electronic Engineering, Shizuoka University, Hamamatsu 432 8561, Japan.

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and (2) where $A_F = A + B_2F$. Then we have the following result.

Lemma 2.1 ([2]): The system (25) and (2) is FTB with respect to (δ_x, δ_w, ε, Γ, Π, T) if there exist positive definite matrices $Q_1 \in \mathbf{R}^{n\times n}$, $Q_2 \in \mathbf{R}^{m_1\times m_1}$, a matrix $L \in \mathbf{R}^{m_2\times n}$ and a scalar $\alpha \geq 0$ such that

$$\mathsf{He}\left\{\begin{bmatrix} AQ_1 + B_2L & B_1 \\ 0 & Q_2S \end{bmatrix}\right\} - \alpha\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} < 0, \quad (6)$$

$$\frac{\delta_x}{\lambda_{\min}(\tilde{Q}_1)} + \lambda_{\max}(\tilde{Q}_2)\delta_w < \frac{\varepsilon\ e^{-\alpha T}}{\lambda_{\max}(\tilde{Q}_1)} \quad (7)$$

where $\tilde{Q}_1 = \Gamma^{\frac{1}{2}}Q_1\Gamma^{\frac{1}{2}}$ and $\tilde{Q}_2 = \Pi^{-\frac{1}{2}}Q_2\Pi^{-\frac{1}{2}}$. In this case the feedback gain F is given by $F = LQ_1^{-1}$.

Remark 2.1: If we set $S = 0$ and $\delta_w = 0$ in (10) and (11), then we can also derive sufficient conditions for FTS [4].

For the system (1)-(3), we consider output feedback controllers of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + B_2u(k) - K[y(t) - C\hat{x}(t)], \quad (8) \\ &\hat{x}(0) = 0, \\ u(t) &= F\hat{x}(t) \end{aligned}$$

where F and K are matrices of compatible dimensions. In the finite-time control problems, boundedness of the physical state of the system is of interest from the practical point of view, we want to find a controller (8) such that the system

$$\begin{aligned} \dot{x}(t) &= A_Fx(t) + B_1w(t) - B_2Fe(t), \\ \dot{w}(t) &= Sw(t) \end{aligned}$$

is FTB with respect to $(\delta_x, \delta_w, \epsilon, \Gamma, \Pi, T)$ where $A_F = A + B_2F$ and $e = x - \hat{x}$. To find such observer-based output feedback controllers, we assume that a state feedback controller $u(t) = Fx(t)$, which makes the system (1) and (2) FTB with respect to $(\delta_x, \delta_w, \epsilon, \Gamma, \Pi, T)$ (or the system (1) with $w \equiv 0$ FTS with respect to $(\delta_x, \epsilon, \Gamma, T)$) exists and has been designed.

The closed-loop system (1)-(3) with (8) can be written as

$$\begin{aligned} \dot{x}(t) &= A_Fx(t) + B_F\tilde{w}(t),\ x(0) = x_0, \\ \dot{\tilde{w}}(t) &= \tilde{S}\tilde{w}(t),\ \tilde{w}(0) = \begin{bmatrix} w_0^T & x_0^T \end{bmatrix}^T \end{aligned} \quad (9)$$

where

$$B_F = \begin{bmatrix} B_1 & -B_2F \end{bmatrix},\ \tilde{w} = \begin{bmatrix} w \\ e \end{bmatrix},\ \tilde{S} = \begin{bmatrix} S & 0 \\ B_1 & A_K \end{bmatrix}$$

and $A_K = A + KC$. For $e = 0$, the system (9) is FTB (or FTS) while the presence of a nonzero e may bring the norm of the state $x(t)$ outside the bound ϵ. Hence we want to design an observer gain K in (8) such that the FTB property of the system (25) and (2) is not lost in the presence of the estimation error. Note that the bound on the initial condition of the exosystem in (9) satisfies

$$w_0^T\Pi w_0 + x_0^T\Gamma x_0 \leq \delta_w + \delta_x.$$

Then if the system (9) is FTB with respect to $(\delta_x, \delta_w + \delta_x, \Gamma, \mathrm{diag}\{\Pi, \Gamma\}, T)$, then the closed-loop system (1)-(3) and (8) is FTB with respect to $(\delta_x, \delta_w, \Gamma, \Pi, T)$.

Lemma 2.2 ([2]): If there exist positive definite matrices P_1, $R \in \mathbf{R}^{n\times n}$, $P_2 \in \mathbf{R}^{m_1\times m_1}$ and a matrix $M \in \mathbf{R}^{n\times p}$ and a scalar $\alpha \geq 0$ such that

$$\mathsf{He}\left\{\begin{bmatrix} P_1A_F & P_1B_F \\ 0 & H_{22}^1 \end{bmatrix}\right\} - \alpha\begin{bmatrix} P_1 & 0 \\ 0 & H_{22}^2 \end{bmatrix} < 0, \quad (10)$$

$$\left[\lambda_{\max}(\tilde{P}_1) + \lambda_{\max}(\tilde{R})\right]\delta_x + \lambda_{\max}(\tilde{P}_2)\delta_w < \varepsilon e^{-\alpha T}\lambda_{\min}(\tilde{P}_1) \quad (11)$$

then an observer-based output feedback controller (8) makes the system (1)-(3) FTB with respect to $(\delta_x, \delta_w, \Gamma, \Pi, T)$ where

$$H_{22}^1 = \begin{bmatrix} P_2S & 0 \\ RB_1 & RA + MC \end{bmatrix},\ H_{22}^2 = \begin{bmatrix} P_2 & 0 \\ 0 & R \end{bmatrix},$$

$$\tilde{P}_1 = \Gamma^{-\frac{1}{2}}P_1\Gamma^{-\frac{1}{2}},\ \tilde{P}_2 = \Pi^{-\frac{1}{2}}P_2\Pi^{-\frac{1}{2}},\ \tilde{R} = \Gamma^{-\frac{1}{2}}R\Gamma^{-\frac{1}{2}}.$$

In this case the observer gain K is given by $K = R^{-1}M$.

Remark 2.2: If we set $S = 0$ and $\delta_w = 0$ in (10) and (11), then we can also derive sufficient conditions for the existence of observer-based output feedback FTS controllers [2].

III. State Feedback Controller Design with Input Constraints

Consider $\mathcal{T}$-period input constraints such that

$$|u_j(t)| \leq u_j^{\max},\ j = 1, \ldots, m_2,\ t \in [0, \mathcal{T}] \quad (12)$$

where $\mathcal{T}$ is a design parameter satisfying $0 \leq \mathcal{T} \leq T$. Then we want to design FTB state feedback controllers (4) satisfying (12) for the system (1) and (2). Using reachable set of the state given by Lyapunov-like functions, we estimate the maximum magnitude of the input signals.

Theorem 3.1: There exist state feedback FTB controllers that satisfy (12) for the system (1) and (2) if there exist positive definite matrices $Q_1 \in \mathbf{R}^{n\times n}$, $Q_2 \in \mathbf{R}^{m_1\times m_1}$, a matrix $L \in \mathbf{R}^{m_2\times n}$ and a scalar $\alpha \geq 0$ such that (6), (7) and

$$\begin{bmatrix} (u_j^{\max})^2/d_{\mathcal{T}} & L_j \\ L_j^T & Q_1 \end{bmatrix} \geq 0,\ j = 1, \ldots, m_2 \quad (13)$$

where

$$d_{\mathcal{T}} = \begin{cases} \dfrac{\delta_x}{\lambda_{\min}(\tilde{Q}_1)}, & \mathcal{T} = 0, \\ e^{\alpha\mathcal{T}}\left[\dfrac{\delta_x}{\lambda_{\min}(\tilde{Q}_1)} + \lambda_{\max}(\tilde{Q}_2)\delta_w\right], & \mathcal{T} > 0, \end{cases}$$

$\tilde{Q}_1 = \Gamma^{\frac{1}{2}}Q_1\Gamma^{\frac{1}{2}}$ and $\tilde{Q}_2 = \Pi^{-\frac{1}{2}}Q_2\Pi^{-\frac{1}{2}}$. In this case the feedback gain F is given by $F = LQ_1^{-1}$.

Proof: See the appendix. ∎

To obtain more numerically tractable sufficient conditions for FTB with $\mathcal{T}$-period input constraints, we assume Q_1 and Q_2 in Theorem 3.1 satisfy

$$\lambda_1 I < \tilde{Q}_1 < I,\ \tilde{Q}_2 < \lambda_2 I \quad (14)$$

and

$$\begin{bmatrix} \varepsilon e^{-\alpha T} - \lambda_2 \delta_w & \delta_x^{\frac{1}{2}} \\ \delta_x^{\frac{1}{2}} & \lambda_1 \end{bmatrix} > 0. \quad (15)$$

Then using Shur complement formula for (15), we obtain

$$\frac{\delta_x}{\lambda_1} + \lambda_2 \delta_w < \varepsilon e^{-\alpha T}.$$

Hence

$$\begin{aligned} & \frac{\delta_x}{\lambda_{\min}(\tilde{Q}_1)} + \lambda_{\max}(\tilde{Q}_2)\delta_w \\ < \quad & \frac{\delta_x}{\lambda_1} + \lambda_2 \delta_w < \varepsilon e^{-\alpha T} < \frac{\varepsilon e^{-\alpha T}}{\lambda_{\max}(\tilde{Q}_1)}, \end{aligned} \quad (16)$$

which satisfy (11). Using (16), we can take an upper bounds on $d_{\mathcal{T}}$ such as

$$\bar{d}_{\mathcal{T}} := \varepsilon e^{\alpha(\mathcal{T}-T)} > d_{\mathcal{T}}$$

for $\mathcal{T} > 0$ and

$$\bar{d}_0 := \frac{\delta_x}{\lambda_1} > \frac{\delta_x}{\lambda_{\min}(\tilde{Q}_1)} = d_0.$$

If we assume Q_1 and L satisfy

$$\begin{bmatrix} (u_j^{\max})^2/\bar{d}_{\mathcal{T}} & L_j \\ L_j^T & Q_1 \end{bmatrix} > 0, \quad j = 1, \ldots, m_2, \quad (17)$$

then (13) holds. In particular, for $\mathcal{N} = 0$, we may assume Q_1, L and λ_1 satisfy

$$\begin{bmatrix} \lambda_1 (u_j^{\max})^2/\delta_x & L_j \\ L_j^T & Q_1 \end{bmatrix} > 0, \quad j = 1, \ldots, m_2. \quad (18)$$

Corollary 3.1: There exist state feedback FTB controllers that satisfy (12) for the system (1) and (2) if there exist positive definite matrices $Q_1 \in \mathbf{R}^{n \times n}$, $Q_2 \in \mathbf{R}^{m_1 \times m_1}$, a matrix $L \in \mathbf{R}^{m_2 \times n}$, scalars $\alpha \geq 0$ and $\lambda_i > 0$, $i = 1, 2$ such that (6), (14), (15) and (17) where

$$\bar{d}_{\mathcal{T}} = \begin{cases} \delta_x/\lambda_1, \ \mathcal{N} = 0, \\ \varepsilon e^{\alpha(\mathcal{T}-T)}, \ \mathcal{N} \geq 1, \end{cases}$$

$\tilde{Q}_1 = \Gamma^{\frac{1}{2}} Q_1 \Gamma^{\frac{1}{2}}$ and $\tilde{Q}_2 = \Pi^{-\frac{1}{2}} Q_2 \Pi^{-\frac{1}{2}}$. In this case the feedback gain F is given by $F = LQ_1^{-1}$.

Corollary 3.2: There exist state feedback FTB controllers that satisfy (12) for the system (1) with $w \equiv 0$ if there exist a positive definite matrix $Q \in \mathbf{R}^{n \times n}$, a matrix $L \in \mathbf{R}^{m_2 \times n}$, scalars $\alpha \geq 0$ and $\lambda > 0$ such that

$$\begin{aligned} & \mathsf{He}\{AQ + B_2 L\} - \alpha Q < 0, \\ & \lambda I < \tilde{Q} < I, \quad 1 < e^{-\alpha T} \frac{\varepsilon}{\delta_x} \lambda, \\ & \begin{bmatrix} \lambda (u_j^{\max})^2/(e^{\alpha \mathcal{T}} \delta_x) & L_j \\ L_j^T & Q \end{bmatrix} \geq 0, \quad j = 1, \ldots, m_2 \end{aligned}$$

where $\tilde{Q} = \Gamma^{\frac{1}{2}} Q \Gamma^{\frac{1}{2}}$. In this case F is given by $F = LQ_1^{-1}$.

IV. Observer-Based Output Feedback Controller Design with Input Constraints

Here the input constraints are discussed for observer-based output feedback controller design. Due to the output feedback controllers (8), reachable sets of the state are not available to estimate the maximum magnitude of the inputs. In contrast with the state feedback case, we focus on searching reachable sets of the estimated state. We assume that a state feedback controller $u(t) = Fx(t)$, which makes the system (1) and (2) FTB with respect to $(\delta_x, \delta_w, \varepsilon, \Gamma, \Pi, T)$ (or the system (1) with $w \equiv 0$ FTS with respect to $(\delta_x, \varepsilon, \Gamma, T)$) and satisfies (12) exists and has been designed. Thanks to the observer-based output feedback controllers (8), it is not required to impose the 0-period input constraints because

$$u_j(0) = F_j \hat{x}(0) = 0, \ j = 1, \ldots, m.$$

Thus we may consider $\mathcal{T}$-period input constrains for $\mathcal{T} > 0$. The subsequent results represent design methods of observer-based output feedback FTB (FTS) controllers with $\mathcal{T}$-period input constrains for $\mathcal{T} > 0$.

Theorem 4.1: There exist output feedback FTB controllers (8) that satisfy (12) for the system (1)-(3) if there exist positive definite matrices P_1, $R \in \mathbf{R}^{n \times n}$, $P_2 \in \mathbf{R}^{m_1 \times m_1}$, a matrix $M \in \mathbf{R}^{n \times p}$, scalars $\alpha \geq 0$ and $\mu > 1$ such that (10), (11),

$$\begin{bmatrix} (u_j^{\max})^2/(\mu d_{\mathcal{T}}) & F_j \\ F_j^T & R \end{bmatrix} \geq 0, \ j = 1, \ldots, m_2, \quad (19)$$

$$\mu P_1 \succeq P_1 + R, \quad (20)$$

where

$$d_{\mathcal{T}} = e^{\alpha \mathcal{T}} \left\{ \left[\lambda_{\max}(\tilde{P}_1) + \lambda_{\max}(\tilde{R}) \right] \delta_x + \lambda_{\max}(\tilde{P}_2)\delta_w \right\},$$

$\tilde{P}_1 = \Gamma^{-\frac{1}{2}} P_1 \Gamma^{-\frac{1}{2}}$, $\tilde{P}_2 = \Pi^{-\frac{1}{2}} P_2 \Pi^{-\frac{1}{2}}$ and $\tilde{R} = \Gamma^{-\frac{1}{2}} R \Gamma^{-\frac{1}{2}}$. In this case the observer gain K is given by $K = R^{-1}M$.

Proof: The proof is in the appendix. ■

To obtain more numerically tractable sufficient conditions for FTB with $\mathcal{T}$-period input constraints, we assume P_1, P_2 and R in Theorem 4.1 satisfy

$$\lambda_1 I < \tilde{P}_1 < \lambda_2 I, \ 0 < \tilde{R} < \lambda_3 I, \ 0 < \tilde{P}_2 < \lambda_4 I, \quad (21)$$

$$(\lambda_2 + \lambda_3)\delta_x + \lambda_4 \delta_w < \varepsilon e^{-\alpha T} \lambda_1 \quad (22)$$

and

$$\mu \lambda_1 > \lambda_2 + \lambda_3. \quad (23)$$

Then we have

$$\begin{aligned} & \left[\lambda_{\max}(\tilde{P}_1) + \lambda_{\max}(\tilde{R}) \right] \delta_x + \lambda_{\max}(\tilde{P}_2)\delta_w \\ < \quad & (\lambda_2 + \lambda_3)\delta_x + \lambda_4 \delta_w < \varepsilon e^{-\alpha T} \lambda_1 \\ < \quad & \varepsilon e^{-\alpha T} \lambda_{\min}(\tilde{P}_1) \end{aligned} \quad (24)$$

and

$$\mu \tilde{P}_1 > \mu \lambda_1 I > (\lambda_2 + \lambda_3) I > \tilde{P}_1 + \tilde{R},$$

which satisfy (11) and (20), respectively. Using (24), we can take an upper bounds on $d_{\mathcal{T}}$ such as

$$\begin{aligned}\lambda_1 \bar{d}_{\mathcal{T}} &= \lambda_1 \varepsilon e^{\alpha(\mathcal{T}-T)} = e^{\alpha\mathcal{T}}(\varepsilon e^{-\alpha T}\lambda_1) \\ &> e^{\alpha\mathcal{T}}\left\{\left[\lambda_{\max}(\tilde{P}_1)+\lambda_{\max}(\tilde{R})\right]\delta_x + \lambda_{\max}(\tilde{P}_2)\delta_w\right\} \\ &= d_{\mathcal{T}}\end{aligned}$$

for $\mathcal{T} > 0$ where $\bar{d}_{\mathcal{T}} := \varepsilon e^{\alpha(\mathcal{T}-T)}$. For this upper bound, if we assume R and β satisfy

$$\begin{bmatrix}\beta(u_j^{\max})^2/\bar{d}_{\mathcal{T}} & \beta F_j \\ \beta F_j^T & R\end{bmatrix} \geq 0,\ j = 1,\ldots,m_2 \qquad (25)$$

then (19) holds where $\beta = \mu\lambda_1$.

Corollary 4.1: There exist output feedback FTB controllers (8) that satisfy (12) for the system (1)-(3) if there exist positive definite matrices P_1, $R \in \mathbf{R}^{n\times n}$, $P_2 \in \mathbf{R}^{m_1\times m_1}$, a matrix $M \in \mathbf{R}^{n\times p}$, scalars $\alpha \geq 0$, $\beta > 0$ and $\lambda_i > 0$, $i = 1,2,3,4$, such that (10), (21)-(23), and (25) where $\bar{d}_{\mathcal{T}} = \varepsilon e^{\alpha(\mathcal{T}-T)}$, $\tilde{P}_1 = \Gamma^{-\frac{1}{2}}P_1\Gamma^{-\frac{1}{2}}$, $\tilde{P}_2 = \Pi^{-\frac{1}{2}}P_2\Pi^{-\frac{1}{2}}$ and $\tilde{R} = \Gamma^{-\frac{1}{2}}R\Gamma^{-\frac{1}{2}}$. In this case the observer gain K is given by $K = R^{-1}M$.

Remark 4.1: If the LMI problem in Corollary 4.1 has a feasible solution, then $\mu > 1$ holds since $\beta > \lambda_2 + \lambda_3 > \lambda_1$ from (21) and (23).

Corollary 4.2: There exist output feedback FTS controllers (8) that satisfy (12) for the system (1) with $w \equiv 0$ and (3) if there exist positive definite matrices P_1, $R \in \mathbf{R}^{n\times n}$, a matrix $M \in \mathbf{R}^{n\times p}$, scalars $\alpha \geq 0$, $\beta > 0$ and $\lambda_i > 0$, $i = 1,2,3$, such that

$$\text{He}\left\{\begin{bmatrix}P_1A_F & -P_1B_2F \\ 0 & H\end{bmatrix}\right\} - \alpha\begin{bmatrix}P_1 & 0 \\ 0 & R\end{bmatrix} < 0,$$

(21), (22) with $\delta_w = 0$, (23), and (25) where $H = RA+MC$, $\bar{d}_{\mathcal{T}} = \varepsilon e^{\alpha(\mathcal{T}-T)}$, $\tilde{P}_1 = \Gamma^{-\frac{1}{2}}P_1\Gamma^{-\frac{1}{2}}$ and $\tilde{R} = \Gamma^{-\frac{1}{2}}R\Gamma^{-\frac{1}{2}}$. In this case the observer gain K is given by $K = R^{-1}M$.

V. Numerical Example

Consider the system (1)-(3) where

$$A = \begin{bmatrix} 0 & 1 \\ 0.1 & -0.1\end{bmatrix},\ B_1 = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix},\ B_2 = \begin{bmatrix}0 \\ 0.1\end{bmatrix},$$
$$S = \begin{bmatrix}0 & 1 \\ -0.5 & 0\end{bmatrix},\ C = \begin{bmatrix}1 & 0.5\end{bmatrix}.$$

We shall design state feedback FTB controllers by Corollary 3.1 and output feedback FTB controllers by Corollary 4.1, respectively. To find feasible solutions of LMIs in the corollaries, we use YALMIP [8] and SeDuMi [9] on Matlab.

We first design state feedback controllers (4) which make the system FTB with respect to $(\delta_x, \delta_w, \varepsilon, \Gamma, \Pi, T) = (1, 1, 50, I_2, I_2, 4.0)$. We set $\alpha = 0.5$ in Corollary 3.1 and we design two cases $\mathcal{T} = 0$ and 4.0: We shall design state feedback FTB controllers with 0-period and 4.0-period input constraint. In the 0-period design with $u_{\max} = 1.0$, we obtain $F = \begin{bmatrix}-0.7395 & -0.5121\end{bmatrix}$. In the 4.0-period design with $u_{\max} = 4.5$, we obtain $F = \begin{bmatrix}-0.5795 & -0.5322\end{bmatrix}$. The reachable sets and the input constraints of the both designs are shown in Fig. 1. We can see that the reachable sets are inside of the input constraints.

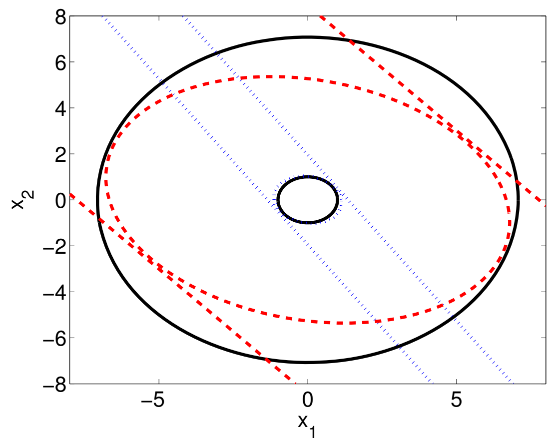

Fig. 1. Reachable sets and input constraints: The inner and outer solid ellipsoids represent $x^T\Gamma x \leq \delta_x$ and $x^T\Gamma x < \varepsilon$, respectively. The dotted ellipsoid and lines represent a reachable set $\mathcal{E}_0$ from (26) and an input constraint $|Fx| \leq 1.0$ by Corollary 3.1 at $\mathcal{T} = 0$. The dashed ellipsoid and lines represent $\mathcal{E}_{4.0}$ and $|Fx| \leq 4.5$ by Corollary 3.1 at $\mathcal{T} = 4.0$.

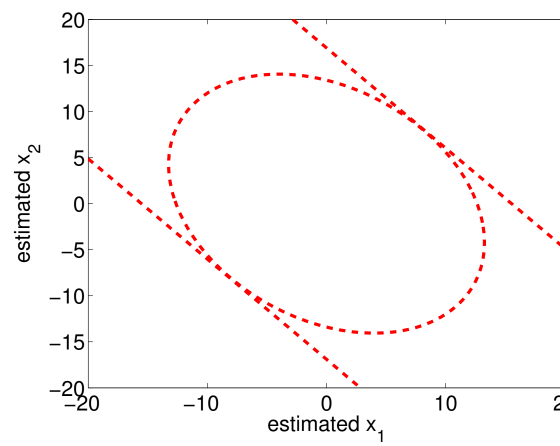

Fig. 2. Reachable sets of the estimated state and input constraints: The dashed ellipsoid and lines represent a reachable set $\mathcal{E}_{4.0}$ from (33) and an input constraint $|Fx| \leq 100.0$ by Corollary 4.1 at $\mathcal{T} = 4.0$.

We also design output feedback controllers (8) which make the system FTB with respect to $(1, 1, 100, I_2, I_2, 4.0)$. From the above result, we adopt a state feedback gain F by $F = \begin{bmatrix}-0.5795 & -0.5322\end{bmatrix}$ which make the system FTB with respect to $(1, 1, 100, I_2, I_2, 4.0)$ by state feedback control. We set $\alpha = 0.5$ in Corollary 3.1 and we shall design output feedback FTB controllers with 4.0-period input constraint. In the 4.0-period design with $u_{\max} = 100.0$, we obtain $K = \begin{bmatrix}-5.7712 & -3.7445\end{bmatrix}^T$ and $\mu = 5.0327$. The reachable sets and the input constraints of the both designs are shown in Fig. 2. We can see that the reachable sets of the estimated state are inside of the input constraints.

VI. Conclusions

We have proposed FTS/FTB control with input constraints for linear time-invariant systems. In order to evaluate the maximum magnitude of the control input signals, we estimate reachable set of the state from the initial time to a required finite-time periods. On the basis of such knowledge, we have proposed design methods of state feedback FTS/FTB controllers design whose design conditions are reduced to LMIs. We have also proposed design methods of observer-based output feedback FTS/FTB controllers based on reachable sets of the estimated state. Numerical examples have been shown to illustrate the proposed state/output feedback FTB design methods.

APPENDIX

PROOF OF THEOREM 3.1

From (6) and (7), the system (25) and (2) is FTB with respect to $(\delta_x,\ \delta_w,\ \varepsilon,\ \Gamma,\ \Pi,\ T)$. Then for

$$V(x(t),w(t)) = x^T(t)Q_1^{-1}x(t) + w^T(t)Q_2w(t),$$

$\dot{V}(x(t),w(t)) < \alpha V(x(t),w(t))$ and $V(x(t),w(t)) < e^{\alpha t}V(x(0),w(0))$ for $t \in (0,\mathcal{T}]$. Then we have

$$x^T(t)Q_1^{-1}x(t) < e^{\alpha t}\left[x_0^TQ_1^{-1}x_0 + w_0^TQ_2w_0\right] \le d_t$$

for $t \in (0,\mathcal{T}]$. For $t=0$, we have

$$x_0^TQ_1^{-1}x_0 \le \frac{\delta_x}{\lambda_{\min}(\tilde{Q}_1)} = d_0.$$

Since the control input depends on the state, it is required to analyze reachable sets of the state for every time in $[0,\mathcal{T}]$. Define the reachable set by

$$\begin{array}{l}\mathcal{E}_t := \left\{\ z \in \mathbf{R}^n \mid z^TQ_1^{-1}z < d_t\right\},\ t \in (0,\mathcal{T}],\\ \mathcal{E}_0 := \left\{\ z \in \mathbf{R}^n \mid z^TQ_1^{-1}z \le d_0\right\}.\end{array} \tag{26}$$

Using $\alpha \ge 0$ and

$$z^TQ_1^{-1}z \le d_0 \le e^{\alpha\cdot 0}\left[\frac{1}{\lambda_{\min}(\tilde{Q}_1)}\delta_x + \lambda_{\max}(\tilde{Q}_2)\delta_w\right],$$

$d_{t_1} \le d_{t_2}$ holds for $t_1,t_2 \in (0,\mathcal{T}]$ with $t_1 \le t_2$. Then we can see the relation

$$\mathcal{E}_{t_1} \subseteq \mathcal{E}_{t_2}.$$

Thus we obtain

$$\cup_{t\in[0,\mathcal{T}]}\mathcal{E}_t = \mathcal{E}_{\mathcal{T}}. \tag{27}$$

Using (27), for $\mathcal{T} > 0$, we have

$$\begin{array}{rl} & \max_{t\in[0,\mathcal{T}]} |u_j(t)|^2 \\ = & \max_{t\in[0,\mathcal{T}]} |(LQ_1^{-1}x(t))_j|^2 \\ \le & \max_{z\in\cup_{t\in[0,\mathcal{T}]}\mathcal{E}_t} |(LQ_1^{-1}z)_j|^2 \\ = & \max_{z\in\mathcal{E}_{\mathcal{T}}} |(LQ_1^{-1}z)_j|^2 \\ \le & \max_{\|(d_{\mathcal{T}}Q_1)^{-\frac{1}{2}}z\|_2=1} |(LQ_1^{-1}z)_j|^2 \\ = & \max_{\|(d_{\mathcal{T}}Q_1)^{-\frac{1}{2}}z\|_2=1} \left|\left(d_{\mathcal{T}}^{\frac{1}{2}}LQ_1^{-\frac{1}{2}}\left[(d_{\mathcal{T}}Q_1)^{-\frac{1}{2}}z\right]\right)_j\right|^2 \\ = & \left\|\left[d_{\mathcal{T}}^{\frac{1}{2}}L_jQ_1^{-\frac{1}{2}}\right]^T\right\|_2^2 \\ = & d_{\mathcal{T}}L_jQ_1^{-1}L_j^T. \end{array}$$

Applying Schur complement formula to (13), we obtain

$$\max_{t\in[0,\mathcal{T}]} |u_j(t)|^2 \le d_{\mathcal{T}}L_jQ_1^{-1}L_j^T \le (u_j^{\max})^2$$

for $j = 1,\dots,m_2$. For $\mathcal{T}=0$, we have

$$|u_j(0)|^2 = |(F\hat{x}(0))_j|^2 \le \max_{z\in\mathcal{E}_0}|(Fz)_j|^2 \le d_0L_jQ_1^{-1}L_j^T$$

for $j = 1,\dots,m_2$. Using Shur complement formula to (13) again, we obtain

$$|u_j(0)|^2 \le d_0L_jQ_1^{-1}L_j^T \le (u_j^{\max})^2,$$

thus we have the assertion.

PROOF OF THEOREM 4.1

From (10) and (11), the system (9) is FTB with respect to $(\delta_x, \delta_w+\delta_x, \epsilon, \Gamma, \mathrm{diag}\{\Pi,\Gamma\}, T)$. Then for

$$\begin{array}{l}V(x(t),w(t),e(t)) \\ = x^T(t)P_1x(t) + w^T(t)P_2w(t) + e^T(t)Re(t),\end{array}$$

we have $\dot{V}(x(t),w(t),e(t)) < \alpha V(x(t),w(t),e(t))$ and

$$V(x(t),w(t),e(t)) < e^{\alpha t}V(x(0),w(0),e(0)),\ t \in (0,\mathcal{T}].$$

Then, for $t \in (0,\mathcal{T}]$, we have

$$\begin{array}{rl} & x^T(t)P_1x(t) + w^T(t)P_2w(t) \\ & \quad +(x(t)-\hat{x}(t))^TR(x(t)-\hat{x}(t)) \\ < & e^{\alpha t}\left[x_0^TP_1x_0 + w_0^TP_2w_0 + x_0^TRx_0\right] \le d_t. \end{array} \tag{28}$$

Since the control input depends on the estimated state, we need to analyze a reachable set of the estimated state for every time in $[0,\mathcal{T}]$. For $t \in (0,\mathcal{T}]$, using (33), we may consider an optimization problem as follows:

$$\begin{array}{ll} \inf_{x,\hat{x},w} & -\hat{x}^TR\hat{x} \\ \text{s.t.} & f(x,\hat{x},w) := x^TP_1x + w^TP_2w \\ & \qquad +(x-\hat{x})^TR(x-\hat{x}) - d_t' \le 0 \end{array} \tag{29}$$

where $d_t' < d_t$. If we have a solution of the problem (29), then we can obtain an upper bound on $\hat{x}(t)^TR\hat{x}(t)$ for $t \in (0,\mathcal{T}]$. It is known that strong duality holds for non-convex quadratic optimization problem with single quadratic constraint and their Lagrange dual problems [10] under Slater's constraint qualification. In this case, the constraint in (29) is a convex set, so that there exists a $(x,\hat{x},w)$ with $f(x,\hat{x},w) < 0$. Thus the constraint satisfies the qualification. The Lagrangian of (29) is

$$L(x,\hat{x},w,\mu) = -\hat{x}^TR\hat{x} + \mu f(x,\hat{x},w),\ \mu \ge 0$$

and the dual function is

$$\begin{array}{rl} g(\mu) = & \inf_{x=x^*,\hat{x}=\hat{x}^*,w=w^*} L(x,\hat{x},w,\mu) \\ = & \left\{\begin{array}{l} -\mu d_k,\ (20), \\ -\infty,\ \text{otherwise}\end{array}\right. \end{array}$$

where (20) and $\mu > 1$ make

$$\frac{\partial^2}{\partial p^2}L(x,\hat{x},w,\mu) = \left[\begin{array}{ccc} 2\mu(P_1+R) & -2\mu R & 0 \\ -2\mu R & 2(\mu-1)R & 0 \\ 0 & 0 & 2\mu P_2 \end{array}\right]$$

positive semidefinite. Also, the points

$$p^* = \left[\begin{array}{ccc} (x^*)^T & (\hat{x}^*)^T & (w^*)^T \end{array} \right]^T,$$

which achieve the infimum of L, satisfy

$$\begin{aligned} P_1 x^* + R(x^* - \hat{x}^*) &= 0, & (30)\\ \hat{x}^* + \mu(x^* - \hat{x}^*) &= 0, & (31)\\ w^* &= 0 & (32) \end{aligned}$$

from $\partial L/\partial p = 0$. Using (30)-(32), we have

$$\begin{aligned} & L(x^*, \hat{x}^*, w^*, \mu) \\ &= -(\hat{x}^*)^T R\hat{x}^* + \mu(x^*)^T P_1 x^* + \mu(w^*)^T P_2 w^* \\ &\qquad + \mu(x^* - \hat{x}^*)^T R(x^* - \hat{x}^*) - \mu d_t' \\ &= -(\hat{x}^*)^T R\hat{x}^* + \mu(x^*)^T P_1 x^* - (x^* - \hat{x}^*)^T R\hat{x}^* - \mu d_t' \\ &= \mu(x^*)^T P_1 x^* - (x^*)^T R\hat{x}^* - \mu d_t' \\ &= \mu(x^*)^T P_1 x^* - (x^*)^T(\mu P_1 x^*) - \mu d_t' \\ &= -\mu d_t'. \end{aligned}$$

Then the Lagrange dual problem of the problem (29) is

$$\max_{\mu=\mu^*} \ -\mu d_t \ \text{ s.t. } \ (20).$$

Since the optimal value of the dual problem is $-\mu^* d_t$, it is also the optimal value of the problem (29). Thus we obtain

$$\hat{x}^T(t) R\hat{x}(t) \le \mu^* d_t < \mu^* d_t, \ t \in (0, \mathcal{T}].$$

Define reachable set of the estimated state by

$$\mathcal{E}_t := \left\{ \ z \in \mathbf{R}^n \mid z^T R z < \mu^* d_t \right\}, \ t \in (0, \mathcal{T}]. \qquad (33)$$

Since $\alpha \ge 0$, $d_{t_1} \le d_{t_2}$ holds for $0 < t_1 \le t_2$. Then we have

$$\mathcal{E}_{t_1} \subseteq \mathcal{E}_{t_2}$$

Thus we obtain $\cup_{t \in (0,\mathcal{T}]} \mathcal{E}_t = \mathcal{E}_{\mathcal{T}}$. Then for $\mathcal{T} > 0$ we have

$$\begin{aligned} \max_{t\in[0,\mathcal{T}]} |u_j(t)|^2 &= \max_{t\in[0,\mathcal{T}]} |(F\hat{x}(t))_j|^2 \\ &\le \max_{z\in\cup_{t\in(0,\mathcal{T}]}\mathcal{E}_t} |(Fz)_j|^2 \\ &= \max_{z\in\mathcal{E}_{\mathcal{T}}} |(Fz)_j|^2 \\ &= \max_{\|(\mu^* d_{\mathcal{T}})^{-\frac{1}{2}} R^{\frac{1}{2}} z\|_2 = 1} |(Fz)_j|^2 \\ &\le \mu^* d_{\mathcal{T}} F_j R^{-1} F_j^T. \end{aligned}$$

Applying Schur complement formula to (19), we obtain

$$\max_{t\in[0,\mathcal{T}]} |u_j(t)|^2 < \mu^* d_{\mathcal{T}} F_j R^{-1} F_j^T \le (u_j^{\max})^2$$

for $j = 1, \ldots, m_2$. Hence we have the assertion.

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